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Numerical approximation of finite life-span age-structured population models

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Abstract

Considering unbounded mortality rates causes difficulties in the numerical simulation of finite life-span population models. In this work, a suitable numerical technique is introduced in order to approximate the density solution of this kind of problems. This procedure isolates the principal complication for the problem: the approximation of the survival probability. Therefore, assuming second-order approximations to the survival probability, we propose a numerical method for the age-structured population model with the same order of convergence. The accuracy of the numerical approximations is proved and numerically illustrated.

KEYWORDS:
Age-structured population; finite life-span; survival probability; numerical methods; characteristics method; convergence analysis.

1 | INTRODUCTION

Numerical methods are a valuable tool for the analysis of physiologically structured population models. In this sense, many different procedures have been proposed for the numerical solution of deterministic age-structured population models in the literature (see [1], for example, and the references therein as a general overview). However, incorporating a finite life-span into the model produces additional problems in its numerical treatment, which are associated with the use of unbounded mortality rates. Moreover, the few attempts carried out to adapt to this particular situation only cover very particular cases.

Our starting point is the Lotka-McKendrick model describing the time evolution of a single one-sex population structured by age (for a valuable historical presentation of population models, see [8]). It is the most basic extension of Malthus model which structures individuals by their chronological age. The model assumes that the population is isolated and lives in an invariant environment with unlimited resources. All individuals are identical, except for their age. The stable population model, as it is also known, has become a crucial tool in Demography and Ecology, and it is used as a baseline model in epidemic systems (see the recent monographs [12,13,18]).

Let $u(a,t)$ represent the age density of individuals at time t. Here a denotes age, and $a_\dagger$ is its maximum value. The vital functions $\mu(a)$ and $\beta(a)$ represent, respectively, the age-specific mortality and fertility rates. Then, we consider that u is the solution of the following linear initial-boundary value problem

$$\frac{\partial u(a,t)}{\partial t} + \frac{\partial u(a,t)}{\partial a} + m(a)\, u(a,t) = 0, \qquad 0 < a < a_\dagger, \quad t > 0, \tag{1}$$

$$u(0,t) = \int_0^{a_\dagger} \beta(a)\, u(a,t)\, da, \qquad t > 0, \tag{2}$$

$$u(a,0) = u^0(a), \qquad 0 \le a < a_\dagger. \tag{3}$$

The problem consists of a first-order partial differential equation, the balance law, describing aging and dying; a boundary condition representing the inflow of newborns; and an initial condition providing the initial age distribution $u^0(a)$.

In the theoretical description of the solution, the survival probability

$$\pi(a) = \exp\left\{ -\int_0^a m(\sigma)\, d\sigma \right\}, \qquad 0 \le a < a_\dagger, \tag{4}$$

representing the probability at birth of living to age a, plays an important role [12]. In order to assure that the survival probability vanishes at the maximum age, we must assume

$$\int_0^{a_\dagger} m(a)\, da = +\infty. \tag{5}$$

So, if $a_\dagger$ is finite, then the mortality rate must be unbounded near the maximum age.

The numerical solution of (1)-(3) can be approached by different methods. However, proofs of convergence of the numerical approximations typically need some derivatives of the vital functions (the degree of accuracy depends on the order of the derivatives) to be bounded over the age interval. In any case, as Iannelli and Milner pointed out in their work [11], the usual finite-difference methods break down near the maximum age when the mortality is unbounded. They focused the difficulties on the approximation of the survival probability. In their work, they faced this problem by modifying some classical techniques to adequately approximate this probability function, and showing their deficiencies in certain situations associated with the growth of the mortality rate. In a recent work [2], we confirmed such difficulties and provided an alternative procedure for an efficient approximation of the survival function based on the use of proper quadrature rules to approximate the integral in (4)..

For the numerical approximation of the finite life-span age-structured population models, different approaches have been proposed. Some of them require an exact knowledge of the survival function [20,19,10,3,4,5]. Other works [15,7,14,16,6,9,17] (and more specifically, for the non-linear case with separable mortality) assume an age-specific part with a particular expression near the maximum age. In any case, all those methods are only useful for very specific kinds of mortality rates.

In Section 2, we propose a suitable numerical technique to approximate the age density of the finite life-span population model (1)-(3). The procedure splits the approach to the problem isolating the principal complication: the approximation of the survival probability. The technique we contemplate involves two numerical problems: the estimation of the survival probability (already considered in the numerical literature) and the discretization of a simpler model than the original one. In particular, assuming second-order approximations to the survival probability such as, for example, those obtained in [11,2], we propose a numerical method for the age-structured population model with the same order of convergence. In Section 3, we prove the convergence of the numerical scheme and, in Section 4, we show a numerical experiment that revalidates the previous theoretical result. Also, we show the consequences of considering lower-order approximations of the survival probability. Section 5 presents the conclusions.

2 | THE NUMERICAL METHOD

We introduce a change of variable very common in the theoretical analysis of the model [12]: we consider a new function $v(a,t)$, related to the original density $u(a,t)$, (the solution of (1)-(3)), by means of the survival probability (4), as

$$u(a,t) = \pi(a)\, v(a,t). \tag{6}$$

This new unknown function satisfies the following simplified problem

$$\frac{\partial v(a,t)}{\partial t} + \frac{\partial v(a,t)}{\partial a} = 0, \qquad 0 < a < a_\dagger, \quad t > 0, \tag{7}$$

$$v(0,t) = \int_0^{a_\dagger} \pi(a)\,\beta(a)\,v(a,t)\,da, \qquad t > 0, \tag{8}$$

$$v(a,0) = v^0(a), \qquad 0 \le a < a_\dagger, \tag{9}$$

where

$$v^0(a) = \frac{u^0(a)}{\pi(a)}, \qquad 0 \le a < a_\dagger. \tag{10}$$

In order to ensure that v^0 in (10), the initial data of the new problem, is a bounded function, we impose that

$$\lim_{a \to a_\dagger} \frac{u^0(a)}{\pi(a)} < +\infty, \tag{11}$$

which is a typical condition for the initial condition of the original problem, imposed in the theoretical analysis of the model.

Note that the change of variable factors out the unbounded mortality m from the partial differential equation (7); the new problem only depends on this function through the bounded survival probability in the boundary condition (8), representing the newborns, and the initial configuration (9).

Some authors have used this change to simplify the model and, then, approach it numerically: for example, in order to describe the evolution of the age profile of a general population [20], or to approximate the dynamics of a particular population [3,4,5]. However, in all these cases, it is essential to know the exact values of π to produce a convergent numerical method approximating the density function. Here, we will face the more general case in which we only have suitable estimations, instead of the exact expression, of the survival probability.

If we had approximations to the survival probability Π_j, on a discrete grid a_j, $j = 0, 1, \ldots, J$, inside the age interval, we would consider connecting numerical approximations U_j^n and V_j^n, to the solutions of problems (1)-(3) and (7)-(9), respectively, along the prescribed time levels t^n, $n = 0, 1, \ldots, N$, taking into account (6). Therefore, starting from an initial approximation U_j^0, $j = 0, 1, \ldots, J$, to the initial data (3), we propose to obtain numerical approximations U_j^n, at a positive time level, t^n, $n = 1, \ldots, N$, from the numerical approximations V_j^n, at the same time level, as follows:

- Preprocessing: Calculate the initial approximations V_j^0 from U_j^0 by means of

$$V_j^0 = \frac{U_j^0}{\Pi_j}, \qquad j = 0, 1, \ldots, J.$$

- Time evolution: Numerically solve the problem (7)-(9) to obtain V_j^n, $j = 0, 1, \ldots, J$, $n = 1, \ldots, N$.

- Postprocessing: Calculate U_j^n by means of the change

$$U_j^n = \Pi_j\, V_j^n, \qquad j = 0, 1, \ldots, J, \quad n = 1, \ldots, N.$$

The particular procedure we will design will depend on the properties of the estimations to the survival probability we have. On the one hand, as we mention in Section 1, different techniques have been proposed to approximate the survival probability in the finite life-span case [11,2]. All of them have proved useful when we assume some particular profiles, frequently used in Biology problems, for the mortality rate that appears in (4). However, for the theoretical analysis of convergence of these procedures, a specific behaviour of the mortality rate is assumed near the maximum age, that is, over an age interval $[a_*, a_\dagger)$, with $0 < a_* < a_\dagger$. So, the numerical method that we propose to approximate the solution of (1)-(3) (and, therefore, (7)-(9)), will take into account this age a_*, that marks the change in the particular properties of the mortality function, in the construction of the discretization grid associated to the age variable. On the other hand, if we consider second-order approximations to the survival probability, it is reasonable to propose a second-order numerical method for the numerical solution of (7)-(9). To this end, we will consider a method based on the integration along the characteristic lines: a standard procedure for the numerical approximation of age-structured population models [1].

Regarding to the discretization of the age variable, given a positive integer J^*, we define the step size as $h = a_*/J^*$, and introduce the discrete ages $a_j = j\,h$, $j = 0, 1, \dots, J$, where J is the nearest integer strictly less than $a_\dagger/h$. Then, we are guaranteed that $a_{J^*} = a_*$ and $0 < a_\dagger - a_J \leq h$ (which prevents the maximum age, where π vanishes, from being a grid point). If we consider the numerical integration along the time interval $[0, T]$, we will use the same step size h for the discretization of the time variable. So, we introduce the discrete times $t^n = n\,h$, $n = 0, 1, \dots N$, where $N = [T/h]$.

For the description and analysis of the method we use the following vector notation: at each time level t^n, $n = 0, 1, \dots, N$, the numerical solution is described by a $(J+1)$-dimensional vector $\mathbf{U}^n = [U_0^n, U_1^n, \dots, U_J^n]$, where U_j^n represents the numerical approximation to $u(a_j, t^n)$, $j = 0, 1, \dots, J$, solution of (1)-(3). In the same way, $\mathbf{V}^n = [V_0^n, V_1^n, \dots, V_J^n]$, where V_j^n represents the numerical approximation to $v(a_j, t^n)$, $j = 0, 1, \dots, J$, solution of (7)-(9). Vector $\boldsymbol{\Pi} = [\Pi_0, \Pi_1, \dots, \Pi_J]$ contains the approximations to the survival probability $\boldsymbol{\pi} = [\pi(a_0), \pi(a_1), \dots, \pi(a_J)]$. Also, we use this vector notation to represent the evaluations of the fertility rate $\boldsymbol{\beta} = [\beta(a_0), \beta(a_1), \dots, \beta(a_J)]$.

In our particular case, the solution of the simplified version of the partial differential equation (7) satisfies that, for each a, with $0 < a < a_\dagger$, and each $h > 0$, such that $a + h < a_\dagger$,

$$v(a+h, t+h) = v(a, t). \tag{12}$$

We will use this equality, instead of (7), for the numerical approximation of the solution to the problem at positive ages.

To approximate the integral that appears in the boundary condition (8), we use an open quadrature formula based on the composite trapezoidal quadrature rule: for the vector $\mathbf{Y} = [Y_0, Y_1, \dots, Y_J]$, that collects the function evaluations, we define

$$\mathcal{Q}_h(\mathbf{Y}) = hY_1 + \sum_{j=1}^{J-1} \frac{h}{2}\left(Y_j + Y_{j+1}\right) + (a_\dagger - a_J)Y_J. \tag{13}$$

Note that, at the first age subinterval, we use the rectangle rule based on the right node in order to obtain an explicit numerical method and; at the last one, the rectangle rule based on the left node in order to avoid the maximum age.

With this notation, the method providing approximations to the original problem (1)-(3) is described by the following stages:

- Initial datal: $u^0 \longrightarrow \mathbf{U}^0$

$$U_j^0 = u^0(a_j), \qquad j = 0, 1, \dots J. \tag{14}$$

- Preprocessing: $\mathbf{U}^0 \longrightarrow \mathbf{V}^0$

$$V_j^0 = \frac{U_j^0}{\Pi_j}, \qquad j = 0, 1, \dots J. \tag{15}$$

- Time evolution: $\mathbf{V}^n \longrightarrow \mathbf{V}^{n+1}$, $n = 0, 1, \dots, N-1$

$$V_{j+1}^{n+1} = V_j^n, \qquad j = 0, 1, \dots J-1, \tag{16}$$

$$V_0^{n+1} = \mathcal{Q}_h(\boldsymbol{\Pi} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^{n+1}). \tag{17}$$

- Postprocessing: $\mathbf{V}^n \longrightarrow \mathbf{U}^n$, $n = 1, 2, \dots, N$

$$U_j^n = \Pi_j V_j^n, \qquad j = 0, 1, \dots J. \tag{18}$$

Note that the survival probability (4) only vanishes at the maximum age, but $a_\dagger$ is not a grid point. Therefore, to be able to carry out (15), the approximations Π_j, $j = 0, 1, \dots, J$ must be non-zero. Equation (16) is the grid restriction of (12), and the vector products in (17), numerical approximation of (8), must be interpreted component-wise. Finally, in practice, we only calculate the approximations to the original problem when necessary: typically, (18) is only applied at $n = N$ to approximate the solution at the final time T.

3 | CONVERGENCE ANALYSIS

In this section we study the convergence of the numerical approximations obtained in the previous section over the fixed time interval $[0, T]$.

In order to compare the numerical and the theoretical solutions, at each time level t^n, $n = 0, 1, \ldots N$, we represent the grid restriction of the solution u of (1)-(3) as the vector $\mathbf{u}^n = [u(a_0, t^n), u(a_1, t^n), \ldots, u(a_J, t^n)]$. Similarly, for the grid restriction of the solution v of (7)-(9), we use the vector $\mathbf{v}^n = [v(a_0, t^n), v(a_1, t^n), \ldots, v(a_J, t^n)]$, $n = 0, 1, \ldots, N$.

For a $(J + 1)$-dimensional vector $\mathbf{Y} = [Y_0, Y_1, \ldots, Y_J]$, we consider the norms

$$||\mathbf{Y}||_1 = \sum_{j=0}^{J} h\,|V_j|, \qquad ||\mathbf{Y}||_\infty = \max_{j=0,\ldots,J} |V_j|.$$

Note that, for a vector vector $\mathbf{Y}$,

$$||\mathbf{Y}||_1 \leq \sum_{j=0}^{J} h\,||\mathbf{Y}||_\infty \leq (J+1)\,h\,||\mathbf{Y}||_\infty,$$

and then

$$||\mathbf{Y}||_1 \;\leq\; (a_\dagger + a_*)\,||\mathbf{Y}||_\infty. \tag{19}$$

On the other hand, for any two vectors $\mathbf{Y}$ and $\mathbf{Z}$, and assuming that $\mathbf{Y} \cdot \mathbf{Z}$ represents element-wise multiplication of the vectors, then

$$\big|\,Q_h(\mathbf{Y} \cdot \mathbf{Z})\,\big| \leq \frac{3\,h}{2}\,|Y_1\,Z_1| + \sum_{j=2}^{J-1} h\,|Y_j\,Z_j| + \frac{3\,h}{2}\,|Y_J\,Z_J|,$$

and then

$$\big|\,Q_h(\mathbf{Y} \cdot \mathbf{Z})\,\big| \;\leq\; \frac{3}{2}\,||\mathbf{Y}||_\infty\,||\mathbf{Z}||_1. \tag{20}$$

In the following result, assuming we have second-order approximations to the survival probability, we establish the same order of convergence of the numerical procedure. From now on, C will denote a positive constant which is independent of h, and possibly has different values in different places.

Theorem 1. Consider nonnegative vital functions $\beta \in C^2([0, a_\dagger])$, and $m \in C^2([0, a_\dagger))$ satisfying the divergence of its integral (5). Let $u \in C^2([0, a_\dagger] \times [0, T])$ be the solution of (1)-(3), and let v, a bounded function on $[0, a_\dagger) \times [0, T]$, be the solution of (7)-(9). Assume that Π_j, $j = 0, 1, \ldots, J$, are approximations to the survival probability π in (4) at the previously defined meshpoints $a_j = j\,h$, $j = 0, 1, \ldots, J$, such that, as $h \to 0$,

$$\Big|\,\Pi_j - \pi(a_j)\,\Big| = \mathcal{O}(h^2), \qquad j = 0, 1, \ldots, J, \tag{21}$$

and

$$\left|\,\frac{v^0(a_j)}{\Pi_j}\,\right| = \mathcal{O}(1), \qquad j = 0, 1, \ldots, J. \tag{22}$$

Then, the numerical approximations $\mathbf{U}^n$ and $\mathbf{V}^n$, $n = 0, 1, \ldots, N$, associated to u and v, respectively, obtained by means of the numerical method (14)-(18), satisfy, as $h \to 0$

$$\max_{n=0,1,\ldots,N} ||\,\mathbf{V}^n - \mathbf{v}^n\,||_\infty = \mathcal{O}(h^2), \tag{23}$$

and

$$\max_{n=0,1,\ldots,N} ||\,\mathbf{U}^n - \mathbf{u}^n\,||_\infty = \mathcal{O}(h^2). \tag{24}$$

Proof. Note that, from the convergence in the approximation to the survival probability values (21), the division that appears in (22) is well defined for h small enough because the survival probability only vanishes at the maximum age, and $a_\dagger$ is not a grid point.

Trivially, from the regularity conditions, the quadrature error produced by (13) satisfies that, for $n = 0, 1, \dots, N$

$$\varepsilon(t^n) = \left| \mathcal{Q}_h(\boldsymbol{\beta} \cdot \mathbf{u}^n) - \int_0^{a_\dagger} \beta(a)\, u(a, t^n)\, da \right| = \mathcal{O}(h^2), \quad \text{as} \quad h \to 0. \tag{25}$$

First of all, we analyze the error $||\mathbf{V}^n - \mathbf{v}^n||_1$, $n = 1, \dots, N$. On the one hand, by using the recursive formulas (16) and (12) about the numerical and theoretical solutions, respectively, we have

$$\left| V_j^n - v(a_j, t^n) \right| = \left| V_{j-1}^{n-1} - v(a_{j-1}, t^{n-1}) \right|, \quad j = 1, \dots, J. \tag{26}$$

On the other hand, from the boundary expressions (17), (8), and using the regularity conditions, the bound (20) associated to the quadrature rule, and the change of variable (6)

$$\begin{aligned}
\left| V_0^n - v(0, t^n) \right| &= \left| \mathcal{Q}_h(\boldsymbol{\Pi} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^n) - \int_0^{a_\dagger} \pi(a)\, \beta(a)\, v(a, t^n)\, da \right| \\
&\leq \left| \mathcal{Q}_h(\boldsymbol{\Pi} \cdot \boldsymbol{\beta} \cdot \mathbf{V}^n) - \mathcal{Q}_h(\boldsymbol{\Pi} \cdot \boldsymbol{\beta} \cdot \mathbf{v}^n) \right| \\
&+ \left| \mathcal{Q}_h(\boldsymbol{\Pi} \cdot \boldsymbol{\beta} \cdot \mathbf{v}^n) - \mathcal{Q}_h(\boldsymbol{\pi} \cdot \boldsymbol{\beta} \cdot \mathbf{v}^n) \right| \\
&+ \left| \mathcal{Q}_h(\boldsymbol{\pi} \cdot \boldsymbol{\beta} \cdot \mathbf{v}^n) - \int_0^{a_\dagger} \pi(a)\, \beta(a)\, v(a, t^n)\, da \right| \\
&\leq C\, ||\mathbf{V}^n - \mathbf{v}^n||_1 + C\, ||\boldsymbol{\Pi} - \boldsymbol{\pi}||_\infty + \varepsilon(t^n).
\end{aligned} \tag{27}$$

Therefore, from (26) and (27)

$$\begin{aligned}
||\mathbf{V}^n - \mathbf{v}^n||_1 &= \sum_{j=0}^{J} h \left| V_j^n - v(a_j, t^n) \right| = h \left| V_0^n - v(0, t^n) \right| + \sum_{j=0}^{J-1} h \left| V_{j+1}^n - v(a_{j+1}, t^n) \right| \\
&\leq C\, h\, ||\mathbf{V}^n - \mathbf{v}^n||_1 + C\, h\, ||\boldsymbol{\Pi} - \boldsymbol{\pi}||_\infty + h\, \varepsilon(t^n) + ||\mathbf{V}^{n-1} - \mathbf{v}^{n-1}||_1.
\end{aligned}$$

Taking h sufficiently small (such that, $1 - C\, h > 0$) and using a recursive argument like the discrete Gronwall Lemma, it follows that

$$||\mathbf{V}^n - \mathbf{v}^n||_1 \leq C \left(||\mathbf{V}^0 - \mathbf{v}^0||_1 + \sum_{m=1}^{N} h \left(||\boldsymbol{\Pi} - \boldsymbol{\pi}||_\infty + \varepsilon(t^n) \right) \right).$$

As a consequence of the order of convergence (21) and (25), in the approximation to the survival probability and the integral, respectively, for $n = 0, 1, \dots, N$, we have

$$||\mathbf{V}^n - \mathbf{v}^n||_1 = \mathcal{O}\left(||\mathbf{V}^0 - \mathbf{v}^0||_1 + h^2 \right), \quad \text{as} \quad h \to 0. \tag{28}$$

Now, we analyze the error $||\mathbf{V}^n - \mathbf{v}^n||_\infty$, $n = 1, \dots, N$. From the error bound (27) in the approach associated with births, and taking into account the behaviour of the error in (28), the relation between the different norms presented in (19) and, again, (21) and (25), we can write

$$\left| V_0^n - v(0, t^n) \right| = \mathcal{O}\left(||\mathbf{V}^0 - \mathbf{v}^0||_\infty + h^2 \right), \quad \text{as} \quad h \to 0. \tag{29}$$

On the other hand, if we apply recursively the evolutions described in (16) and (12), for $j = 1, 2, \dots, J$, we can write

$$\left| V_j^n - v(a_j, t^n) \right| = \begin{cases} \left| V_{j-n}^0 - v^0(a_{j-n}) \right| & \text{if } j \geq n, \\ \left| V_0^{n-j} - v(0, t^{n-j}) \right| & \text{if } j < n. \end{cases}$$

Therefore, in any case, by using (29), we have

$$\left| V_j^n - v(a_j, t^n) \right| = \mathcal{O}\left(||\mathbf{V}^0 - \mathbf{v}^0||_\infty + h^2 \right), \quad \text{as} \quad h \to 0,$$

and we conclude, for $n = 0, 1, \dots, N$,

$$||\mathbf{V}^n - \mathbf{v}^n||_\infty = \mathcal{O}\left(||\mathbf{V}^0 - \mathbf{v}^0||_\infty + h^2 \right), \quad \text{as} \quad h \to 0. \tag{30}$$

However, from the initial expressions (15), (10) and (14), and taking into account hypothesis (22) and (21),

$$||\mathbf{V}^0 - \mathbf{v}^0||_\infty = \mathcal{O}\left(h^2\right), \quad \text{as} \quad h \to 0, \tag{31}$$

because

$$\left| V_j^0 - v^0(a_j) \right| = \left| \frac{U_j^0}{\Pi_j} - \frac{u^0(a_j)}{\pi(a_j)} \right| = \left| u^0(a_j) \right| \left| \frac{1}{\Pi_j} - \frac{1}{\pi(a_j)} \right| = \left| \frac{v^0(a_j)}{\Pi_j} \right| \left| \Pi_j - \pi(a_j) \right|, \quad j = 0, 1, \dots, J.$$

Hence, if we put (30) and (31) together, we arrive at the convergence result (23) about the approximation to the solution of the model after the change of variable.

Finally, the relations (18) and (6), which provide the connections with the original variables, give for $n = 0, 1, \dots, N$, and $j = 0, 1, \dots, J$,

$$\left| U_j^n - u(a_j, t^n) \right| = \left| \Pi_j V_j^n - \pi(a_j)\, v(a_j, t^n) \right| \le \left| \Pi_j \right| \left| V_j^n - v(a_j, t^n) \right| + \left| v(a_j, t^n) \right| \left| \Pi_j - \pi(a_j) \right|,$$

then, from (23) and (21) (that give the behaviour of the error in v and π, respectively), we conclude (24), the behaviour of the error in u. □

4 | NUMERICAL EXPERIMENTS

In order to validate the effectiveness of the new numerical technique, we consider a theoretical test. Taking into account the important role of accuracy in the approximation to the survival probability, for the simulation we consider one of the most representative unbounded mortality rates that appears in the literature about the numerical solution of finite life-span age-structured population models (see [11,15], for example):

$$m(a) = \frac{\lambda}{a_\dagger - a}, \quad 0 \le a < a_\dagger, \tag{32}$$

with $\lambda > 0$, which is a typical example of a function that possesses a divergent improper integral (5). For the numerical comparison, it is important to note that, in this case, the exact expression of the survival probability is

$$\pi(a) = \left(1 - \frac{a}{a_\dagger}\right)^\lambda, \quad 0 \le a < a_\dagger. \tag{33}$$

However, we want to use approximations to the survival probability associated with (32), and not the exact values of π, for experimental purposes. Here, we consider the numerical procedure [2] based on the approximation of the integral in (4), by means of the composite trapezoidal quadrature rule, to estimate (33) over the age grid. It has been shown that for values of $\lambda \ge 3$, these are second-order approximations. Furthermore, this optimal rate of convergence is still observed experimentally when $2 \le \lambda < 3$. Otherwise, the order of convergence does deteriorate. In fact, when $0 < \lambda < 2$, the experimental order of convergence is actually given by λ. Note that these theoretical and experimental conclusions about the order of convergence are just the same to those ones presented with the second-order numerical procedure presented in other works [11].

With respect to the fertility function, we take

$$\beta(a) = \begin{cases} 0 & \text{if } 0 \le a < a_\dagger/10, \\ \beta\,(a - a_\dagger/10)^3\,(a_\dagger - a)^3 & \text{if } a_\dagger/10 \le a \le a_\dagger, \end{cases} \tag{34}$$

where β is a constant taken to get that the net reproduction number

$$R = \int_0^{a_\dagger} \beta(a)\,\pi(a)\,da,$$

is greater than 1, and therefore we can expect the population to show an increasing trend. Note that there exists an initial age subinterval where the fertility function vanishes which supports the existence of a maturity age (in this case, $a_\dagger/10$).

On the other hand, with respect to the initial condition u^0 in (3), and the corresponding v^0 in (9) (obtained by using the change of variable (10)), we will pay attention to the behaviour of these initial data with respect to the survival probability and its approximation (hypotheses (11) and (22), respectively). However these conditions are not very restrictive: any initial data with compact support verifies both of them, so, we take

$$u^0(a) = \begin{cases} \rho\, a^3\,(a_\dagger/10 - a)^3 & \text{if } 0 \le a \le a_\dagger/10, \\ \\ 0 & \text{if } a_\dagger/10 < a < a_\dagger. \end{cases}$$

Taking into account how $\beta(a)$ is defined in (34), choosing the interval $[0, a_\dagger/10]$ as the support of the initial condition guarantees sufficient compatibility between the initial condition (3) and the boundary condition (2), for the solution of (1) to be sufficiently smooth. Therefore, the population starts with an initial distribution across the nonreproductive ages that can be interpreted as the insertion of a juvenile population in an empty environment. Constant ρ is chosen to normalize the initial data: that is, so that $\max_{0\le a\le a_\dagger} u^0(a) = 1$ (in this case, $\rho = (20/a_\dagger)^6$). Considering that the problem is not based on a specific biological application, from now on, we take the maximum age $a_\dagger = 1$ (this value does not affect the accuracy conclusions).

In order to test the numerical method, in our first experiment we consider $\lambda = 2$ in the mortality rate (32), so we are assured that the numerical approximation to the survival probability is second-order accurate (that is, (21) is satisfied). For the fertility function (34), we choose the constant $\beta = 2000$. We consider the numerical evolution of the previous initial data till $T = 4$.

Note that we do not know the exact solution to the problem (1)-(3) nor the problem (7)-(9). So, to estimate the errors produced by the numerical approximations obtained with different values of h, we take the numerical solutions computed with a sufficiently small value of the size step ($h^* = 1/409600$) as valid references of the exact solutions. But, to eliminate sources of error, in (15), (17) and (18) we use the exact values of the survival probability $\boldsymbol{\pi} = [\pi(a_0), \pi(a_1), \dots, \pi(a_J)]$ (obtained with (33)), instead of their approximations $\boldsymbol{\Pi} = [\Pi_0, \Pi_1, \dots, \Pi_J]$. Therefore, we denote as $\widetilde{\mathbf{U}}^N$ and $\widetilde{\mathbf{V}}^N$ such representations of the exact solutions $\mathbf{u}^N$ and $\mathbf{v}^N$, respectively, at the final time.

Figure 1 shows, in logarithmic scale, the different errors involved in the proposed numerical procedure, for different values of the step size ($h = 1/800$, $h = 1/1600$, $h = 1/3200$, $h = 1/6400$, $h = 1/12800$, $h = 1/25600$ and $h = 1/51200$). For each h, we plot (dashed line) the error produced by the approximation to the survival probability, measured in the maximum norm,

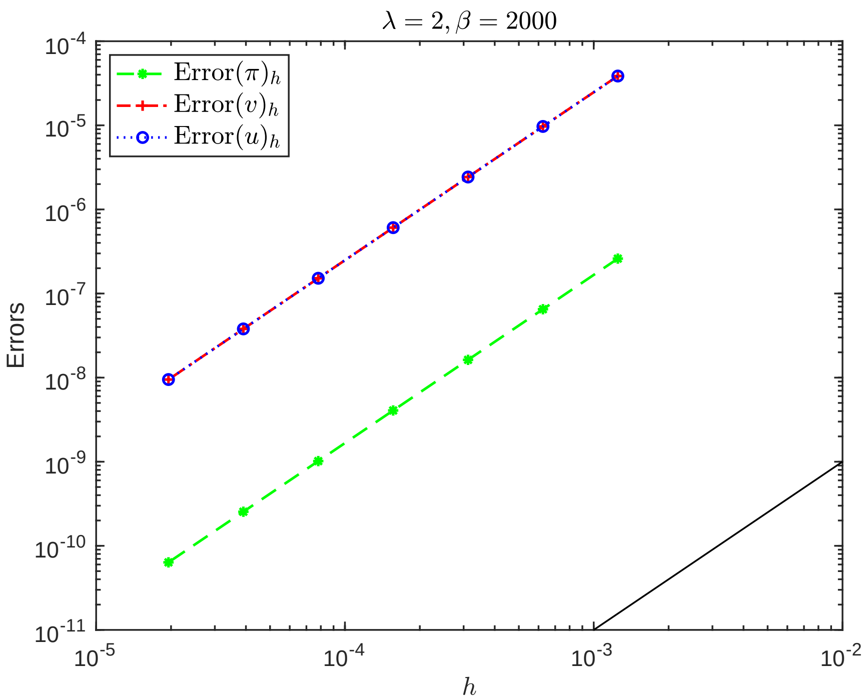

FIGURE 1 Numerical errors involved in the simulation ($\lambda = 2$, $\beta = 2000$). Line plotted in the lower right-hand corner represents quadratic slope.

that we denote

$$\text{Error}(\pi)_h = ||\boldsymbol{\Pi} - \boldsymbol{\pi}||_\infty.$$

Also, at the final time T, we compare the computed numerical approximation $\mathbf{V}^N$, with $\widetilde{\mathbf{V}}^N$ at the coarsest grid (obtained with h). Then, we plot (dash-dotted line) the error measured in the maximum norm: that is,

$$\text{Error}(v)_h = ||\mathbf{V}^N - \widetilde{\mathbf{V}}^N||_\infty.$$

Finally, we compare the approximations $\mathbf{U}^N$ to $\widetilde{\mathbf{U}}^N$

$$\text{Error}(u)_h = ||\mathbf{U}^N - \widetilde{\mathbf{U}}^N||_\infty,$$

and we plot (dotted line) this error.

We observe that, in this case, $\text{Error}(v)_h$ and $\text{Error}(u)_h$ coincide. Figure 1 clearly confirms, from the line plotted in the lower right-hand corner (solid line) that represents quadratic slope, the expected second-order of convergence in the three approximations.

Remember that, for a smaller value of the parameter λ in the mortality rate (32), the order observed in the numerical approximation to the survival probability (by means of the numerical methods that we have considered) deteriorates. In such a case, if we follow the proof of Theorem 1, we could conclude that the approximations to u and v provided by the numerical method show, at least, the order of convergence exhibited by the approximations to the function π. In the next experiment, we test this situation. For example, assuming now that $\lambda = 1$, it is known that we can only obtain first-order approximations to the survival probability by means of the previous numerical method. We take the same fertility rate (34) but, to achieve a similar value of the net reproduction number as in the previous simulation ($R \approx 1.5374$), now we take $\beta = 1000$. Also, we consider the same initial data u^0 as in the previous experiment, with the same value of ρ. In Figure 2 we can compare the numerical approximations to the density function u, obtained at $T = 4$ by means of the numerical method: dashed line corresponds to $\lambda = 2$ and $\beta = 2000$; dash-dotted lined to $\lambda = 1$ and $\beta = 1000$.

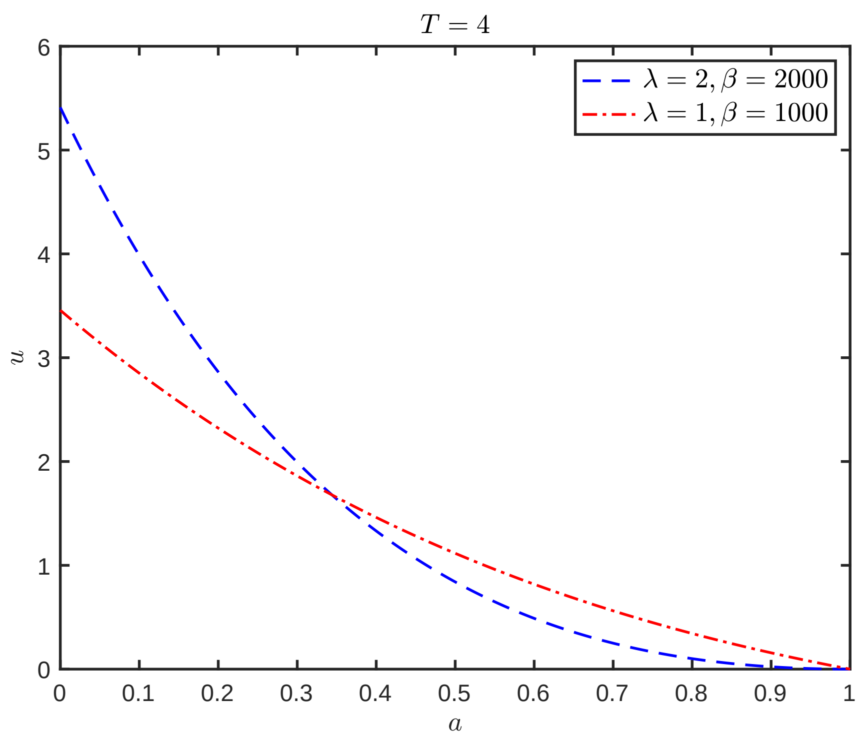

FIGURE 2 Numerical approximation to the density at $T = 4$, depending on the values of the parameters λ, β.

The different errors involved in this case are presented in Figure 3 . For the values of the step size $h = 1/800$, $h = 1/1600$, $h = 1/3200$, $h = 1/6400$, $h = 1/12800$, $h = 1/25600$ and $h = 1/51200$, we plot the errors $\mathrm{Error}(\pi)_h$ (dashed line), $\mathrm{Error}(\upsilon)_h$ (dash-dotted line) and $\mathrm{Error}(u)_h$ (dotted line).

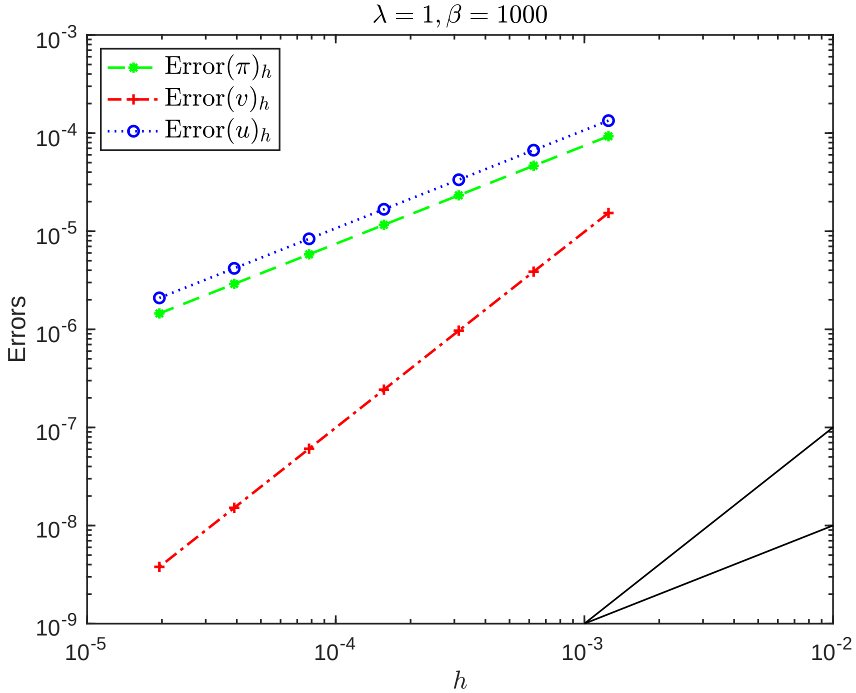

FIGURE 3 Numerical errors involved in the simulation ($\lambda = 1$, $\beta = 1000$). Lines plotted in the lower right-hand corner represent linear and quadratic slopes.

As we previously mentioned, from the plotted line that represents the linear slope (the bottom solid line in the lower right-hand corner), we observe that the approximation to π is only first-order accurate. But, as we expected, so is the approximation to the density u of the original problem (1)-(3). However, the approximation to the density υ of the modified problem (7)-(9) is still second-order accurate (see the other solid line that represents the quadratic slope).

To better understand this situation, in Figure 4 we show, for $h = 1/51200$, how each of the errors that appear in the simulation is distributed over the age interval: the dashed line corresponds to $\lambda = 2$, $\beta = 2000$, and the dash-dotted line to $\lambda = 1$, $\beta = 1000$. The upper picture presents the errors in the approximation to the survival probability (33); the middle one, the errors in the solution of the modified problem (7)-(9) at $T = 4$; and the lower one, the errors in the solution of the original problem (1)-(3), also at the final time. For $\lambda = 2$, as we previously observed, the approximation to the survival probability has order 2 when the error is measured in the maximum norm. In the upper picture in Figure 4 , we observe that the maximum error occurs near the maximum age. However, for $\lambda = 1$, the order of convergence of the approximation to the survival probability is only 1, and this picture shows that the error is maximum at the last grid point a_J. In the middle picture, we observe that the maximum error in the approximation to υ appears, for both values of λ, in the computation of newborns by means of the quadrature rule (here, the order is two). Finally, from the lower picture we conclude that, for $\lambda = 2$, the maximum error in the approximation to u also occurs at the first grid point (and, again, this approximation has order 2). However, for $\lambda = 1$, the maximum error in the approximation to u is found as in the survival probability: at the last grid point a_J. In this case, it seems that the approximation to the survival function generates the main source of the error and, therefore, the approximation to the density function is only order 1.

So that, as we conclude from the last experiment, when we only have first-order approximations to the survival probability, the order of the computed approximations to the density of the original problem is only 1, too. Consequently, in such a case, it would be enough to consider a first-order convergent quadrature rule for the approximation of the newborns. On the other

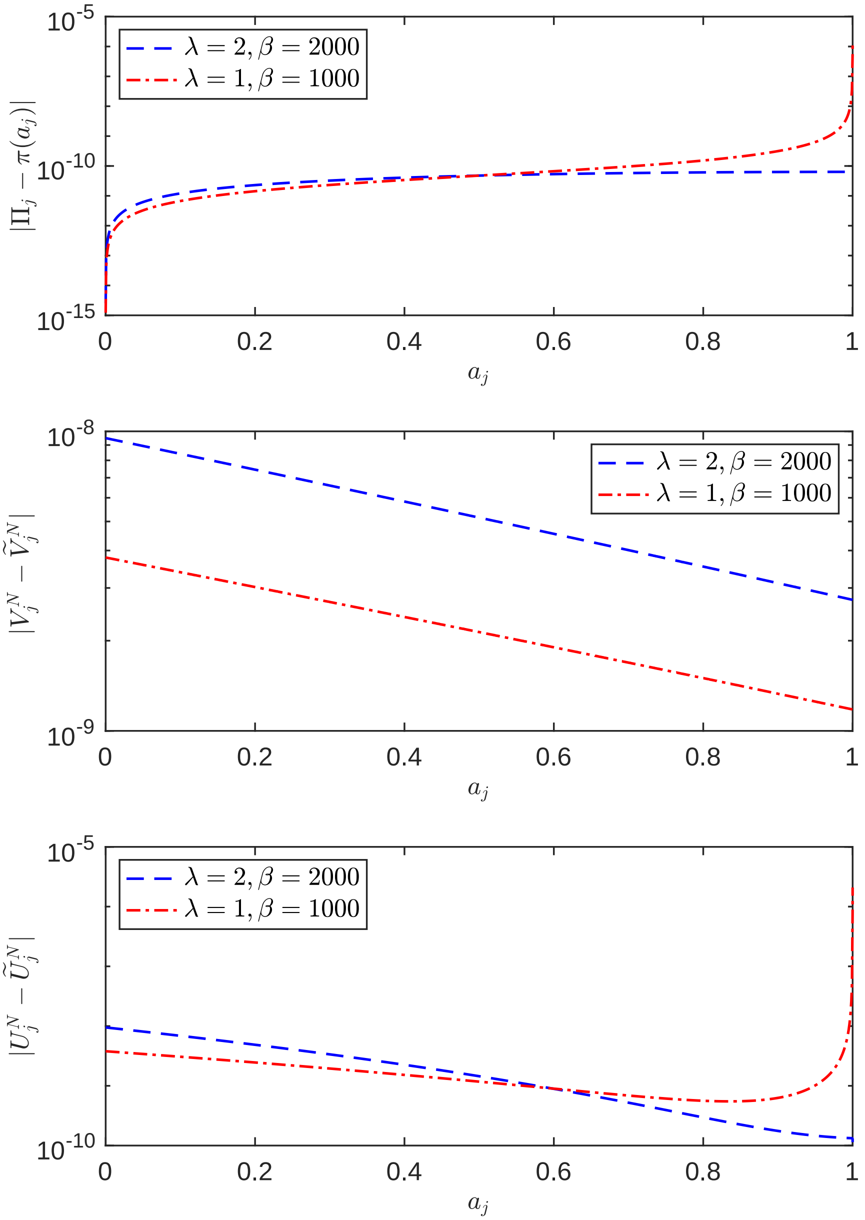

FIGURE 4 Depending on the values of the parameters λ, β, distribution over the age interval of the errors that appear in the simulation: in the approximation to the survival probability (upper), in the approximation to the solution of the modified problem (middle), and in the approximation to the solution of the original problem (lower).

hand, assuming higher-order estimations to π, we could obtain approximations to u with the same high-order if we modify the procedure using appropriate quadrature formulas.

5 | CONCLUSIONS

It is known that for age-structured populations models, the incorporation of a finite life-span entails important difficulties in the design of numerical methods for the numerical approximation of their solution.

In this work, we propose a numerical procedure based on a change of variable widely used in the theoretical analysis of the model. This change offers a new approach that allows us to isolate the principal complication in this kind of problems: the approximation of the survival probability. The technique we contemplate involves two numerical problems: the estimation of the survival probability and the discretization of a simpler model than the original one. Therefore, one important advantage is that the method only requires approximations to the survival probability instead of the exact values, as other schemes do.

Assuming second-order approximations to this survival function we consider a specific numerical method. We prove that, the order of convergence of the approximations to the density function is also 2. Our numerical simulation ratifies this fact and shows the sensitivity of the technique with respect to the degree of precision in the estimation of the survival probability. We can conclude that, in the numerical approximation of the density function, the accuracy reached estimating the survival probability should be taken into account.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

References

1. Abia LM, Angulo O, López-Marcos JC. Age-structured population dynamics models and their numerical solutions. *Ecol Model*. 2005; 188:112–136. DOI: 10.1016/j.ecolmodel.2005.05.007.

2. Abia LM, Angulo O, López-Marcos JC, López-Marcos MA. Approximating the survival probability in finite life-span population models. *J Comput Appl Math*. 2018; 330:783–793. DOI: 10.1016/j.cam.2017.05.004.

3. Angulo O, López-Marcos JC, López-Marcos MA, Martínez-Rodríguez J. Numerical analysis of an open marine population model with spaced-limited recruitment. *Math Comput Model*. 2010; 52:1037–1044. DOI: 10.1016/j.mcm.2010.02.020.

4. Angulo O, López-Marcos JC, López-Marcos MA, Martínez-Rodríguez J. Numerical investigation of the recruitment process in open marine population models. *J Stat Mech-Theory E*. 2011; 2011:P01003. DOI: 10.1088/1742-5468/2011/01/P01003.

5. Angulo O, López-Marcos JC, López-Marcos MA, Martínez-Rodríguez J. Numerical analysis of a population model of marine invertebrates with different life stages. *Commun Nonlinear Sci*. 2013; 18:2153–2163. DOI: 10.1016/j.cnsns.2013.01.009.

6. Angulo O, López-Marcos JC, López-Marcos MA, Milner FA. A numerical method for nonlinear age-structured population models with finite maximum age. *J Math Anal Appl*. 2010; 361:150–160. DOI: 10.1016/j.jmaa.2009.09.001.

7. Angulo O, López-Marcos JC, Milner FA. The application of an age-structured model with unbounded mortality to demography. *Math Biosci*. 2008; 208:495–520. DOI: 10.1016/j.mbs.2006.11.001.

8. Bacaër N. *A Short History of Mathematical Population Dynamics*. Springer; 2011.

9. Coyle C, Nigam N. High-order Discontinuous Galerkin Methods for a class of transport equations with structured populations. *Comput Math Appl*. 2016; 72:768–784. DOI: 10.1016/j.camwa.2016.05.024.

10. Cusulin C, Gerardo-Giorda L. A numerical method for spatial diffusion in age-structured populations. *Numer Meth Part D E*. 2010; 26:253–273. DOI: 10.1002/num.20425.

11. Iannelli M, Milner FA. On the approximation of the Lotka-McKendrick equation with finite life-span. *J Comput Appl Math*. 2001; 136:245–254. DOI: 10.1016/S0377-0427(00)00616-6. DOI: 10.1002/num.20425.

12. Iannelli M, Milner FA. *The Basic Approach to Age-Structured Population Dynamics: Models, Methods and Numerics*. Springer; 2017.

13. Inaba H. *Age-Structured Population Dynamics in Demography and Epidemiology*. Springer; 2017.

14. Kim MY. Discontinuous-Continuous Galerkin Methods for a Structured Model of a Biological System. *SIAM J Sci Comput*. 2008; 31:913–938. DOI: 10.1137/070706094.

15. Kim MY, Kwon Y. A Collocation Method for the Gurtin-MacCamy Equation with Finite Life-Span. *SIAM J Numer Anal*. 2002; 39:1914–1937. DOI: 10.1137/S0036142900370927.

16. Kim MY, Selenge T. Age-time continuous Galerkin methods for a model of population dynamics. *J Comput Appl Math*. 2009; 223:659–671. DOI: 10.1016/j.cam.2008.02.004.

17. Kim MY, Selenge T. Discontinuous-continuous Galerkin methods for population diffusion with finite life span. *Math Popul Stud*. 2016; 23:17–36. DOI: 10.1080/08898480.2013.836428.

18. Li X-Z, Yang J, Martcheva M. *Age Structured Population Epidemic Modeling*. Springer; 2020.

19. Pelovska G. An improved explicit scheme for age-dependent population model with spatial diffusion. *Int J Numer Anal Model*. 2008; 5:466–490.

20. Pelovska G, Iannelli M. Numerical methods for the Lotka-McKendrick's equation. *J Comput Appl Math*. 2006; 197:534–557. DOI: 10.1016/j.cam.2005.11.033.

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