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The miniversal deformation of certain complete intersection monomial curves

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Abstract

The aim of this paper is to provide an explicit basis of the miniversal deformation of a monomial curve defined by a free semigroup—these curves make up a notable family $\mathscr{C}$ of complete intersection monomial curves. First, we dispense a general decomposition result of a basis B of the miniversal deformation of any complete intersection monomial curve. As a consequence, we explicitly calculate B in the particular case of a monomial curve defined from a free semigroup. This direct computation yields some estimates for the dimension of the moduli space of the family $\mathscr{C}$.

Keywords Monomial curves · complete intersection · deformation theory · moduli space · curve singularities

Mathematics Subject Classification 14B07 · 14H10

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1 Introduction

Let $(C, 0) \subset (\mathbb{C}^g, 0)$ be a germ of irreducible complex monomial curve singularity. This means that C can be defined through a parameterization $C : (t^{a_0}, t^{a_1}, \ldots, t^{a_g})$ where the set of exponents $\{a_0, a_1, \ldots, a_g\} \subset \mathbb{N}$ generates a semigroup with finite complement over $\mathbb{N}$ and satisfies $\gcd(a_0, a_1, \ldots, a_g) = 1$. In particular, this is equivalent to the fact that the coordinate ring of the curve is isomorphic to the semigroup algebra $\mathbb{C}[\Gamma] = \bigoplus_{s\in\Gamma} \mathbb{C}t^s \subset \mathbb{C}[t]$ defined by the numerical semigroup

$$\Gamma = a_0\mathbb{N} + a_1\mathbb{N} + \cdots + a_g\mathbb{N} = \langle a_0, a_1, \ldots, a_g\rangle.$$

The module T^1_C of first order infinitesimal deformations of a monomial curve C plays a central role in the study of two important moduli problems. It has a natural $\mathbb{Z}$–graded structure, i.e. $T^1_C = \bigoplus_{n\in\mathbb{Z}} T^1_C(n)$. For the definition of first order infinitesimal deformations and further details, see for example Greuel et al.[13, Chapter II, Sect. 1.4]; see also the first author [1, Sect. 2.3] and Buchweitz [3] for the specific case of monomial curves and the definition of the natural grading of T^1_C. Let us briefly summarize the context and state of the art of each problem.

Regarding the first problem, consider a smooth algebraic curve X with genus g and fix a point $p \in X$. The Weierstraß semigroup of X at p is defined as

$$\Gamma_p = \Gamma := \left\{ h \in \mathbb{N} : \begin{array}{l} \text{there exists a meromorphic function defined on } X, \\ \text{holomorphic on } X \setminus p \text{ with a pole of degree } h \text{ at } p \end{array} \right\}.$$

One can then define the moduli space $\mathscr{M}_{g,1}$ of pointed smooth algebraic curves (X, p) of genus g (see for example Deligne & Mumford [8]), i.e. the set of isomorphism classes of pointed smooth algebraic curves (X, p) with its natural scheme structure. In his seminal work [17], Pinkham showed that the subscheme $\mathscr{M}^{\Gamma}_{g,1}$ of $\mathscr{M}_{g,1}$ of isomorphism classes of pointed smooth algebraic curves (X, p) with prescribed Weierstraß semigroup Γ is mapped bijectively to the negatively graded part of the module of infinitesimal deformations of the monomial curve defined by the semigroup Γ. Let us be more precise; it is well known that the Weierstraß semigroup Γ_p is a numerical semigroup (for a basic insight into Weierstraß semigroups the reader is referred e.g. to the second author [14] and the references therein). Thus, Pinkham's result [17, Theorem 13.9] provides a bijection between $\mathscr{M}^{\Gamma}_{g,1}$ and the $\mathbb{C}$–vector space $\bigoplus_{n\in\mathbb{Z}_{<0}} T^1_C(n)$.

The study of $\mathscr{M}^{\Gamma}_{g,1}$ has led to numerous results, far too many to list comprehensively. In what follows, we summarize the main known results and refer to the cited literature for further information, while acknowledging that some contributions may not be mentioned, not deliberately. In [22], Stöhr provided a description of $\mathscr{M}^{\Gamma}_{g,1}$ for a symmetric semigroup Γ, i.e. the monomial curve is a Gorenstein curve, in terms of Gröbner bases and the analysis of the syzygies of the defining ideal of the monomial curve. Deepening in that description, Contiero and Stöhr [6] provided a method to obtain upper bounds for the dimension of this moduli space for symmetric semigroups in terms of the combinatorics of the semigroup. In [20], Rim and Vitulli provided a classification of negatively graded semigroups, which is a particular family of numerical semigroups for which T^1_C has no positively graded part, and provided some formulas for the dimension of $\mathscr{M}_{g,1}^{\Gamma}$ in those cases. Some other noteworthy results to mention are those by Nakano [15], Polishchuk [18], and Stevens [21], which follow similar ideas to those provided in Stöhr's work. Lastly, the work by Buchweitz [3] computes the dimensions of the corresponding graded components of T_C^1 using a rather intricate combinatorial formula in terms of some combinatorial invariants of the semigroup.

For the description of the second problem, consider a germ $(C, 0) \subset \left(\mathbb{C}^2, 0\right)$ of an irreducible complex plane curve singularity. Let R denote its local ring at the origin and let $\overline{R} \simeq \mathbb{C}[t]$ be its normalization. The normalization morphism $R \hookrightarrow \overline{R}$ induces a discrete valuation $v : R \to \mathbb{Z}$ from which $S = v(R)$ has a natural structure of a finitely generated subsemigroup of the semigroup $(\mathbb{N}, +)$ of natural numbers with 0 element, which is in fact a complete topological invariant of the curve (see Zariski [25] for details). The moduli space of irreducible plane curve singularities with fixed semigroup is the set of analytic classes (modulo biholomorphisms of $\mathbb{C}\{x, y\}$) of irreducible plane curves with fixed semigroup. Teissier [23] proved that any analytic class can be realized as a fibre of a positively graded deformation of the monomial curve C^S with semigroup algebra $\mathbb{C}[S]$; this means that the understanding of $\bigoplus_{n\in\mathbb{Z}_{>0}} T^1_{C^S}(n)$ is a good tool to understand that moduli space. He also showed that, in this particular case, C^S is in fact a complete intersection monomial curve. Following these ideas, Cassou-Noguès [5] provided a monomial basis of $T^1_{C^S}$ in an iterative way using the combinatorial properties of the semigroup of an irreducible plane curve. Moreover, she supplied a closed formula for the dimension of the positively graded part of $T^1_{C^S}$ in terms of the generators of the semigroup.

Our work can be understood as a natural continuation of Cassou-Noguès paper [5], as our initial aim is to address the following question: how general can her results [5, Theorems 2 and 3] be? As already said, those theorems are stated for the monomial curve associated to the semigroup of an irreducible plane curve singularity, which is a particular example of a complete intersection monomial curve. In order to generalize these theorems, we introduce a new ingredient in the topic which up to the authors' knowledge has not been exploited yet. The new ingredient builds upon the approach introduced by Delorme [9], who proposed a combinatorial decomposition of the semigroup algebra of a complete intersection monomial curve, yielding a particularly useful ordering of its implicit equations.

A numerical semigroup $\Gamma = \langle a_0, a_1, \ldots, a_g\rangle$ is said to be a complete intersection semigroup if its semigroup algebra is a complete intersection. Set the ring $R = \mathbb{C}[u_0, u_1, \ldots, u_g]$ and write C^Γ for the complete intersection monomial curve with 'coordinate ring' R/I, where I is the ideal of R generated by $f_1, \ldots, f_g$. Setting appropriate coordinates in a neighborhood of $\mathbf{0} \in \mathbb{C}^{g+1}$, the curve C^Γ is parametrically defined by $u_i = t^{a_i}$ for $i = 0, 1, \ldots, g$. For complete intersection curves, Tjurina [24] (see also [13]) showed that $T^1_{C^\Gamma}$ is isomorphic as $\mathbb{C}$–algebra to

$$\frac{\mathbb{C}[u_0, \ldots, u_g]^g}{\left(\frac{\partial f_i}{\partial u_j}\right)_{i,j} \mathbb{C}[u_0, \ldots, u_g]^{g+1} + (f_1, \ldots, f_g)\mathbb{C}[u_0, \ldots, u_g]^g}.$$

We realized that the order provided by Delorme of the implicit equations of the curve reflects in a very good structure of the Jacobian matrix. In fact, it provides a block decomposition on it (see Sect. 2 for further details). In light of these considerations, the **purpose and main results of this paper** can be summarized in three main points:

(1) To give a general decomposition in "separate variables" of a basis B of the miniversal deformation of any *complete intersection* monomial curve following Delorme's decomposition of the semigroup algebra. This is achieved in Theorem 2.3.
(2) To study the previous decomposition in order to compute a monomial basis of the miniversal deformation of a complete intersection. Such a basis is obtained for the particular case of a *free* monomial curve, see Theorem 2.7. This constitutes a specific, yet sufficiently general, family within the category of complete intersection monomial curves in which the combinatorics inherent to the problem becomes more accessible.
(3) Using Pinkham's bijection, to study recursive formulas for the dimension of the moduli space $\mathscr{M}_{g,1}^{\Gamma}$ of a *free* monomial curve from the generators of the free numerical semigroup Γ associated to the curve. Sharp upper and lower bounds for this dimension are obtained in Theorem 3.4 as well as closed formula in some particular cases: Theorem 3.6 and Theorem 3.8.

We conclude this introduction by setting forth the conventions and notation that will be useful to the reader in the remainder of the paper.

Conventions and notation.

- The set of natural numbers $\mathbb{N}$ consists of nonnegative integer numbers. For every $a, b \in \mathbb{N}$ we define
$$[a, b] := \{n \in \mathbb{N} : a \leq n \leq b\}.$$
- We write $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ for the multiplicative group of the units of the field $\mathbb{C}$.
- The vectors $e_1, \ldots, e_g$ denote the canonical basis of $\mathbb{C}[u_0, \ldots, u_g]^g$ as well as the standard basis of $\mathbb{Z}^g$.
- Usually we will work in a graded polynomial ring $\mathbb{C}[u_0, \ldots, u_g] = \bigoplus_{j \in S} \mathbb{C}[u_0, \ldots, u_g]_j$ with $S \subset \mathbb{N}$ and $\deg(u_i) = a_i$ for some $a_i \in \mathbb{N}$. For a homogeneous element $f \in \mathbb{C}[u_0, \ldots, u_g]_j$ for some j we denote $\deg(f) = j = \alpha_0 a_0 + \cdots + \alpha_g a_g$ for some $\alpha_i \in \mathbb{N}$.

2 Deformations of complete intersection monomial curves

This section will be devoted to provide two of our main results. We will start with some basic definitions as well as recall the decomposition theorem of Delorme [9] about the semigroup algebra of a complete intersection monomial curve. Moreover, we will prove the decomposition theorem of the basis of the miniversal deformation of a complete intersection monomial curve. Finally, we will use the decomposition Theorem 2.3 to provide an explicit basis of the miniversal deformation of a monomial curve associated to a free numerical semigroup.

2.1 Complete intersection monomial curves

Let $\Gamma = \langle a_0, a_1, \ldots, a_g \rangle$ be a numerical semigroup. Let $t \in \mathbb{C}$ be a local coordinate of the germ $(\mathbb{C}, 0)$ and let $(u_0, u_1, \ldots, u_g) \in \mathbb{C}^{g+1}$ be local coordinates of the germ $\left(\mathbb{C}^{g+1}, \mathbf{0}\right)$. The monomial curve $\left(C^\Gamma, \mathbf{0}\right) \subset \left(\mathbb{C}^{g+1}, \mathbf{0}\right)$ defined via the parameterization

$$C^\Gamma : u_i = t^{a_i}, \qquad i \in [0, g]$$

is called the monomial curve associated to Γ. Write $\mathbb{C}[C^\Gamma] := \mathbb{C}[t^\nu : \nu \in \Gamma]$, which coincides with the semigroup algebra $\mathbb{C}[\Gamma]$ associated to Γ. We will use either notation depending on whether we want to highlight the geometric or algebraic interpretation. The numerical semigroup Γ is said to be complete intersection if $\mathbb{C}[\Gamma]$ is a complete intersection; this means that, if $\Gamma = \langle a_0, \ldots, a_g \rangle$ is generated by $g + 1$ elements, then we have an exact sequence

$$\mathbb{C}[u_0, \ldots, u_g]^g \to \mathbb{C}[u_0, \ldots, u_g] \xrightarrow{\varphi} \mathbb{C}[C^\Gamma] \to 0, \tag{2.1}$$

with $\ker \varphi = (f_1, \ldots, f_g)$ and $f_1, \ldots, f_g$ defining a regular sequence. (Observe that the mapping $\mathbb{C}[u_0, \ldots, u_g]^g \to \mathbb{C}[u_0, \ldots, u_g]$ is just $e_i \mapsto f_i$ for every $i \in [1, g]$). In particular, this implies that the monomial curve $(C^\Gamma, \mathbf{0})$ is a complete intersection. For a comprehensive synthesis of the existing results on monomial curves that arise as complete intersections (and, in particular, the details pertaining to complete intersection curves) we guide readers to the survey [1] and its cited bibliography for additional references and context.

In 1976, Delorme [9, Lemme 7] showed the following combinatorial characterization of a complete intersection numerical semigroup. Set $A := \{a_0, \ldots, a_g\}$; for $\Gamma = \langle A \rangle$ we have that Γ is a complete intersection numerical semigroup if and only if there exists a partition $A = A_1 \sqcup A_2$ of the set of generators A with $A_1 \neq \emptyset \neq A_2$ such that the following two conditions holds:

(1) $\mathbb{C}[\Gamma_i]$ are complete intersections defined by $I_i := \ker \varphi_i$, where Γ_i stands for the numerical semigroup Γ_{A_i/d_i} generated by the elements of A_i divided by $d_i := \gcd(A_i)$, for $i = 1, 2$;
(2) $\mathbb{C}[\Gamma]$ is defined by $I_1 + I_2 + \langle \rho \rangle$, where $\langle \rho \rangle$ is a binomial ideal whose generator has degree $\deg(\rho) = \deg(f_g) = \operatorname{lcm}(d_1, d_2)$.

More precisely, if we set $g_i := |A_i| - 1$, then $g_1 + g_2 = g - 1$ and we have the exact sequences

$$\begin{aligned} &\mathbb{C}[x_0, \ldots, x_{g_1}]^{g_1} \to \mathbb{C}[x_0, \ldots, x_{g_1}] \xrightarrow{\varphi_1} \frac{\mathbb{C}[x_0, \ldots, x_{g_1}]}{I_1} \simeq \mathbb{C}[\Gamma_1] \to 0 \\ &\mathbb{C}[y_0, \ldots, y_{g_2}]^{g_2} \to \mathbb{C}[y_0, \ldots, y_{g_2}] \xrightarrow{\varphi_2} \frac{\mathbb{C}[y_0, \ldots, y_{g_2}]}{I_2} \simeq \mathbb{C}[\Gamma_2] \to 0 \end{aligned} \tag{2.2}$$

In addition —as pointed out before— we can define a binomial ρ in separated variables with total degree $\operatorname{lcm}(d_1, d_2)$. Thus there is a natural decomposition of the semigroup algebra of Γ in the form

$$\mathbb{C}[\Gamma] = \frac{\mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2]}{\langle \rho \rangle}, \tag{2.3}$$

where we write $\mathbb{C}[u_0, \ldots, u_g] = \mathbb{C}[x_0, \ldots, x_{g_1}; y_0, \ldots, y_{g_2}]$.

Remark 2.1 The set of generators A of Γ does not need to be a minimal generating set.

2.2 Deformations

Before moving forward, we briefly recall some fundamental concepts and essential terminology from the theory of curve deformations. A *deformation* of an isolated singularity $(X, \mathbf{0})$ over a complex germ space $(S, \mathbf{0})$ is a pair (ϕ, i), where

(1) $\phi : (\mathscr{X}, \mathbf{0}) \to (S, \mathbf{0})$ is a germ of flat morphism.
(2) $i : (X, \mathbf{0}) \to (\phi^{-1}(\mathbf{0}), \mathbf{0})$ is an isomorphism onto the *special fibre*.

Here, we say that $(\mathscr{X}, \mathbf{0})$ is the total space, $(S, \mathbf{0})$ is the base space, and $(\mathscr{X}_s, \mathbf{0}) \cong (X, \mathbf{0})$ is the special fibre of the deformation.

In a somewhat informal sense, one says that $\phi : (\mathscr{X}, \mathbf{0}) \to (S, \mathbf{0})$ constitutes a deformation of $(X, \mathbf{0})$. Notice that the point $\mathbf{0}$ around which we define the germs might not have been chosen, so that we can think of a commutative diagram as

$$\begin{array}{ccc} (X, x) & \xrightarrow{\ i\ } & (\mathscr{X}, x) \\ \downarrow & & \downarrow \phi \\ \{\mathrm{pt}\} & \longrightarrow & (S, s) \end{array}$$

The deformation ϕ is called *versal* if any other deformation $\psi : (\mathscr{Y}, \mathbf{0}) \to (T, \mathbf{0})$ is induced by ϕ by base change $(T, \mathbf{0}) \to (S, \mathbf{0})$. A versal deformation is called *miniversal* if it is versal and the base space S has minimal possible dimension. The existence of a miniversal deformation for isolated singularities is a celebrated result by Grauert [10]. The reader is refereed to [13, Part II, Sect. 1] for further details.

A useful point of view to study the singularity $(X, \mathbf{0})$ is the one given by the concept of "first order deformations", encoded as vector spaces $T^1_{(X,\mathbf{0})}$ which can be understood as linearizations of the deformations of a germ $(X, \mathbf{0})$. The writing $T^1_{(X,\mathbf{0})}$ refers to the fact that it can be identified —whenever it exists— with the Zariski tangent space to the semiuniversal base of $(X, \mathbf{0})$; for further details, interested readers are referred to [13]. Under the previous notation and in the particular case of a complete intersection singularity $(X, \mathbf{0})$, the base space of the deformation $(S, \mathbf{0})$ is smooth so we can identify it with the Zariski tangent space. Moreover, as we are only interested in the dimension of the base space and the section is usually fixed, we will abuse notation to refer to $T^1_{(X,0)} = T^1(X)$.

In the particular case of a complete intersection monomial curve $C^\Gamma : (t^{a_0}, \ldots, t^{a_g})$ defined by $I = (f_1, \ldots, f_g)$, we can apply Tjurina's theorem [24] which states that the base space of the miniversal deformation of $\left(C^{\Gamma},\mathbf{0}\right)$ is

$$T^{1}\left(C^{\Gamma}\right):=\frac{\mathbb{C}[u_0,\ldots,u_g]^g}{\left(\frac{\partial f_i}{\partial u_j}\right)_{i,j}\mathbb{C}[u_0,\ldots,u_g]^{g+1}+(f_1,\ldots,f_g)\mathbb{C}[u_0,\ldots,u_g]^g}.$$

Moreover, Pinkham [16] (see also Buchweitz & Greuel [4], Buchweitz [3], Deligne [7], Rim [19]) showed that, for a complete intersection monomial curve C^{Γ}, the dimension of $T^1(C^{\Gamma})$ as a $\mathbb{C}$–vector space equals the conductor of the semigroup, i.e.

$$\dim_{\mathbb{C}} T^{1}\left(C^{\Gamma}\right)=c(\Gamma):=\min\{\nu\in\Gamma\ :\ \nu+\mathbb{N}\subset\Gamma\}.$$

Even more, by a result of Greuel [11], as C^{Γ} is a quasi-homogeneous complete intersection singularity, then $c(\Gamma)=\dim_{\mathbb{C}}T^{1}\left(C^{\Gamma}\right)=\mu\left(C^{\Gamma}\right)$ where $\mu\left(C^{\Gamma}\right)$ is the Milnor number associated to C^{Γ} (see also [4]).

Consider a basis $s_1,\ldots,s_\tau\in\mathbb{C}[u_0,\ldots,u_g]^g$ of $T^1\left(C^{\Gamma}\right)$, where $s_i=\left(s_i^1,\ldots,s_i^g\right)$ for $i\in[1,\tau]$. Then the miniversal deformation of C^{Γ} can be described as follows: For $\boldsymbol{u}=(u_0,u_1,\ldots,u_g)$, $\boldsymbol{w}=(w_1,w_2,\ldots,w_\tau)$ we define

$$\begin{aligned} F_1(\boldsymbol{u},\boldsymbol{w}) &= f_1(\boldsymbol{u})+\sum_{j=1}^{\tau}w_j s_j^1(\boldsymbol{u}),\\ &\ \ \vdots \qquad\qquad\quad \vdots\\ F_g(\boldsymbol{u},\boldsymbol{w}) &= f_g(\boldsymbol{u})+\sum_{j=1}^{\tau}w_j s_j^g(\boldsymbol{u}) \end{aligned} \tag{2.4}$$

and let $(\mathscr{X},\mathbf{0}):=V(F_1,\ldots,F_g)\subset(\mathbb{C}^{g+1}\times\mathbb{C}^t,\mathbf{0})$ be the zero set of $F_1,\ldots,F_g$; then the deformation defined by $\left(C^{\Gamma},\mathbf{0}\right)\xrightarrow{i}(\mathscr{X},\mathbf{0})\xrightarrow{\phi}(\mathbb{C}^{\tau},\mathbf{0})$ is the miniversal deformation of $\left(C^{\Gamma},\mathbf{0}\right)$, where i is induced by the inclusion and ϕ by the natural projection. In fact, if one chooses $\deg(w_j)=-\deg(s_j)$, then we endow the algebra $\mathbb{C}[u_0,\ldots,u_g,w_1,\ldots,w_\tau]$ with the unique grading for which $\deg(u_i)=a_i$ and the F_i are homogeneous with $\deg(F_i)=\deg(f_i)$. Under this grading, we obtain a partition of the base space $\mathbb{C}^{\tau}$ into two parts. Define the sets

$$\begin{aligned} P_+ &:=\{j\in\{1,\ldots,\tau\}:\ \deg(w_j)<0\}\\ P_- &:=\{j\in\{1,\ldots,\tau\}:\ \deg(w_j)>0\}. \end{aligned}$$

Remark 2.2 It is worth noting that—when dealing with a deformation—we have the parameter space with coordinates $w_1,\ldots,w_\tau$ on the one hand, and the basis of T^1 given by Theorem 2.7 on the other hand. Therefore, from the choice of the grading, a parameter with negative weight provides a monomial with positive grading in T^1. This is the reason that motivates the definition of P_+ and P_- as we will use them to refer to positive/negative weight deformations. Observe that the way we have defined these sets is then the opposite of the one given by Teissier in [23].

Denote by $\tau_+(\Gamma) := \tau_+ := |P_+|$ and $\tau_-(\Gamma) := \tau_- := |P_-|$. Since there are no w_j of degree zero, we have the equality $\tau = \tau_+ + \tau_-$. Moreover, there is a natural action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ over $(\mathscr{X}, \mathbf{0})$ which is compatible with the previous construction and that induces the natural action on $\phi^{-1}(\mathbf{0}) \cong C^\Gamma$. The notations introduced here will play a significant role in Sect. 3.

2.3 Decomposition of the base space *à la Delorme*

After this brief digression into deformation theory, we now return to our objective. From the decomposition (2.3) of the semigroup algebra $\mathbb{C}[\Gamma]$ it is easily deduced that the Jacobian matrix presents a simple-to-describe block decomposition. Indeed, if we set $\Gamma_i := \Gamma_{A_i/d_i}$, $(h_1^1, \ldots, h_{g_1}^1) = I_1$ and $(h_1^2, \ldots, h_{g_2}^2) = I_2$, then the Jacobian matrix of the defining equations of C^Γ has a block decomposition in terms of the Jacobian matrices of C^{Γ_i} and an extra row in terms of the extra new relation as follows:

$$\left(\frac{\partial f_i}{\partial u_j}\right)_{\substack{1\le i\le g\\ 0\le j\le g}} = \begin{pmatrix} \left(\frac{\partial h_i^1}{\partial x_j}\right) & 0 \\ 0 & \left(\frac{\partial h_i^2}{\partial y_j}\right) \\ \rho_1 & \rho_2 \end{pmatrix},$$

where we identify

$$f_1 = h_1^1,\; f_2 = h_2^1,\; \ldots,\; f_{g_1} = h_{g_1}^1,\; f_{g_1+1} = h_1^2,\; f_{g_1+2} = h_2^2,\; \ldots,\; f_{g_1+g_2} = h_{g_2}^2,$$

and $f_g = \rho_1 + \rho_2$ for $\rho_1 = (\partial\rho/\partial u_0, \ldots, \partial\rho/\partial u_{g_1})$ and $\rho_2 = (\partial\rho/\partial u_{g_1+1}, \ldots, \partial\rho/\partial u_g)$. The consideration of this block decomposition leads to the proof of the fact that the base spaces of the miniversal deformations of C^{Γ_i} are contained in the base space of the miniversal deformation of C^Γ.

To do so, first observe that $\left(\frac{\partial f_i}{\partial u_j}\right)_{i,j} \mathbb{C}[u_0, \ldots, u_g]^{g+1}$ is just the $\mathbb{C}[u_0, \ldots, u_g]^g$–submodule

$$\overline{N}_\Gamma = \left\langle \left(\frac{\partial f_1}{\partial u_0}, \ldots, \frac{\partial f_g}{\partial u_0}\right), \ldots, \left(\frac{\partial f_1}{\partial u_g}, \ldots, \frac{\partial f_g}{\partial u_g}\right) \right\rangle \subset \mathbb{C}[u_0, \ldots, u_g]^g. \tag{2.5}$$

Let us now define the $\mathbb{C}[u_0, \ldots, u_g]^{g_i}$–submodules

$$\overline{N}_{\Gamma_1} := \left\langle \left(\frac{\partial h_1^1}{\partial x_0}, \ldots, \frac{\partial h_{g_1}^1}{\partial x_0}\right), \ldots, \left(\frac{\partial h_1^1}{\partial x_{g_1}}, \ldots, \frac{\partial h_{g_1}^1}{\partial x_{g_1}}\right) \right\rangle \subset \mathbb{C}[x_0, \ldots, x_{g_1}]^{g_1},$$

$$\overline{N}_{\Gamma_2} := \left\langle \left(\frac{\partial h_1^2}{\partial y_0}, \ldots, \frac{\partial h_{g_2}^2}{\partial y_0}\right), \ldots, \left(\frac{\partial h_1^2}{\partial y_{g_2}}, \ldots, \frac{\partial h_{g_2}^2}{\partial xy_{g_2}}\right) \right\rangle \subset \mathbb{C}[y_0, \ldots, y_{g_2}]^{g_2}.$$

For each $i = 1, 2$ we have the canonical projections

$$\tau_1 : \mathbb{C}[\Gamma]^g = \bigoplus_{j=1}^{g} \mathbb{C}[\Gamma]e_j \to \mathbb{C}[\Gamma]^{g_1} = \bigoplus_{j=1}^{g_1} \mathbb{C}[\Gamma]e_j,$$
$$\tau_2 : \mathbb{C}[\Gamma]^g \to \mathbb{C}[\Gamma]^{g_2} = \bigoplus_{j=g_1+1}^{g_1+g_2} \mathbb{C}[\Gamma]e_j,$$

where the e_j build the standard $\mathbb{Z}$-basis. We shall denote by τ_i^{-1} the maps

$$\tau_1^{-1} : \bigoplus_{j=1}^{g_1} \mathbb{C}[\Gamma]e_j \ni v \mapsto \tau_1^{-1}(v) := (v, \mathbf{0}) \in \bigoplus_{j=1}^{g} \mathbb{C}[\Gamma]e_j$$
$$\tau_2^{-1} : \bigoplus_{j=g_1+1}^{g_1+g_2} \mathbb{C}[\Gamma]e_j \ni v \mapsto \tau_2^{-1}(v) := (\mathbf{0}, v) \in \bigoplus_{j=1}^{g} \mathbb{C}[\Gamma]e_j$$

We are now ready to prove the first main result of the paper.

Theorem 2.3 *Let Γ be a complete intersection numerical semigroup. Under the previous notation, write $N_\Gamma := \varphi(\overline{N}_\Gamma)$ and $N_i := \varphi_i(\overline{N}_{\Gamma_i})$ for $i = 1, 2$. Then, the linear maps τ_1, τ_2 induce the following injective morphisms*

$$\Phi_1 : \mathbb{C}[\Gamma_1]^{g_1}/N_1 \longrightarrow \mathbb{C}[\Gamma]^g/N_\Gamma, \quad \Phi_2 : \mathbb{C}[\Gamma_2]^{g_2}/N_2 \longrightarrow \mathbb{C}[\Gamma]^g/N_\Gamma,$$

Proof First we observe that

$$N_\Gamma = \left\langle \left(\sum_{j=1}^{g} \varphi\Big(\frac{\partial f_j}{\partial u_0}\Big) e_j, \ldots, \sum_{j=1}^{g} \varphi\Big(\frac{\partial f_j}{\partial u_g}\Big) e_j \right) \right\rangle$$

is a submodule of $\bigoplus_{j=1}^{g} \mathbb{C}[\Gamma]e_j$. Hence, in order to compare N_i with N_Γ we need first to understand the relation between the maps φ_i and the map φ. Thanks to the tensor product decomposition of $\mathbb{C}[\Gamma]$ we can write

$$\varphi = \pi \circ \lambda_\rho \circ (\varphi_1 \otimes \varphi_2), \tag{2.6}$$

where λ_ρ is the multiplication by ρ in $\mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2]$ and π is the canonical projection. By the hypothesis, ρ is a regular element of $\mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2]$ and thus yields the exact sequence

$$0 \to \mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2] \xrightarrow{\lambda_\rho} \mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2] \xrightarrow{\pi} \frac{\mathbb{C}[\Gamma_1] \otimes \mathbb{C}[\Gamma_2]}{\langle \rho \rangle} \to 0.$$

Therefore the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{C}[u_0,\dots,u_g] & \xrightarrow{\quad\varphi\quad} & \mathbb{C}[C^\Gamma] & \longrightarrow & 0\\ \wr\| & & \uparrow{\scriptstyle \pi\circ\lambda_\rho} & & \\ \mathbb{C}[x_0,\dots,x_{g_1}]\otimes\mathbb{C}[y_0,\dots,y_{g_2}] & \xrightarrow{\varphi_1\otimes\varphi_2} & \mathbb{C}[\Gamma_1]\otimes\mathbb{C}[\Gamma_2] & \longrightarrow & 0. \end{array}$$

Now we are ready to prove the statement. We will only check that Φ_1 is injective, as the result for Φ_2 may be handled in much the same way.

Let us denote by $\mathcal{N} := \tau_1(N_\Gamma)$, the submodule of $\bigoplus_{j=1}^{g_1}\mathbb{C}[\Gamma]e_j$ which is generated as

$$\begin{aligned}\mathcal{N} &= \left\langle \sum_{j=1}^{g_1}\varphi\Big(\frac{\partial f_j}{\partial u_0}\Big)e_j,\dots,\sum_{j=1}^{g_1}\varphi\Big(\frac{\partial f_j}{\partial u_g}\Big)e_j\right\rangle\\ &= \left\langle \sum_{j=1}^{g_1}\varphi\Big(\frac{\partial h_j^1}{\partial x_0}\Big)e_j,\dots,\sum_{j=1}^{g_1}\varphi\Big(\frac{\partial h_j^1}{\partial x_{g_1}}\Big)e_j\right\rangle.\end{aligned}$$

The equality follows from the identification $x_i = u_i$ for $i \in [0, g_1]$ and the fact that $\partial h_j^1/\partial u_j = 0$ for $j \in [g_1+1, g]$.

The previous considerations yield the commutative diagram

$$\begin{array}{ccccccccc} \bigoplus_{j=0}^{g_1}\mathbb{C}[u_0,\dots,u_g]e_j & \xrightarrow{\varphi_1\otimes\varphi_2} & \bigoplus_{j=0}^{g_1}\mathbb{C}[\Gamma_1]\otimes\mathbb{C}[\Gamma_2]e_j & \xrightarrow{\pi\circ\lambda_\rho} & \bigoplus_{j=0}^{g_1}\mathbb{C}[\Gamma]e_j & \xrightarrow{\pi_1} & \dfrac{\bigoplus_{j=0}^{g_1}\mathbb{C}[\Gamma]e_j}{\mathcal{N}} & \longrightarrow & 0\\ \Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i} & & & & \nearrow{\scriptstyle \Phi'_1} & & \\ \bigoplus_{j=0}^{g_1}\mathbb{C}[x_0,\dots,x_{g_1}]e_j & \xrightarrow{\varphi_1} & \bigoplus_{j=0}^{g_1}\mathbb{C}[\Gamma_1]e_j & \xrightarrow{\pi_2} & \dfrac{\bigoplus_{j=0}^{g_1}\mathbb{C}[\Gamma_1]e_j}{N_1} & & \longrightarrow & & 0 \end{array}$$

As $\Phi_1 = \tau_1^{-1}\circ\Phi'_1$, for the proof of the injectivity of Φ_1 it is enough to prove the injectivity of Φ'_1. In order to show the injectivity of Φ'_1 it is enough to prove that

$$\pi\circ\lambda_\rho\circ i(N_1) = \mathcal{N},$$

however this follows by definition of N_1, which is

$$N_1 = \left\langle \sum_{j=1}^{g_1}\varphi_1\Big(\frac{\partial h_j^1}{\partial x_0}\Big)e_j,\dots,\sum_{j=1}^{g_1}\varphi_1\Big(\frac{\partial h_j^1}{\partial x_{g_1}}\Big)e_j\right\rangle,$$

and the fact that the map i is defined as $z(x_0,\dots,x_{g_1}) \mapsto z\otimes 1 \in \mathbb{C}[u_0,\dots,u_g]$. □

Remark 2.4 Observe that Theorem 2.3 implies that the base spaces of the miniversal deformations of C^{Γ_1} and C^{Γ_2} are embedded in the miniversal deformation of C^{Γ}. Recall that $T^1\left(C^{\Gamma_1}\right) \simeq \mathbb{C}[\Gamma_1]^{g_1}/N_1$ as $\mathbb{C}[u_0, \dots, u_{g_1}]$–algebra and $T^1(C^{\Gamma_2}) \simeq \mathbb{C}[\Gamma_2]^{g_2}/N_2$ as $\mathbb{C}[u_{g_1+1}, \dots, u_g]$–algebra. As the maps Φ_1, Φ_2 are injective, we can identify $T^1\left(C^{\Gamma_1}\right)$ (resp. $T^1\left(C^{\Gamma_2}\right)$) with its image by Φ_1 (resp. Φ_2) so that $T^1(C^{\Gamma_1}) \oplus T^1\left(C^{\Gamma_2}\right)$ is a $\mathbb{C}[u_0, \dots, u_g]$–submodule of $T^1\left(C^{\Gamma}\right)$. Hence it is provided an embedding of the corresponding miniversal deformations.

2.4 Miniversal deformation of a free semigroup curve

In this part, we will consider a particular class of complete intersection numerical semigroups which are called 'free semigroups'. Consider a numerical semigroup Γ generated (not necessarily minimally) by $G := \{a_0, a_1, \dots, a_g\}$. Assume that G satisfies the condition

$$n_i a_i \in \langle a_0, a_1, \dots, a_{i-1} \rangle, \tag{2.7}$$

where $n_i := \gcd(a_0, a_1, \dots, a_{i-1})/\gcd(a_0, a_1, \dots, a_i)$, for all $i \in [1, g]$. A numerical semigroup admitting a set of generators G satisfying (2.7) for all $i \geq 1$ was named *free numerical semigroup* by Bertin and Carbonne [2]. Moreover, without loss of generality we can further assume that $n_i > 1$ for all $i \in [1, g]$. We therefore define:

Definition 2.5 A numerical semigroup $\Gamma = \langle G \rangle$ generated by a set G satisfying the condition (2.7) for all $i \geq 1$ and $n_i > 1$ for all $i \in [1, g]$ is called *free*. The monomial curve C^{Γ} corresponding to a free numerical semigroup Γ will be called free numerical semigroup curve —or *free semigroup curve* in short.

Let $\Gamma = \langle a_0, a_1, \dots, a_g \rangle$ be a free semigroup. From the condition (2.7), for each i there exist numbers $\ell_0^{(i)}, \dots, \ell_{i-1}^{(i)} \in \mathbb{N}$ such that

$$n_i a_i = \ell_0^{(i)} a_0 + \cdots + \ell_{i-1}^{(i)} a_{i-1}, \ i \in [1, g]. \tag{2.8}$$

Therefore, it is easy to see that the equations

$$f_i = u_i^{n_i} - u_0^{\ell_0^{(i)}} u_1^{\ell_1^{(i)}} \cdots u_{i-1}^{\ell_{i-1}^{(i)}} = 0 \quad \text{for} \quad i \in [1, g] \tag{2.9}$$

define the curve C^{Γ}. In this case, we will explicitly describe the $\mathbb{C}$–basis of $T^1(C^{\Gamma})$. To do so, we need first to define some auxiliary sets parametrized by some distinguished coefficients in the identities (2.8), as the following diagram points out:

$$
\begin{aligned}
n_1 a_1 &= \boxed{\ell_0^{(1)}} a_0 \\
n_2 a_2 &= \ell_0^{(2)} a_0 + \ell_1^{(2)} a_1 \\
n_3 a_3 &= \ell_0^{(3)} a_0 + \ell_1^{(3)} a_1 + \ell_2^{(3)} a_2 \\
n_4 a_4 &= \ell_0^{(4)} a_0 + \ell_1^{(4)} a_1 + \ell_2^{(4)} a_2 + \ell_3^{(4)} a_3 \\
&\vdots \\
n_g a_g &= \ell_0^{(g)} a_0 + \ell_1^{(g)} a_1 + \ell_2^{(g)} a_2 + \cdots + \ell_{g-2}^{(g)} a_{g-2} + \ell_{g-1}^{(g)} a_{g-1}.
\end{aligned}
$$

To $\ell_0^{(1)}$ we assign the set

$$
E_{\ell_0^{(1)}, n_1} = \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ 0 \leq k_0 \leq \ell_0^{(1)} - 2 \ \wedge \ k_1 \in [0, n_1 - 2] \right\},
$$

which depends on $\ell_0^{(1)}$ and n_1; this corresponds to the simply plain box in the diagram above.

In a next step, we need an iteration process to introduce a second auxiliary family of sets as follows: for $s \in [2, g]$ we consider

$$
\begin{aligned}
I^{(1)}_{\ell_0^{(s)}} &= \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ k_0 \in \left[0, \ell_0^{(s)} - 1\right] \ \wedge \ k_1 \in [0, n_1 - 1] \right\} \\
I^{(1)}_{\ell_1^{(s)}} &= \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ k_0 - \ell_0^{(s)} \in \left[0, \ell_0^{(1)} - 1\right] \ \wedge \ k_1 \in \left[0, \ell_1^{(s)} - 1\right] \right\}
\end{aligned}
$$

(These sets depend on the coefficients enclosed in a grey-shaded square and the one with a yellow background corresponding to $\ell_1^{(2)}$ in the preceding diagram). Now we define the sets $D'_{\ell_1^{(s)}}$ depending on the annihilation of $\ell_0^{(s)}$, namely

- If $\ell_1^{(s)} = 0$, then $D'_{\ell_1^{(s)}} = I^{(1)}_{\ell_0^{(i)}}$ for $i \in [2, g]$.
- If $\ell_1^{(i)} \neq 0$, then $D'_{\ell_1^{(i)}} = I^{(1)}_{\ell_0^{(i)}} \bigcup I^{(1)}_{\ell_1^{(i)}}$, for $i \in [2, g]$.

These base cases allow us to define iteratively the sets $D'_{\ell_{s-1}^{(s)}}$ for $s \in [3, g]$ from the construction of sets $D'_{\ell_i^{(j)}}$ for $i \in [1, g-1]$, $j \in [i+2, g]$ with $j - i > 1$.

To do so, we set for $t \in [3, g-1]$, $s \in [t+1, g]$

$$
I^{(t-1)}_{\ell_{t-2}^{(s)}} = \left\{ (k_0, \ldots, k_{t-1}) \in \mathbb{N}^t \ : k_{t-1} \in [0, n_{t-1} - 1] \ \wedge \ (k_0, \ldots, k_{t-2}) \in D'_{\ell_{t-2}^{(s)}} \right\}
$$

$$
I^{(t-1)}_{\ell_{t-1}^{(s)}} = \left\{ (k_0, \ldots, k_{t-1}) \in \mathbb{N}^t \ : k_{t-1} \in \left[0, \ell_{t-1}^{(s)} - 1\right] \ \wedge \right.
\left. \left(k_0 - \ell_0^{(s)}, \ldots, k_{t-2} - \ell_{t-2}^{(s)}\right) \in D'_{\ell_{t-2}^{(t-1)}} \right\}
$$

For $t \in [3, g-1]$ and $s \in [t+1, g]$ we finally define

$$D'_{\ell^{(s)}_{t-1}} = \begin{cases} I^{(t-1)}_{\ell^{(s)}_{t-2}} & \text{if } \ell^{(s)}_{t-1} = 0, \\ I^{(t-1)}_{\ell^{(s)}_{t-2}} \cup I^{(t-1)}_{l^{(s)}_{t-1}} & \text{if } \ell^{(s)}_{t-1} \neq 0. \end{cases}$$

We focus on the sets $D'_{\ell^{(s)}_{s-1}}$ (those corresponding to coefficients highlighted in yellow in the diagram above); they have the following interpretation:

Lemma 2.6 *For every $s \in [2, g]$, the set*

$$\left\{ \bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_{s-1}^{k_{s-1}} \ : \ (k_0, k_1, \ldots, k_{s-1}) \in D'_{\ell^{(s)}_{s-1}} \right\}$$

is a system of generators of $\mathbb{C}[u_0, \ldots, u_s]/(f_1, \ldots, f_s)$; the writing $\bar{u}_0, \bar{u}_1, \ldots, \bar{u}_s$ points out the classes by the canonical projection.

Proof The statement can be proved in much the same way as [5, Lemme 4]. □

Finally, depending on the vanishing of $\ell^{(s)}_{s-1}$, we define for $s \in [2, g]$

$$D_{\ell^{(s)}_{s-1}} := \begin{cases} D'_{\ell^{(s)}_{s-1}} \setminus \max \left\{ (k_0, k_1, \ldots, k_{s-1}) \in I^{(s-1)}_{\ell^{(s)}_{s-2}} \right\} & \text{if } \ell^{(s)}_{s-1} = 0, \\ D'_{\ell^{(s)}_{s-1}} \setminus \max \left\{ (k_0, k_1, \ldots, k_{s-1}) \in I^{(s-1)}_{\ell^{(s)}_{s-1}} \right\} & \text{if } \ell^{(s)}_{s-1} \neq 0, \end{cases}$$

where the maximal point is considered with regard to the lexicographical order in $\mathbb{N}^s$.

The sets $D_{\ell^{(s)}_{s-1}}$ allow us to describe a basis of the quotient $\mathbb{C}$-vector space $\mathbb{C}[\Gamma]^g/N_\Gamma$:

Theorem 2.7 *Let $\Gamma = \langle a_0, a_1, \ldots, a_g \rangle$ be a free numerical semigroup. Consider the standard basis given by the (column) unit vectors $e_1, \ldots, e_g$. A basis of the $\mathbb{C}$-vector space $T^1 = \mathbb{C}[\Gamma]^g/N_\Gamma$ consists of the images by the $\mathbb{C}$-linear map $\mathbb{C}[u_0, \ldots, u_g]^g/\overline{N}_{\Gamma_g} \to \mathbb{C}[\Gamma_{g+1}]^{g+1}/N_{\Gamma_{g+1}}$ where $\overline{N}_{\Gamma_g}$ is defined in eqn. (2.5) (and similarly for N_{Γ_g}) of the following column vectors of monomials:*

- ◇ $\left(\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_g^{k_g}\right) e_1$, *where $(k_0, k_1) \in E_{\ell^{(1)}_0, n_1}$, and $k_m \in [0, n_m - 1]$, $m = 2, \ldots, g$.*
- ◇ $\left(u_0^{k_0} u_1^{k_1} u_2^{k_2} \cdots u_g^{k_g}\right) e_2$, *where $(k_0, k_1) \in D_{\ell^{(2)}_1}$, and $k_2 = 0, \ldots, n_2 - 2$, $k_m = 0, \ldots, n_m - 1$ for $m = 3, \ldots, g$.*
- ◇ $\left(\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_g^{k_g}\right) e_m$, *where $(k_0, k_1, \ldots, k_{m-1}) \in D_{\ell^{(m)}_{m-1}}$; here $k_m \in [0, n_m - 2]$ for $m = 3, \ldots, g-1$ and $k_{m'} \in [0, n_{m'} - 1]$ for $m' \in [m+1, g]$.*
- ◇ $\left(\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_g^{k_g}\right) e_g$, *where $(k_0, k_1, \ldots, k_{g-1}) \in D_{\ell^{(g)}_{g-1}}$ and $k_g \in [0, n_g - 2]$.*

Proof We will proceed by induction on the number of generators of the numerical semigroup Γ. Write $\Gamma_h = \langle a_0, a_1, \dots, a_h \rangle$ for $h = 1, \dots, g$.

As base case, assume $h = 1$; here the claim is easy since the monomial curve is the plane curve with equation $f_1(u_0, u_1) = u_1^{a_0} - u_0^{a_1}$. Therefore, as f_1 is quasihomogeneous it belongs to the Jacobian ideal $(\partial f_1/\partial u_0, \partial f_1/\partial u_1) = (u_0^{a_1-1}, u_1^{a_0-1})$ and then

$$T^1\left(C^{\Gamma_1}\right) = \frac{\mathbb{C}[u_0, u_1]}{\left(u_0^{a_1-1}, u_1^{a_0-1}\right)}.$$

In this case we denote $\ell_0^{(1)} = a_1$ and a $\mathbb{C}$–basis of $T^1(C^{\Gamma_1})$ is the set $\left\{\bar{u}_0^{k_0}\bar{u}_1^{k_1} \ : \ (k_0, k_1) \in E_{\ell_0^{(1)}, n_1}\right\}$.

Now suppose by induction that the result is true for $h < g$ and let us assume we are in the case $h + 1$. Recall that, since Γ_{h+1} is a free semigroup, we may write $\Gamma_{h+1} = n_{h+1}\Gamma_h + a_{h+1}\mathbb{N}$. Also, by Delorme [9, Proposition 10] we have $c(\Gamma_{h+1}) = n_{h+1}c(\Gamma_h)+(n_{h+1}-1)(a_{h+1}-1)$, which according to Pinkham [16, Sect. 10] coincides with the dimension of $T^1(C^{\Gamma_{h+1}})$; observe that $n_{h+1} \neq 1$ since Γ_{h+1} is free.

From now on, we will denote $N_h := N_{\Gamma_h}$. Let us denote by $\mathscr{B}_h$ the basis of the $\mathbb{C}$-vector space $\mathbb{C}[\Gamma_h]^h/N_h$ provided by induction hypothesis. Theorem 2.3 yields the injective $\mathbb{C}$-linear map

$$\begin{array}{ccc} \mathbb{C}[\Gamma_h]^h/N_h & \xrightarrow{\Phi} & \mathbb{C}[\Gamma_{h+1}]^{h+1}/N_{h+1} \\ \bar{z} & \mapsto & (\bar{z}, 0) \end{array}$$

Then,

$$\left\{\bar{u}_{h+1}^{k}\Phi(\bar{z}) \ : \ \bar{z} \in \mathscr{B}_h, \ ^\circ \ k \in [0, n_{h+1} - 1]\right\}$$

is a set of $\mathbb{C}$–linearly independent non-zero elements of $\mathbb{C}[\Gamma_{h+1}]^{h+1}/N_{h+1}$ whose cardinality is $n_{h+1}c(\Gamma_h)$. Moreover, by induction hypothesis those are precisely the set of vectors defined by parts (1), (2) and (3). Therefore, it remains to show that

$$\left(\bar{u}_0^{k_0}\bar{u}_1^{k_1}\bar{u}_2^{k_2}\cdots\bar{u}_h^{k_{h+1}}\right)\vec{e}_{h+1}$$

for $(k_0, k_1, \dots, k_h) \in D_{\ell_h^{(h+1)}}$ with $k_h \in [0, n_h - 2]$ are $(n_{h+1} - 1)(a_{h+1} - 1)$ non-zero elements which together with the previous vectors build a system of generators of $\mathbb{C}[\Gamma_{h+1}]^{h+1}/N_{h+1}$.

By Lemma 2.6 we have that $\bar{u}_0^{k_0}\cdots\bar{u}_h^{k_h}\bar{u}_{h+1}^{k_{h+1}}$ with $(k_0, \dots, k_h) \in D'_{\ell_h^{(h+1)}}$ is a system of generators of $\mathbb{C}[u_0, \dots, u_{h+1}]/(f_1, \dots, f_{h+1})$. Observe that, as each f_i is homogeneous of degree $n_i a_i$, our definition of $D_{\ell_h^{(h+1)}}$ from $D'_{\ell_h^{(h+1)}}$ eliminates the unique element of $D'_{\ell_h^{(h+1)}}$ that goes to 0 after taking quotient with N_{h+1}. Therefore, to conclude the proof we only need to show that the number of elements of $D'_{\ell_h^{(h+1)}}$ is a_{h+1}.

To prove that, we recall the definition of the sets $D'_{\ell^{(s)}_{t-1}}$, namely

$$D'_{\ell^{(s)}_{t-1}} = \begin{cases} I^{(t-1)}_{\ell^{(s)}_{t-2}} & \text{if } \ell^{(s)}_{t-1} = 0. \\ I^{(t-1)}_{\ell^{(s)}_{t-2}} \cup I^{(t-1)}_{l^{(s)}_{t-1}} & \text{if } \ell^{(s)}_{t-1} \neq 0. \end{cases}$$

for $s \geq 2$ and $t = 3, \ldots, s$. From this definition it is easily seen that

$$\left|D'_{\ell^{(s)}_{t-1}}\right| = n_{t-1}\left|D'_{\ell^{(s)}_{t-2}}\right| + \ell^{(s)}_{t-1}\left|D'_{\ell^{(t-1)}_{t-2}}\right|. \tag{2.10}$$

Recall also that $\ell^{(1)}_0 = a_1/e_1 = a_1/(n_2 \cdots n_{h+1})$ and then

$$\left|D'_{\ell^{(h+1)}_1}\right| = \ell^{(h+1)}_0 n_1 + \ell^{(h+1)}_1 \ell^{(1)}_0 = \frac{\ell^{(h+1)}_0 a_0 + \ell^{(h+1)}_1 a_1}{n_2 \cdots n_{h+1}} \quad \text{and}$$

$$\left|D'_{\ell^{(2)}_1}\right| = \frac{a_2}{e_2} = \frac{a_2}{n_3 \cdots n_{h+1}}.$$

Recursively, we can use eqn. (2.10) to show

$$\left|D'_{\ell^{(s)}_{t-2}}\right| = \frac{\ell^{(s)}_0 a_0 + \cdots + \ell^{(s)}_{t-2} a_{t-2}}{n_{t-1} \cdots n_{h+1}} \quad \text{and} \quad \left|D'_{\ell^{(t-1)}_{t-2}}\right| = \frac{a_{t-1}}{e_{t-1}} = \frac{a_{t-1}}{n_{t-2} \cdots n_{h+1}} \tag{2.11}$$

Finally, applying the previous computations to the case $s = h+1$ and $t = h+1$ and the identity

$$n_{h+1} a_{h+1} = \ell^{(h+1)}_0 a_0 + \cdots + \ell^{(h+1)}_{h-1} a_{h-1} + \ell^{(h+1)}_h a_h$$

we obtain $\left|D'_{\ell^{(h+1)}_h}\right| = a_{h+1}$. In this way, the number of vectors of the form (4) is exactly the product $(n_{h+1} - 1)(a_{h+1} - 1)$, which is the desired conclusion. □

Remark 2.8 (1) Theorem 2.7 is a generalization of [5, Théorème 3] in the sense that, if Γ is the value semigroup associated to a plane branch (hence irreducible), then our Theorem 2.7 recovers [5, Théorème 3].
(2) If we allow $n_i = 1$ for some $i \in [1, g+1]$, then observe that there is no loss of generality in the proof of Theorem 2.7. In that case the conductor in the iteration remains constant and the induction is trivial by Theorem 2.3.

We illustrate Theorem 2.7 by showing an explicit construction of a basis of the $\mathbb{C}$-vector space $\mathbb{C}[\Gamma_g]^g/N_{\Gamma_g}$.

Example 2.9 Set the sequence of positive integers $A = (18, 27, 21, 32)$. It is easily seen that the sequence of (n_1, n_2, n_3) associated to A is $(2, 3, 3)$ and that the numerical semigroup $\Gamma_A = \langle A \rangle$ generated by A is free, since

$$\begin{aligned} n_1\, a_1 &= 2 \cdot 27 = 3 \cdot 18 = \ell_0^{(1)} a_0, \\ n_2\, a_2 &= 3 \cdot 21 = 2 \cdot 18 + 1 \cdot 27 = \ell_0^{(2)} a_0 + \ell_1^{(2)} a_1 \\ n_3\, a_3 &= 3 \cdot 32 = 3 \cdot 18 + 0 \cdot 27 + 2 \cdot 21 = \ell_0^{(3)} a_0 + \ell_1^{(3)} a_1 + \ell_2^{(3)} a_2. \end{aligned} \tag{2.12}$$

Now, let us describe a basis of the quotient $\mathbb{C}$-vector space $\mathbb{C}[\Gamma_A]^3/N_{\Gamma_A}$ taking into account both Theorem 2.7 and eqns. (2.12). First we calculate the elements belonging to the set $E_{\ell_0^{(1)}, n_1}$; this is

$$E_{\ell_0^{(1)}, n_1} = \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ k_0 = 0, 1 \ \text{ and } \ k_1 = 0 \right\} = \{(0, 0), (1, 0)\}.$$

After this base case, the following two steps are the computation of the elements in the sets $D_{\ell_1^{(2)}}$ and $D_{\ell_2^{(3)}}$. But before that, we need to calculate the corresponding sets $I^{(1)}_{\ell_0^{(2)}}$, $I^{(1)}_{\ell_1^{(2)}}$, $I^{(2)}_{\ell_1^{(3)}}$, $I^{(2)}_{\ell_2^{(3)}}$ (along with the set $I^{(1)}_{\ell_0^{(3)}}$, which is necessary for defining the set $D'_{\ell_1^{(3)}}$, itself required in the definition of $I^{(2)}_{\ell_1^{(3)}}$) giving rise to them. We start computing the elements in the set $D_{\ell_1^{(2)}}$. In this particular case, as $\ell_1^{(2)} = 1 \neq 0$, we have to calculate $I^{(1)}_{\ell_0^{(2)}}$, $I^{(1)}_{\ell_1^{(2)}}$, $D'_{\ell_1^{(2)}}$ and $\mathbf{h}\big(I^{(1)}_{\ell_1^{(2)}}\big) := \max \left\{ (k_0, k_1) \in I^{(1)}_{\ell_1^{(2)}} \right\}$:

$$\begin{aligned} I^{(1)}_{\ell_0^{(2)}} &= \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ 0 \leq k_0 \leq \ell_0^{(2)} - 1 = 1 \text{ and } 0 \leq k_1 \leq n_1 - 1 = 1 \right\} \\ &= \{(0, 0), (0, 1), (1, 0), (1, 1)\}. \\ I^{(1)}_{\ell_1^{(2)}} &= \left\{ (k_0, k_1) \in \mathbb{N}^2 \ : \ 0 \leq k_0 - \ell_0^{(2)} \leq \ell_0^{(1)} - 1 \text{ and } 0 \leq k_1 \leq \ell_1^{(2)} - 1 \right\} \\ &= \{(2, 0), (3, 0), (4, 0)\}. \\ D'_{\ell_1^{(2)}} &= I^{(1)}_{\ell_0^{(2)}} \cup I^{(1)}_{\ell_1^{(2)}} \quad \text{and} \quad \mathbf{h}\big(I^{(1)}_{\ell_1^{(2)}}\big) = (4, 0). \end{aligned}$$

This yields the set $D_{\ell_1^{(2)}}$, namely

$$D_{\ell_1^{(2)}} = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (3, 0)\}.$$

To conclude, we obtain the set $D_{\ell_2^{(3)}}$. In this case, first we have to compute $I^{(1)}_{\ell_0^{(3)}}$ and $D'_{\ell_1^{(3)}}$ since $\ell_1^{(3)} = 0$ and after the sets $I^{(2)}_{\ell_1^{(3)}}$, $I^{(2)}_{\ell_2^{(3)}}$, and $D'_{\ell_2^{(3)}}$, because $\ell_2^{(3)} \neq 0$, and the vector $\mathbf{h}\big(I^{(2)}_{\ell_2^{(3)}}\big) := \max\left\{\left(k_0, k_1, k_2) \in I^{(2)}_{\ell_2^{(3)}}\right)\right\}$:

$$\begin{aligned}
I^{(1)}_{\ell_0^{(3)}} &= \left\{(k_0, k_1) \in \mathbb{N}^2 \ : \ 0 \le k_0 \le \ell_0^{(3)} - 1 = 2 \text{ and } 0 \le k_1 \le n_1 - 1 = 1\right\} \\
&= \{(0,0), (0,1), (1,0), (1,1), (2,0), (2,1)\}.
\end{aligned}$$

$$D'_{\ell_1^{(3)}} = I^{(1)}_{\ell_0^{(3)}}.$$

$$\begin{aligned}
I^{(2)}_{\ell_1^{(3)}} &= \left\{ (k_0, k_1, k_2) \in \mathbb{N}^3 \ : \ 0 \le k_2 \le n_2 - 1 = 2 \text{ and } (k_0, k_1) \in D'_{\ell_1^{(3)}} \right\} \\
&= \{(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,0,1), \\
&\quad (1,0,2), (1,1,0), (1,1,1), (1,1,2), (2,0,0), (2,0,1), (2,0,2), (2,1,0), \\
&\quad (2,1,1), (2,1,2)\}.
\end{aligned}$$

$$\begin{aligned}
I^{(2)}_{\ell_2^{(3)}} &= \left\{(k_0, k_1, k_2) \in \mathbb{N}^3 : 0 \le k_2 \le \ell_2^{(3)} - 1 = 1 \text{ and } \left(k_0 - \ell_0^{(3)}, k_1 - \ell_1^{(3)}\right) \in D'_{\ell_1^{(2)}}\right\} \\
&= \{(3,0,0), (3,0,1), (3,1,0), (3,1,1), (4,0,0), (4,0,1), (4,1,0), \\
&\quad (4,1,1), (5,0,0), (5,0,1), (6,0,0), (6,0,1), (7,0,0), (7,0,1)\}.
\end{aligned}$$

$$D'_{\ell_2^{(3)}} = I^{(2)}_{\ell_1^{(3)}} \cup I^{(2)}_{\ell_2^{(3)}} \quad \text{and} \quad \mathbf{h}\big(I^{(2)}_{\ell_2^{(3)}}\big) = (7,0,1).$$

As a result, we get

$$\begin{aligned}
D_{\ell_2^{(3)}} = \{&(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,0,1), \\
&(1,0,2), (1,1,0), (1,1,1), (1,1,2), (2,0,0), (2,0,1), (2,0,2), (2,1,0), \\
&(2,1,1), (2,1,2), (3,0,0), (3,0,1), (3,1,0), (3,1,1), (4,0,0), (4,0,1), \\
&(4,1,0), (4,1,1), (5,0,0), (5,0,1), (6,0,0), (6,0,1), (7,0,0)\}.
\end{aligned}$$

A straightforward application of Theorem 2.7 shows that the image in $\mathbb{C}[\Gamma_A]^3/N_{\Gamma_A}$ of the following monomials provides a $\mathbb{C}$–basis:

$$\begin{aligned}
&(1,0,0), (u_0,0,0), (u_2,0,0), \left(u_2^2,0,0\right), (u_3,0,0), \left(u_3^2,0,0\right), \\
&(u_2u_3,0,0), \left(u_2u_3^2,0,0\right), \left(u_3^2u_2^2,0,0\right), (0,1,0), (0,u_1,0), (0,u_0,0), \\
&(0,u_0u_1,0), (0,u_0^2,0), \left(0,u_0^3,0\right), (0,u_2,0), (0,u_1u_2,0), (0,u_0u_2,0), \\
&(0,u_0u_1u_2,0), \left(0,u_0^2u_2,0\right), \left(0,u_0^3u_2,0\right), (0,u_3,0), (0,u_1u_3,0), (0,u_0u_3,0), \\
&(0,u_0u_1u_3,0), \left(0,u_0^2u_3,0\right), \left(0,u_0^3u_3,0\right), \left(0,u_3^2,0\right), \left(0,u_1u_3^2,0\right), \left(0,u_0u_3^2,0\right), \\
&\left(0,u_0u_1u_3^2,0\right), \left(0,u_0^2u_3^2,0\right), \left(0,u_0^3u_3^2,0\right), (0,u_2u_3,0), (0,u_1u_2u_3,0), (0,u_0u_2u_3,0), \\
&(0,u_0u_1u_2u_3,0), \left(0,u_0^2u_2u_3,0\right), \left(0,u_0^3u_2u_3,0\right), \left(0,u_2u_3^2,0\right), \\
&\left(0,u_1u_2u_3^2,0\right), \left(0,u_0u_2u_3^2,0\right),
\end{aligned}$$

$$\left(0, u_0u_1u_2u_3^2, 0\right), \left(0, u_0^2u_2u_3^2, 0\right), \left(0, u_0^3u_2u_3^2, 0\right), (0, 0, 1), (0, 0, u_2), \left(0, 0, u_2^2\right),$$
$$(0, 0, u_1), (0, 0, u_1u_2), \left(0, 0, u_1u_2^2\right), (0, 0, u_0), (0, 0, u_0u_2), \left(0, 0, u_0u_2^2\right),$$
$$(0, 0, u_0u_1), (0, 0, u_0u_1u_2), \left(0, 0, u_0u_1u_2^2\right), \left(0, 0, u_0^2\right), \left(0, 0, u_0^2u_2\right), \left(0, 0, u_0^2u_2^2\right),$$
$$\left(0, 0, u_0^2u_1\right), \left(0, 0, u_0^2u_1u_2\right), \left(0, 0, u_0^2u_1u_2^2\right), \left(0, 0, u_0^3\right), \left(0, 0, u_0^3u_2\right), \left(0, 0, u_0^3u_1\right),$$
$$\left(0, 0, u_0^3u_1u_2\right), \left(0, 0, u_0^4\right), \left(0, 0, u_0^4u_2\right), \left(0, 0, u_0^4u_1\right), \left(0, 0, u_0^4u_1u_2\right),$$
$$\left(0, 0, u_0^5\right), \left(0, 0, u_0^5u_2\right),$$
$$\left(0, 0, u_0^6\right), \left(0, 0, u_0^6u_2\right), \left(0, 0, u_0^7\right), (0, 0, u_3), (0, 0, u_2u_3), \left(0, 0, u_2^2u_3\right),$$

$$(0, 0, u_1u_3), (0, 0, u_1u_2u_3), \left(0, 0, u_1u_2^2u_3\right), (0, 0, u_0u_3),$$
$$(0, 0, u_0u_2u_3), \left(0, 0, u_0u_2^2u_3\right),$$
$$(0, 0, u_0u_1u_3), (0, 0, u_0u_1u_2u_3), \left(0, 0, u_0u_1u_2^2u_3\right), \left(0, 0, u_0^2u_3\right),$$
$$\left(0, 0, u_0^2u_2u_3\right), \left(0, 0, u_0^2u_2^2u_3\right),$$
$$\left(0, 0, u_0^2u_1u_3\right), \left(0, 0, u_0^2u_1u_2u_3\right), \left(0, 0, u_0^2u_1u_2^2u_3\right), \left(0, 0, u_0^3u_3\right), \left(0, 0, u_0^3u_2u_3\right),$$
$$\left(0, 0, u_0^3u_1u_3\right), \left(0, 0, u_0^3u_1u_2u_3\right), \left(0, 0, u_0^4u_3\right), \left(0, 0, u_0^4u_2u_3\right),$$
$$\left(0, 0, u_0^4u_1u_3\right), \left(0, 0, u_0^4u_1u_2u_3\right),$$
$$\left(0, 0, u_0^5u_3\right), \left(0, 0, u_0^5u_2u_3\right), \left(0, 0, u_0^6u_3\right), \left(0, 0, u_0^6u_2u_3\right), \left(0, 0, u_0^7u_3\right).$$

3 On the dimension of the moduli space of a monomial curve associated to a free semigroup

Let Γ be a complete intersection numerical semigroup and C^Γ its monomial curve. Let $\tau := \dim_{\mathbb{C}} T^1_{C^\Gamma}$ be the dimension of the base space of the miniversal deformation of C^Γ. Following the notation of Subsection 2.2, we write τ_- for the dimension of the negatively graded part of $T^1_{C^\Gamma}$ (see also eqn. (2.4) and the text following it). At this juncture, we now continue with the contents of Subsection 2.2. According to Pinkham [17], we focus on the negative part P_- of the deformation in order to study the moduli space associated to Γ.

To this purpose we need first to consider the base change in the deformation induced by the inclusion map defined as $V_- := (\mathbb{C}^{\tau_-} \times \{\mathbf{0}\}, 0) \hookrightarrow (\mathbb{C}^\tau, \mathbf{0})$, on account of the diagram

$$\begin{array}{ccccc}
(C^\Gamma, \mathbf{0}) & \longrightarrow & (\mathscr{X}, \mathbf{0}) & \longrightarrow & (\mathbb{C}^\tau, \mathbf{0}) \\
& \searrow & \uparrow & & \uparrow \\
& & (\mathscr{X}_\Gamma, \mathbf{0}) := (\mathscr{X}, \mathbf{0}) \times_{(\mathbb{C}^\tau, \mathbf{0})} (V_-, \mathbf{0}) & \longrightarrow & (V_-, \mathbf{0}).
\end{array}$$

Let us denote by $G_\Gamma : \mathscr{X}_\Gamma \to V_-$ the deformation induced by this base change. Observe that this deformation can be described in terms of the eqns. (2.4) by making $w_j = 0$ for all $j \in P_+$. This is now a negatively graded deformation. Following Pinkham, we must projectivize the fibers of G_Γ without projectivizing the base space V_-. This can be done by replacing s_j with $s_j(u_0, \ldots, u_g)X_{g+1}^{-\deg s_j}$ so that we obtain the projectivization $\overline{\mathscr{X}}_\Gamma$ of $\mathscr{X}_\Gamma$; observe that we have the inclusion $\overline{\mathscr{X}}_\Gamma \subset \mathbb{P}^{g+2} \times V_-$, where the ring $\mathbb{C}[u_0, \ldots, u_g, X_{g+1}]$ has $\deg u_i = a_i$ and $\deg X_{g+1} = 1$. According to Pinkham [16, Proposition 13.4, Remark 10.6] the morphism

$$\pi : \overline{\mathscr{X}}_\Gamma \longrightarrow V_-$$

is flat and proper, and has fibres which are reduced projective curves, and all the fibres lying over a given $\mathbb{C}^*$-orbit of V_- (i.e. orbits under the action of the multiplicative group of units $\mathbb{C}^*$) are isomorphic.

This leads Pinkham [16, Theorem 13.9] to prove the following. (Our formulation sticks to Buchweitz [3, Theorem 3.3.4]).

Theorem 3.1 (Pinkham) *Let $\mathscr{M}_{g,1}$ be the coarse moduli space of smooth projective curves C of genus g with a section i.e. of pointed compact Riemann surfaces of genus g. Let Γ be a numerical semigroup and set the subscheme of $\mathscr{M}_{g,1}$ parameterizing pairs*

$$W_\Gamma = \Big\{ (X_0, p) : X_0 \textit{ is a smooth projective curve of genus} g, \textit{and } p \in X_0 \textit{with} \Gamma_p = \Gamma \Big\},$$

where Γ_p is the Weierstraß semigroup at the point p. Moreover, write V_s^- for the open subset of V^- given by the points $u \in V_-$ such that the fibre of $\overline{\mathscr{X}}_\Gamma \to V_-$ above u is smooth. This is $\mathbb{C}^*$ equivariant, and so there exists a bijection between W_Γ and the orbit space $V_s^-/\mathbb{C}^*$.

As complete intersections can be deformed without obstructions, the following corollary is an easy consequence of Theorem 3.1 and Deligne-Greuel's formula [7, 12]:

Corollary 3.2 *Let Γ be a complete intersection numerical semigroup. Then,*

$$\dim W_\Gamma = \tau_- = c(\Gamma) - \tau_+.$$

3.1 On the recursive computation of the dimension of the moduli space of a free semigroup

Starting with the recursive presentation of a free semigroup, our aim is to compute τ_+, and then the dimension of the moduli space, in a recursive way. We will focus on the particular case of free numerical semigroups $\Gamma_g = \langle a_0, a_1, \ldots, a_g \rangle$. Recall that Γ is a free semigroup if it satisfies the condition:

$$n_i a_i \in \langle a_0, a_1, \ldots, a_{i-1} \rangle \ \text{ for all } \ i \in [1, g] \tag{3.1}$$

where $e_0 = a_0$, $e_i = \gcd(a_0, \ldots, a_i)$ and $n_i = \frac{e_{i-1}}{e_i}$. A useful observation about this class is that can be constructed in an iterative way. Let us denote by $\Gamma_i = \langle a_0, a_1, \ldots, a_i \rangle / e_i$ the "truncated numerical semigroups". Observe that we have $\Gamma_2 = n_2\Gamma_1 + a_2/e_2\mathbb{N}$, $\Gamma_3 = n_3\Gamma_2 + a_3/e_3\mathbb{N}$ and thus we can write

$$S = e_{g-1}\Gamma_{g-1} + \cdots + a_g\mathbb{N}.$$

We will provide sharp upper and lower bounds for the dimension of the moduli space of Γ_g in terms of the dimension of the moduli space of Γ_{g-1}.

Before continuing with the procedure to compute the dimension of the moduli space of a free semigroup, we need the following technical result.

Proposition 3.3 *Let $\langle a_0, a_1, \ldots, a_g \rangle$ be a free numerical semigroup and*

$$A_{s,k} = \Big\{(k_0, k_1) \in E_{\ell_0^{(1)}, n_1} : k_0a_0 + k_1a_1 + ka_s > n_1a_1 \text{ and } a_s < n_1a_1\Big\},$$

for $2 \leq s \leq g$ and $1 \leq k \leq n_s - 1$. Set $b_{s,k} := |A_{s,k}|$; then

$$b_{s,k} = \tau^+_{\langle n_1, \ell_0^{(1)} \rangle} + \left\lfloor \frac{ka_s}{e_1} \right\rfloor - \sigma_{1,k}(a_s) - \gamma_{1,k}(a_s) + 1,$$

where

$$\sigma_{1,k}(t) = \begin{cases} 0, \text{ if } \left\lfloor \frac{kt}{e_1} \right\rfloor < \ell_0^{(1)}, \\ 1, \text{ otherwise}, \end{cases}$$

and

$$\gamma_{1,k}(t) = \begin{cases} 0, & \text{if } \left\lfloor \frac{kt}{e_1} \right\rfloor < n_1 - \left\lfloor \frac{n_1}{\ell_0^{(1)}} \right\rfloor \ell_0^{(1)}, \\ \left\lfloor \frac{n_1}{\ell_0^{(1)}} \right\rfloor + 1, & \text{if } \left\lfloor \frac{kt}{e_1} \right\rfloor \geq n_1, \\ \left\lfloor \frac{kt}{e_1} \right\rfloor - n_1 + \left\lfloor \frac{n_1}{\ell_0^{(1)}} \right\rfloor \ell_0^{(1)}, & \text{if } n_1 - \left\lfloor \frac{n_1}{\ell_0^{(1)}} \right\rfloor \ell_0^{(1)} \leq \left\lfloor \frac{kt}{e_1} \right\rfloor \leq n_1. \end{cases}$$

Proof First of all, observe that we have a partition of $E_{\ell_0^{(1)}, n_1}$ as

$$E_{\ell_0^{(1)}, n_1} = \Big\{(k_0, k_1) \in E_{\ell_0^{(1)}, n_1} \ : \ k_0a_0 + k_1a_1 > n_1a_1\Big\} \sqcup \Big\{(k_0, k_1) \in E_{\ell_0^{(1)}, n_1} \ : \ k_0a_0 + k_1a_1 < n_1a_1\Big\}.$$

As by definition, $\Big\{(k_0, k_1) \in E_{\ell_0^{(1)}, n_1} \ : \ k_0a_0 + k_1a_1 > n_1a_1\Big\} \subset A_{s,k}$ then at least there are $\tau^+_{\langle n_1, \ell_0^{(1)} \rangle}$ points in $A_{s,k}$.

Set the index $s \in [2, g]$ such that $a_s < n_1a_1$ and $k \in [1, n_s - 1]$. Now, assume $(k_0, k_1) \in E_{\ell_0^{(1)}, n_1}$ such that $k_0a_0 + k_1a_1 < n_1a_1$; we want to see how many of those points satisfy $k_0a_0 + k_1a_1 + ka_2 > n_1a_1$ for some $k > 0$. Since $k_0a_0 + k_1a_1 < n_1a_1$, we have that $k_0a_0 + k_1a_1 = n_1a_1 - \varepsilon e_1$ if and only if $k_0n_1 + k_1\ell_0^{(1)} = n_1\ell_0^{(1)} - \varepsilon$, for $\varepsilon \in \mathbb{N}$. We know that, in

$$B = \left\{(k_0, k_1) : k_0 \in \left[0, \ell_0^{(1)} - 1\right],\ k_1 \in [0, n_1 - 1]\right\} \subseteq \mathbb{N}^2,$$

the line $k_0n_1 + k_1\ell_0^{(1)} = n_1\ell_0^{(1)} - \varepsilon$ contains a unique point. Assume for a moment that $(k_0, k_1) \in B$, then

$$k_0a_0 + k_1a_1 + ka_2 - n_1a_1 > 0 \iff n_1a_1 - \varepsilon e_1 + ka_2 - n_1a_1 > 0,$$

which is equivalent to $0 \leq \varepsilon \leq \left\lfloor \frac{ka_2}{e_1} \right\rfloor$. This means that there are, at most,

$$\left\lfloor \frac{ka_s}{e_1} \right\rfloor + 1$$

points such that $(k_0, k_1) \in E_{\ell_0^{(1)}, n_1}$ with $k_0a_0 + k_1a_1 < n_1a_1$ and $k_0a_0 + k_1a_1 + ka_2 > n_1a_1$ for some $k > 0$. At this stage, two distinct cases may be considered, namely:
Case 1: $\varepsilon = n_1 + k_1\ell_0^{(1)}$ with $k_1 \in [0, n_1 - 1]$ and $\ell_0^{(1)}n_1 + k_1\ell_0^{(1)} = n_1\ell_0^{(1)} - \varepsilon$.
Case 2: $\varepsilon = \ell_0^{(1)} - k_0n_1$ with $k_0n_1 + n_1\ell_0^{(1)} - \ell_0^{(1)} = n_1\ell_0^{(1)} - \varepsilon$.

Without loss of generality we can assume that $a_0 > a_1$. Since $\langle a_0, a_1, \ldots, a_g\rangle$ is a free semigroup, we have $n_1 = a_0/n_1$ and $\ell_0^{(1)} = a_1/e_1$, which implies $n_1 > \ell_0^{(1)}$. Thus, in Case 2, as $\varepsilon \geq 0$, the only possibility is $k_0 = 0$. So, under the hypothesis of $\left\lfloor \frac{ka_s}{e_1} \right\rfloor < \ell_0^{(1)}$, there will be no points satisfying the conditions of Case 2, and under the hypothesis of $\left\lfloor \frac{k\delta_s}{e_1} \right\rfloor \geq \ell_0^{(1)}$, there will be only one point satisfying the conditions of Case 2. This justifies the definition of $\sigma_{1,k}(a_s)$. On the other hand, the points satisfying the conditions of Case 1 can be studied through the function $\gamma_{1,k}(a_s)$, so that we obtain

$$b_{s,k} = \tau^+_{\langle n_1, \ell_0^{(1)}\rangle} + \left\lfloor \frac{ka_s}{e_1} \right\rfloor + 1 - \sigma_{1,k}(a_s) - \gamma_{1,k}(a_s),$$

as wished. □

Now, we can proceed with the main result of this section which relates τ_m^+ and τ_{m-1}^+. If $\Gamma = \langle a_0, a_1, \ldots, a_g\rangle$ is a free semigroup, then its monomial curve is defined by the ideal generated by the elements $f_1, \ldots, f_g \in \mathbb{C}[u_0, \ldots, u_g]$ with degrees $\deg(f_i) = n_ia_i$. According to the discussion at the beginning of this section, we can endow T^1 with a grading in such a way $\deg(u_i) = a_i$ and the equations of the deformation

$$F_i(\boldsymbol{u}, \boldsymbol{w}) = f_i(\boldsymbol{u}) + \sum_{j=1}^{\tau} w_j s_j^1(\boldsymbol{u})$$

are homogeneous with $\deg(F_i) = \deg(f_i)$, for $i \in [1, k]$. In this way, the monomial basis of T^1 provided by Theorem 2.7 has the following weights:

(1) $\deg\left(\left(\bar{u}_0^{k_0}, \bar{u}_1^{k_1}, \bar{u}_2^{k_2}\cdots\bar{u}_m^{k_m}\right)e_1\right) = \sum_{i=1}^m k_i a_i - n_1 a_1$, where $(k_0,k_1)\in E_{\ell_0^{(1)},n_1}$, and we have $k_r\in[0,n_r-1]$ for $r\in[2,m]$.

(2) $\deg\left(\left(\bar{u}_0^{k_0}, \bar{u}_1^{k_1}, \bar{u}_2^{k_2}\cdots\bar{u}_m^{k_m}\right)e_2\right) = \sum_{i=1}^m k_i a_i - n_2 a_2$, where $(k_0,k_1)\in D_{\ell_1^{(2)}}$, with $k_2\in[0,n_2-1]$, and $k_r\in[0,n_r-1]$, $r\in[3,m]$.

(3) $\deg\left(\left(\bar{u}_0^{k_0}, \bar{u}_1^{k_1}, \bar{u}_2^{k_2}\cdots\bar{u}_m^{k_m}\right)e_r\right) = \sum_{i=1}^m k_i a_i - n_r a_r$, where $(k_0,k_1,\ldots,k_{r-1})\in D_{\ell_{r-1}^{(r)}}$ with $k_r\in[0,n_r-1]$ for $r\in[3,m-1]$, and $k_{r'}\in[0,n_{r'}-1]$ for $r'\in[r+1,m]$.

(4) $\deg\left(\left(\bar{u}_0^{k_0}, \bar{u}_1^{k_1}, \bar{u}_2^{k_2}\cdots\bar{u}_m^{k_m}\right)e_m\right) = \sum_{i=1}^m k_i a_i - n_m a_m$, where $(k_0,k_1,\ldots,k_{m-1})\in D_{\ell_{m-1}^{(m)}}$ with $k_m\in[0,n_m-2]$.

Set $d_{m,k} = \left|D^+_{\ell_{m-1,k}^{(m)}}\right|$ where $D^+_{\ell_{m-1,k}^{(m)}} = \left\{(k_0,\ldots,k_{m-1})\in D_{\ell_{m-1}^{(m)}} : \sum_{i=1}^{m-1} k_i a_i > (n_m-k)a_m\right\}$ for $k\in[0,n_m-2]$. We have therefore the following.

Theorem 3.4 *Let $\Gamma_m=\langle a_0,a_1,\ldots,a_m\rangle$ be a free numerical semigroup with m generators. Then,*

$$\tau^+_{m-1}+(n_m-1)(\mu_{m-1}+d_{m,0})+\sum_{k=1}^{n_m-2}\left\lfloor\frac{ka_m}{n_m}\right\rfloor \geq \tau^+_m \geq \tau^+_{m-1}+(n_m-1)(\tau^+_{m-1}+d_{m,0}),$$

where μ_{m-1} is the conductor of the semigroup $\Gamma_{m-1}=(\langle a_0,\ldots,a_m\rangle)/(\gcd(a_0,\ldots,a_{m-1})$.

Moreover,

$$\tau^+_m \geq \tau^+_{\langle n_1,\ell_0^{(1)}\rangle}+\sum_{\substack{j\notin L_1\\ j\notin J_m}}(n_j-1)\tau^+_{\langle n_1,\ell_0^{(1)}\rangle}+\sum_{j\in L_1}\left(\sum_{k=1}^{n_j-1}b_{j,k}\right)+(n_m-1)\left(\sum_{j\in J_m}\frac{a_j}{e_j}+d_{m,0}\right),$$

where $J_m:=\left\{j\in[1,m-1]: a_m\geq n_j a_j\right\}$, $L_1:=\left\{i\in[2,m]: a_i<n_1a_1\right\}$ and $b_{j,k}$ is defined as in Proposition 3.3.

Proof As in the proof of Theorem 2.7, let us denote by $\mathscr{B}_{m-1}$ the $\mathbb{C}$–basis of $\mathbb{C}[\Gamma_{m-1}]^{m-1}/N_{m-1}$. By Theorem 2.3 we have the injective map

$$\begin{array}{ccc}\mathbb{C}[\Gamma_{m-1}]^{m-1}/N_{m-1} & \xrightarrow{\Phi} & \mathbb{C}[\Gamma_m]^m/N_m\\ \overline{z} & \mapsto & (\overline{z},\overline{0})\end{array}$$

Then, Theorem 2.7 shows that the basis $\mathscr{B}_m$ decomposes as

$$\mathscr{B}_m = \left\{ u_m^k \Phi(\overline{z}) \ : \ \overline{z} \in \mathscr{B}_{m-1}, \ k \in [0, n_m - 1] \right\}$$
$$\sqcup \left\{ \left(\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_m^{k_m} \right) e_m \ : \ (k_0, k_1, \ldots, k_{m-1}) \in D_{\ell_{m-1}^{(m)}} \right\}.$$

Let us first assume that $a_m \geq a_j n_j$ for all $j \in [1, m-1]$, then for all $k \geq 1$ and denote by $\bar{\omega} = \Phi(\overline{z})$ for $\overline{z} \in \mathscr{B}_{m-1}$. We have

$$\deg\left(\bar{u}_m^k \bar{\omega}\right) = \deg(\bar{\omega}) + k a_m = \sum_{i=0}^{m-1} \alpha_i a_i - n_j a_j + k a_m \geq 0.$$

Therefore, if $a_m \geq a_j n_j$ for all $j \in [1, m-1]$, then the decomposition of the basis implies that

$$\tau_m^+ = \tau_{m-1}^+ + (n_m - 1)\mu_{m-1} + \sum_{k=0}^{n_m-2} d_{m,k}. \tag{3.2}$$

Now, let us assume the existence of an index $j_0 \in [1, m-1]$ such that $a_m < a_{j_0} n_{j_0}$. Thus, there exists $\bar{z} \in \mathscr{B}_{m-1}$ such that $\deg(\bar{u}_m \bar{z}) < 0$. Hence, in this case we obtain the strict inequality

$$\tau_m^+ < \tau_{m-1}^+ + (n_m - 1)\mu_{m-1} + \sum_{k=0}^{n_m-2} d_{m,k}. \tag{3.3}$$

Let us now move to provide a lower bound. Set $J_m = \left\{ j \in [1, m-1] : a_m \geq n_j a_j \right\}$ and assume $[1, m-1] \setminus J_m \neq \emptyset$ as otherwise we are in the previous situation. Observe that, by the decomposition of the basis $\mathscr{B}_m$, any element $\bar{z} \in \mathscr{B}_{m-1}$ with $\deg(\bar{z}) \geq 0$ also satisfies $\deg(a_m^k \bar{z}) \geq 0$ for $k \in [0, n_m - 1]$. This implies the inequality $\tau_m^+ \geq n_m \tau_{m-1}^+$. Analogously, for any $j \in J_m$ and any $z \in D_{\ell_{j-1}^{(j)}}$ such that $\deg(z) \leq 0$ (here we identify monomial residue class and its exponent for brevity), we have $\deg(a_m^k z) \geq 0$ for $k \in [1, n_m - 1]$. Thus, if for $j \in J_m$ we set

$$d_j^- = \left| \left\{ z \in D_{\ell_{j-1}^{(j)}} \mid \deg(z) < 0 \right\} \right|,$$

then

$$\tau_m^+ \geq n_m \tau_{m-1}^+ + (n_m - 1) \left(\sum_{j \in J_m} d_j^- \right) + \sum_{k=0}^{n_m-2} d_{m,k}. \tag{3.4}$$

Independently of the assumptions on $a_m \geq n_j a_j$ or $a_m < n_{j_0} a_{j_0}$, let us now estimate the sum $\sum_{k=0}^{n_m-2} d_{m,k}$. Recall that $d_{m,0} = \left| D^+_{\ell_{m-1,0}^{(m)}} \right|$ and $d'_{m,0} = \left| D'^+_{\ell_{m-1}^{(m)}} \right| = d_{m,0} + 1$, where

$$D'^{+}_{\ell^{(m)}_{m-1,0}} = \left\{ (k_0, \ldots, k_{m-1}) \in D'_{\ell^{(m)}_{m-1}} : \sum_{i=1}^{m-1} k_i a_i > n_m a_m \right\}$$

$$D'^{+}_{\ell^{(m)}_{m-1,k}} = \left\{ (k_0, \ldots, k_{m-1}) \in D'_{\ell^{(m)}_{m-1}} : \sum_{i=1}^{m-1} k_i a_i > (n_m - k) a_m \right\}$$

This implies that

$$0 \leq \left| D'^{+}_{\ell^{(m)}_{m-1,k}} \setminus D'^{+}_{\ell^{(m)}_{m-1,0}} \right| \leq \left\lfloor \frac{k a_m}{n_m} \right\rfloor,$$

therefore $d'_{m,0} \leq d'_{m,k} = d_{m,k} + 1 \leq d'_{m,0} + \lfloor k a_m / n_m \rfloor$. We deduce then that

$$(n_m - 1)(d'_{m,0} - 1) = \sum_{k=0}^{n_m-2} (d'_{m,0} - 1) \leq \sum_{k=0}^{n_m-2} d_{m,k} \leq (n_m - 1)(d'_{m,0} - 1) + \left\lfloor \frac{k a_m}{n_m} \right\rfloor. \tag{3.5}$$

As $d_j^- \geq 0$ for all $j \in J_m$, then a combination of (3.2), (3.3), (3.4) and (3.5) provides the desired inequalities

$$\tau^+_{m-1} + (n_m - 1)(\mu_{m-1} + d_{m,0}) + \sum_{k=1}^{n_m-2} \left\lfloor \frac{k a_m}{n_m} \right\rfloor \geq \tau^+_m \geq \tau^+_{m-1} + (n_m - 1)(\tau^+_{m-1} + d_{m,0}).$$

To finish, let us show the inequality

$$\tau^+_m \geq \tau^+_{\langle n_1, \ell^{(1)}_0 \rangle} + \sum_{\substack{j \notin L_1 \\ j \notin J_m}} (n_j - 1) \tau^+_{\langle n_1, \ell^{(1)}_0 \rangle} + \sum_{j \in L_1} \left(\sum_{k=1}^{n_j - 1} b_{j,k} \right) + (n_m - 1) \left(\sum_{j \in J_m} \frac{a_j}{e_j} + d_{m,0} \right).$$

Observe that the basis $\mathscr{B}_{m-1}$ is computed through the sets $E_{\ell^{(1)}_0, n_1}$ and $D_{\ell^{(s)}_{s-1}}$.

In this way, it is obvious that in $\mathscr{B}_m$ there are at least $\tau^+_{\langle n_1, \ell^{(1)}_0 \rangle}$ positive weight elements which are precisely those of the form $\left(\bar{u}_0^{k_0} \bar{u}_{k_1}^{k_1} \right) e_1$ with $(k_0, k_1) \in E_{\ell^{(1)}_0, n_1}$ such that $k_0 a_0 + k_1 a_1 > n_1 a_1$. Now, under the notations of Proposition 3.3, there are $\sum_{j \in L_1} \sum_{k=1}^{n_j - 1} b_{j,k}$ positive weight elements which are of the form $\left(\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_s^{k} \right) e_1$ such that $k_0 a_0 + k_1 a_1 + k a_s > n_1 a_1$ with $(k_0, k_1) \in E_{\ell^{(1)}_0, n_1}$ and $a_s < n_1 a_1$.

Additionally, for any $r \in J_m$ we have $\deg \left((\bar{u}_0^{k_0} \bar{u}_1^{k_1} \bar{u}_2^{k_2} \cdots \bar{u}_m^{k_m}) e_r \right) > 0$ if $(k_0, k_1, \ldots, k_{r-1}) \in D_{\ell^{(r)}_{r-1}}$ with $k_r \in [0, n_r - 2]$ and $r' \in [r + 1, m - 1]$ so that $k_{r'} \in [0, n_{r'} - 1]$ and $k_m \in [1, n_m - 1]$. Observe that the number of such elements is precisely $(n_m-1)\left|D_{\ell^{(r)}_{r-1}}\right|$. Recall that eqn. (2.11) implies $\left|D_{\ell^{(r)}_{r-1}}\right| = a_r/e_r$. Therefore, the previous discussion provides the desired inequality. □

An immediate consequence of Theorem 3.4 is the following:

Corollary 3.5 *Let* $\Gamma = \langle a_0, a_1, \ldots, a_m\rangle$ *be a free numerical semigroup. Then the dimension* τ^-_m *of the moduli space* W_Γ *satisfies the following inequalities*

$$n_m\tau^-_{m-1} - (n_m-1)(d_{m,0}+a_m-1) \geq \tau^-_m \geq \tau^-_{m-1} - (n_m-1)(d_{m,0}+a_m-1) - \sum_{k=1}^{n_m-2}\left\lfloor \frac{ka_m}{n_m}\right\rfloor.$$

The sets $D_{\ell^{(j)}_{j-1}}$ used to determine the monomial basis of T^1 define recursively a j-dimensional staircase in $\mathbb{N}^m$. Observe that in order to determine $d_{m,k}$ or d^-_j we need to count how many points of this staircase are below and over the hyperplane defined by $n_ja_j = \ell^j_0a_0 + \cdots + \ell^j_{j-1}a_j$. Without any extra assumptions on the generators of Γ_m, it is quite a difficult task from a combinatorial point of view to provide an exact formula for $d_{m,k}$ or d^-_j and hence for τ^+_m. However, we can be more precise if we impose some extra conditions over the generators of the semigroup. In fact, we will show that the bounds of Theorem 3.4 are sharp.

3.2 Some special families of free semigroups

Let us first start with the following new characterization of the semigroup of values of an irreducible plane curve:

Theorem 3.6 *Let* $\Gamma = \langle a_0, a_1, \ldots, a_m\rangle$ *be a free numerical semigroup. Then the following statements are equivalent:*

(1) $n_ia_i < a_{i+1}$ *for all* i, *i.e.* Γ *is the semigroup of values of an irreducible plane curve singularity.*
(2) The dimension of the positive part of T^1 *can be computed recursively via the formula*

$$\tau^+_m = \tau^+_{m-1} + (n_m-1)\mu_{m-1} + \sum_{k=0}^{n_m-2} d_{m,k}.$$

Moreover for a free semigroup satisfying the conditions (1) *we have*

$$\sum_{k=0}^{n_m-2} d_{m,k} = \frac{(n_m-1)\mu_{m-1}}{2} + \frac{(n_m-3)(a_m/e_m-3)}{2} + \left\lfloor \frac{a_m}{e_mn_m}\right\rfloor - 2$$

Proof (1) $\Leftrightarrow$ (2) is a consequence of the first part of the proof of Theorem 3.4 as inductively equations (3.2) and (3.3) shows that the upper bound in Theorem 3.4 is attained if and only if $n_i a_i < a_{i+1}$ for all i. The formula for $\sum_{k=0}^{n_m-2} d_{m,k}$ is obtained by [5, Théorème 8]. □

Before to show that the lower bound is also sharp, let us first prove the following

Proposition 3.7 *Let $\Gamma_m = \langle a_0, a_1, \ldots, a_m \rangle$ be a free numerical semigroup with m generators. Under the previous notation, we have*

$$\frac{\left(\ell_0^{(1)} - 1\right)(n_1 - 2)}{2} + \left\lfloor \frac{kq_2}{n_2} \right\rfloor - 1 \geq \left| D^+_{\ell_1^{(2)},k} \right| \geq \frac{(\ell_0^{(1)} - 1)(n_1 - 2)}{2} - |C| - 1,$$

where C is the triangle defined by the lines $i = 0$, $j = n_1$ and $in_1 + j\ell_0^{(2)} = n_1\ell_0^{(2)}$. In particular, if $n_2 a_2 > n_1 a_1$ then the upper bound is attained. (See Fig. 1)

Proof Let us first denote $q_2 = a_2/e_2$. We have $n_2 a_2 = \ell_0^{(2)} a_0 + \ell_1^{(2)} a_1$.

Assume $\ell_1^{(2)} = 0$, then $D'_{\ell_1^{(2)}} = I_{\ell_0^{(2)}}$; consider

$$A = \left\{ (i, j) \in I_{\ell_0^{(2)}} : in_1 + j\ell_0^{(1)} > q_2 \right\}.$$

In order to count the points in A, we must distinguish two situations:
Case 1: $n_2 a_2 > n_1 a_1$.

The inequality $n_2 a_2 > n_1 a_1$ is equivalent to $\ell_0^{(1)} n_1 < q_2$. The lines

$$r \equiv in_1 + j\ell_0^{(1)} = n_1\ell_0^{(1)}$$
$$s \equiv in_1 + j\ell_0^{(2)} = q_2$$

are parallel. The lines r, $i = \ell_0^{(1)}$ and $j = n_1$ enclose a triangle B, and s, $i = \ell_0^{(2)}$ and $j = n_1$ the triangle A. We have that

$$|A| = |B| = \frac{\left(\ell_0^{(1)} - 1\right)(n_1 - 1)}{2} = \frac{c(\Gamma_1)}{2}.$$

Case 2: $n_2 a_2 < n_1 a_1$.

Consider again the parallel lines

$$r \equiv in_1 + j\ell_0^{(1)} = n_1\ell_0^{(1)}$$
$$s \equiv in_1 + j\ell_0^{(2)} = q_2$$

Now the inequality $n_2 a_2 < n_1 a_1$ is equivalent to $\ell_0^{(1)} n_1 > q_2$ so that a triangle C delimited by the lines r, s and $j = n_1$ appears. The interesting area here is that of the

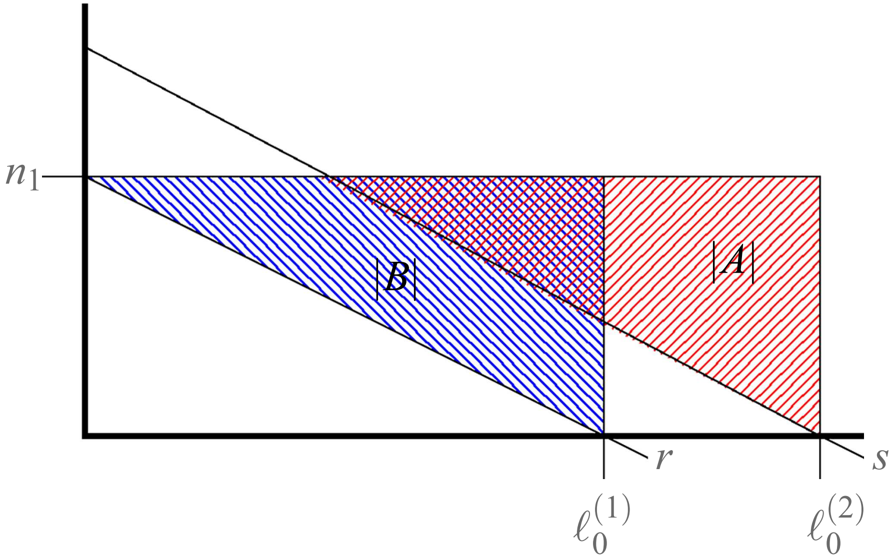

Fig. 1 Case $\ell_1^{(2)} = 0$ and $n_2a_2 > n_1a_1$.

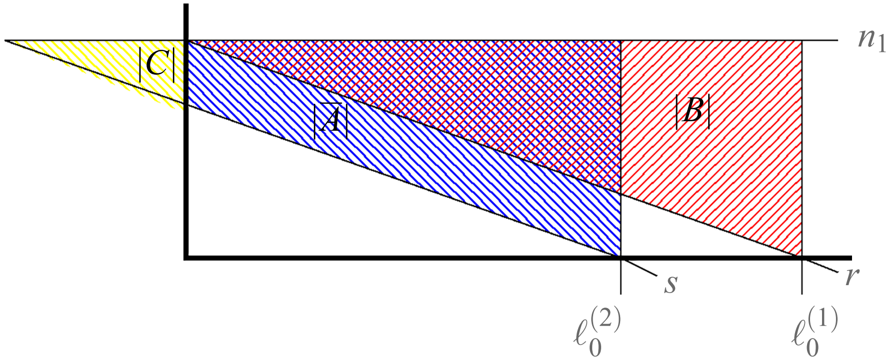

Fig. 2 Case $\ell_1^{(2)} = 0$ and $n_2a_2 < n_1a_1$.

quadrilateral $\overline{A}$ (see Fig. 2), namely

$$|\overline{A}| = |A| - |C| = |B| - |C| = \frac{\left(\ell_0^{(1)} - 1\right)(n_1 - 2)}{2} - |C|.$$

In this way we can bound the area of the region $\overline{A}$ both above and below.

Assume now that $\ell_1^{(2)} \neq 0$, then $D'_{\ell_1^{(2)}} = I_{\ell_0^{(2)}} \cup I_{\ell_1^{(2)}}$, and so the set A can be written as a union $A = A_1 \cup A_2$, where

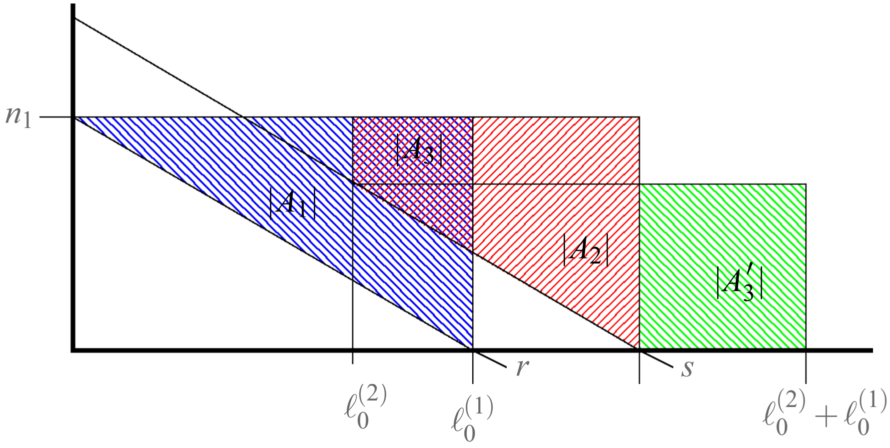

Fig. 3 Case $\ell_1^{(2)} \neq 0$ and $n_2a_2 > n_1a_1$.

$$A_1 = \left\{(i,j) \in I^{(2)}_{\ell_0} : in_1 + j\ell_0^{(2)} > q_2\right\}$$
$$A_2 = \left\{(i,j) \in I^{(2)}_{\ell_1} : in_1 + j\ell_0^{(1)} > q_2\right\}.$$

We must distinguish again two cases:
Case 1: $n_2a_2 > n_1a_1$.
As in the previous case, we have that $|B| = \frac{c(\Gamma_1)}{2} = |A|$. (See Figures 3, 4)

We have to check that $|A_3| = |A_3'|$, which is true, as the comparison of the following two easy computations shows:

$$|A_3| = \left(\frac{q_2}{n_1} - \ell_0^{(2)}\right)\left(n_1 - \ell_1^{(2)}\right) = q_2 - \frac{\ell_1^{(2)}q_2}{n_1} - \ell_0^{(2)}n_1$$
$$|A_3'| = \ell_1^{(2)}\left(\ell_0^{(2)} + \ell_0^{(1)} - \frac{q_2}{n_1}\right)$$

Case 2: $n_2a_2 < n_1a_1$.

As in the previous cases, $|\overline{A}| = |A| - |C| = |B| - |C| = \frac{c(\Gamma_1)}{2} - |C|$, where C is again a triangle. Moreover, as in the case $\ell_1^{(2)} = 0$, the part $|C|$ which is loosed comes from $I_{\ell_0^{(2)}}$.

Finally, observe that the points in

$$A = \left\{(i,j) \in D'_{\ell_1^{(2)}} : ia_0 + ja_1 + ka_2 > n_2a_2\right\}$$

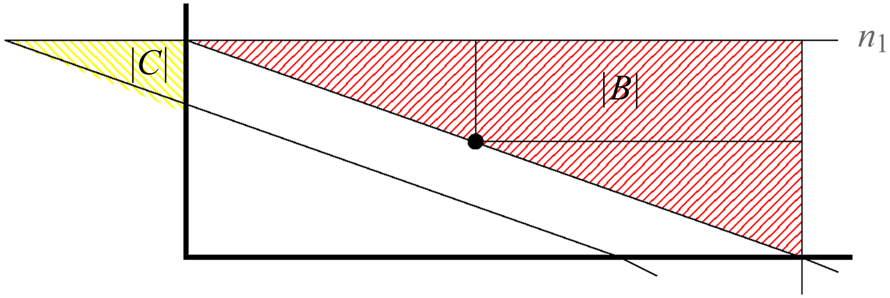

Fig. 4 Case $\ell_1^{(2)} \neq 0$ and $n_2 a_2 < n_1 a_1$.

are those satisfying $in_1 + j\ell_0^{(1)} > q_2 - \frac{kq_2}{n_2}$. From the previous pictures, it is easy to see that if $n_2a_2 > n_1a_1$ then there are exactly $\lfloor \frac{kq_2}{n_2} \rfloor$ points $(i, j) \in D'_{\ell_1^{(2)}}$ satisfying

$$q_2 > in_2 + j\ell_0^{(1)} > q_2 - \frac{kq_2}{n_2}. \tag{3.6}$$

However, if $n_2a_2 < n_1a_1$ it may happen that some of the possible solutions of eqn. (3.6) may belong to the triangle C and thus not in $D'_{\ell_1^{(2)}}$. From which we obtain the desired results. □

Proposition 3.7 will help us to show that —under certain assumption— we can give a precise formula for τ_m^+ and in particular to show that the lower bound is sharp.

Theorem 3.8 *Let $\Gamma = \langle a_0, a_1, a_2 \rangle$ be a free numerical semigroup such that $n_1a_1 > a_2$ and $a_0 > a_1$ such that $n_2a_2 > n_1a_1$. Then,*

$$\tau_2^+ = n_2\tau_1^+ + \frac{(n_2-1)(c(\Gamma_1)-2)}{2} - \frac{(n_2-1)a_2}{e_1} + \sum_{k=1}^{n_2-1}\left(2\left\lfloor \frac{ka_2}{e_1} \right\rfloor - \sigma_{1,k}(a_s) - \gamma_{1,k}(a_s) + 1\right).$$

In particular, if $n_2 = 2$, then the lower bound in Theorem 3.4 is attained.

Proof As in the proof of Theorem 3.4, we have the basis of the miniversal deformation described as

$$\mathscr{B}_2 = \left\{\bar{u}_2^k \Phi(\bar{z}) e_1 \,:\, \bar{z} \in \mathscr{B}_1,\ k \in [0, n_2-1]\right\} \sqcup \left\{\left(\bar{u}_0^{k_0}\bar{u}_1^{k_1}\bar{u}_2^{k_2}\right) e_2 \,:\, (k_0, k_1) \in D_{\ell_1^{(2)}}\right\}.$$

Recall that $\mathscr{B}_1$ is in one to one to correspondence with $E_{\ell_0^{(1)},n_1}$. As by hypothesis we have $n_1a_1 > a_2$, therefore $\deg\left(\bar{u}_2^k\Phi\left(\bar{u}_0^{k_0}\bar{u}_1^{k_1}\right)e_1\right) > 0$ if and only if

$$(k_0, k_1) \in A_{2,k} = \left\{(i, j) \in E_{\ell_0^{(1)},n_1} : ia_0 + ja_1 + ka_2 > n_1a_1 \text{ and } a_2 < n_1a_1\right\}.$$

Finally, the number of positively weighted elements in $D_{\ell_1^{(2)}}$ are those corresponding to the the sets $D^+_{\ell_1^{(2)},k}$ with $k \in [0, n_2 - 2]$. Hence, by using Proposition 3.7 and Proposition 3.3 we obtain

$$\begin{aligned}\tau_2^+ = n_2\tau_1^+ + \frac{(n_2-1)(c(\Gamma_1)-2)}{2} - \frac{(n_2-1)a_2}{e_1}\\ + \sum_{k=1}^{n_2-1}\left(2\left\lfloor\frac{ka_2}{e_1}\right\rfloor - \sigma_{1,k}(a_s) - \gamma_{1,k}(a_s) + 1\right),\end{aligned}$$

which completes the proof. □

As one may observe in Proposition 3.7 and Proposition 3.3, the computation of $d_{m,k}$ for a free numerical semigroup without any restriction can be extremely difficult. The reason is that the numbers $d_{m,k}$ depend on the relations between the different n_ja_j as we realize looking at the proof of Proposition 3.7. Similarly to Proposition 3.7 one should be able to write out an algorithmic formula or estimation for $\sum d_{m,k}$. However, the algorithmic calculation is a hard combinatorial problem. Given the length and tedious nature of these combinatorics for more than three generators, this is clearly a suitable problem to assign to a computer program, since Theorem 2.7 provides an explicit basis for the deformation.

A final remark on the dimension of the moduli space

Throughout the section we have been working without loss of generality with a system of generators $\{a_0, a_1, \ldots, a_g\}$ of the numerical semigroup satisfying the condition $n_ia_i \in \langle a_0, a_1, \ldots, a_{i-1}\rangle$ and $n_i > 1$ for all $i = 1, \ldots, g$. However, our statements encode more general information. More concretely, the monomial basis of T^1 is clearly not unique. Nevertheless, the dimension of the moduli space is intrinsic to the semigroup itself, so in particular it does not depend on the generating set of the semigroup. Our results should then be interpreted as follows: given a free numerical semigroup, there exists a system of generators that allows us to explicitly compute a monomial basis for T^1. This monomial basis leads us to the estimation of the dimension of the moduli space in a recursive way. By looking at Corollary 3.2, a part of this estimation is independent of the system of generators —essentially the one given by Theorem 2.3— and the other part actually depends on $\{a_0, a_1, \ldots, a_g\}$ as it is linked to the obtained monomial basis.

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Declarations

Conflict of Interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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