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ORIGINAL PAPER

On the image of a curve in a normal surface by a plane projection

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Abstract
We consider a finite analytic morphism $\varphi = (f, g)$ defined from a complex analytic normal surface (Z, z) to $\mathbb{C}^2$. We describe the topology of the image by φ of a reduced curve on (Z, z) by means of iterated pencils defined recursively for each branch of the curve from the initial one $\langle f, g\rangle$. This result generalizes the one obtained in a previous paper for the case in which (Z, z) is smooth and the curve irreducible. The methods we use also permit us to describe the topological type of the discriminant curve of φ, in particular, the topological type of each branch of the discriminant can be obtained from the map without previous knowledge of the critical locus.

Keywords Topological type · Pencils · Analytic morphisms · Discriminant curve · Critical locus

Mathematics Subject Classification 14H20 · 32S05 · 32S15 · 32S45 · 32S55

Introduction

Let (Z, z) be the germ of a complex analytic normal surface and let $\varphi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be the germ of a finite complex analytic morphism. Let $\gamma \subset (Z, z)$ be the germ of a reduced curve, $\gamma = \bigcup_{i=1}^{r} \gamma_i$, with γ_i a branch in (Z, z), and $\delta = \bigcup_{i=1}^{r} \delta_i$ with $\varphi(\gamma_i) = \delta_i$, $1 \leq i \leq r$. If (Z, z) is equal to $(\mathbb{C}^2, 0)$ and γ is irreducible, in [4] we have described the topology of the plane branch δ, using iterated pencils of functions defined recursively from the initial one $\Phi = \langle f, g\rangle := \{ag - bf : (a : b) \in \mathbb{CP}^1\}$. The purpose of this article is to show that this result can be generalized to the case of normal surfaces and when the curve

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γ is not necessarily irreducible; it means that we can describe the topological type of each branch δ_i, $1 \leq i \leq r$ (Sect. 1), and moreover give the intersection multiplicities between each pair of branches δ_i, δ_j (Sect. 2), which completely describes the topology of δ.

In Sect. 1, we show that, as in the plane case, the topology of the image by φ of an irreducible curve γ can be determined by the degree of the restriction map $\varphi|_\gamma : \gamma \to \delta = \varphi(\gamma)$ and intersection multiplicities between γ and some particular fibres of a sequence of iterated pencils constructed recursively from $\langle f, g \rangle$ (see 1.5). This result is stated in Theorem 1.8. In Sect. 2 we treat the case where γ is reduced but not more irreducible. It is well-known that the topological type of the plane curve $\delta = \bigcup_{i=1}^r \delta_i$, image by φ of γ, is entirely determined by the topological type of each irreducible component $\delta_i = \varphi(\gamma_i)$ and the intersection multiplicity between each pair (δ_i, δ_j), $j \neq i$. As the topological type of each irreducible component δ_i has been treated in Sect. 1, it leaves to compute the intersection multiplicity between each pair of branches of δ. This result is stated in Theorem 2.12. The proof consists in first computing such intersection multiplicities when φ is the identity map from the plane to the plane (Theorem 2.8), then using the projection formula for the intersection multiplicity (see Proposition 1.1) we can compute the intersection multiplicity between δ_i and δ_j for the general case of a finite map $\varphi : (Z, z) \to (\mathbb{C}^2, 0)$ (see Theorem 2.8).

A remarkable fact is that, for the computation of the intersection multiplicity, one needs more precise information than the one used for the topological type of the branches (see the Example 2.11). Namely, information coming from the branches of the fibres of the pencil Φ and not only the intersection numbers of the branches of γ with the whole fibres of Φ.

In [4] the process was, in particular, applied to describe the topology of the branches of the discriminant locus of $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. The key point was the study of the special fibres of the pencil $\langle f, g \rangle$ and their relations with the critical locus of φ developed in [2] as well as the relationship with the behaviour of the Hironaka quotients established in [9] and [10]. For a normal singularity (Z, z) and a finite map $\varphi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ the relation of the special fibres with the critical locus was generalized in [5]. (For a reduced surface singularity, the behaviour of generic and special fibres has been recently treated in [1]). These relations allow us to extend the results of [4] and so apply the results of Sects. 1 and 2 to describe the topological type of the discriminant curve of φ. This is the purpose of Sect. 3. In relation with the study of the discriminant curve one has to mention the recent paper [7] from García-Barroso and Popescu-Pampu in which they show that the Newton polynomial of the discriminant curve depends only on the curves defined by f and g. (See also [8] for the plane case.) Although the results are different, some of the technics used there are very close to those we use here.

1 Case of an irreducible curve

Let $\gamma \subset (Z, z)$ be an irreducible germ of curve (in short, a *branch*) and let $u : (\mathbb{C}, 0) \to (\gamma, z)$ be a normalization (uniformization) of (γ, z). Let $\{h = 0\} \subset Z$ be the curve defined by the analytic function $h : (Z, z) \to (\mathbb{C}, 0)$. We define the intersection multiplicity of $\{h = 0\}$ and γ at z, denoted $I_z(h, \gamma)$, by:

$$I_z(h, \gamma) = \operatorname{ord}_t((h \circ u)(t)).$$

Notice that the normalization u identifies the local ring $\mathcal{O}_{\gamma,z}$ to be an subring of $\mathbb{C}\{t\}$ and $\nu_\gamma(-) := I_z(-, \gamma)$ is the semi-valuation ν_γ of the local ring $\mathcal{O}_{Z,z}$ defined by γ. Moreover, we write $I_z(h, \gamma) = \infty$ if γ is a branch of $\{h = 0\}$, i.e. if the function h vanishes on γ. If

(Z,z) is smooth then $I_z(h,\gamma)$ coincides with the usual intersection multiplicity between the curve germs $\eta := \{h=0\}$ and γ. In this case we use also the notation $[\eta,\gamma] = [\{h=0\},\gamma]$ (and $[\eta,\gamma]_z$ if we want to specify the point) to indicate the intersection multiplicity $I_z(h,\gamma)$.

Let $\varphi^* : \mathcal{O}_{\mathbb{C}^2,0} \to \mathcal{O}_{Z,z}$, be the ring homomorphism associated to φ, i.e $w \mapsto \varphi^* w := w \circ \varphi$ for $w \in \mathcal{O}_{\mathbb{C}^2,0}$. If $\xi = \{w=0\} \subset (\mathbb{C}^2,0)$ we also use $\varphi^*\xi$ to denote the local curve in (Z,z) defined by $\varphi^* w$. Notice that $\varphi^*\xi$ is a local principal divisor in (Z,z).

Let $\gamma \subset (Z,z)$ be a branch, and $\varphi(\gamma) = \delta$. Let $\mathfrak{p}$ be the prime ideal on $\mathcal{O}_{Z,z}$ defining the curve germ γ on (Z,z), then the prime ideal $\mathfrak{q} = (\varphi^*)^{-1}(\mathfrak{p})$ is the ideal defining $\delta = \varphi(\gamma) \subset (\mathbb{C}^2,0)$ and one has the finite extension of one-dimensional local rings $\mathcal{O}_{\delta,0} = \mathcal{O}_{\mathbb{C}^2,0}/\mathfrak{q} \hookrightarrow \mathcal{O}_{\gamma,0} = \mathcal{O}_{Z,z}/\mathfrak{p}$. The degree k of the above extension is the degree of the restriction map $\varphi|_\gamma : \gamma \to \delta$. We define the *direct image* of γ as the divisor $\varphi_*\gamma := k \cdot \delta$. The integer k plays a relevant role in the paper (see Proposition 1.1 below); however we do not use the non-reduced structure of the direct image curve.

Proposition 1.1 (Projection formula) *One has:*

$$I_z(\varphi^* w,\gamma) = I_0(w,\varphi_*\gamma)(= kI_0(w,\delta)) \tag{1}$$

Proof Let $\mathbb{C}\{t\}$ (respectively $\mathbb{C}\{\tau\}$) be isomorphic to the normalization of $\mathcal{O}_{\gamma,z}$ (respectively to the one of $\mathcal{O}_{\delta,0}$), one has $\mathbb{C}\{\tau\} \subset \mathbb{C}\{t\}$ and $\tau(t) \in \mathbb{C}\{t\}$ has order k. In fact, one could take $\tau = t^k$ with a convenient choice of the uniformizing parameters τ and t. Now, let $w \in \mathcal{O}_{\mathbb{C}^2,0}$ be an holomorphic function on $\mathbb{C}^2$. One has $\mathrm{ord}_t(\varphi^* w(t)) = \mathrm{ord}_t(w(t^k)) = k \cdot \mathrm{ord}_\tau(w(\tau))$.□

Remark 1.2 The above formula (1) extends the classical projection formula for a map φ between smooth surfaces (see, e.g. [6]). Notice also that one could extend the above formula for any local Weil divisor $\gamma = \sum_{i=1}^r n_i\gamma_i$ on (Z,z). Finally, the formula (1) is obviously also true for a finite morphism $\varphi : (Z,z) \to (X,x)$ between normal surface singularities.

1.3 Let $\pi : (X,E) \to (Z,z)$ be a good resolution of (Z,z), i.e. a resolution of the singularity (Z,z) such that the exceptional divisor $E = \bigcup_{\sigma\in\Gamma} E_\sigma$ is a union of smooth projective curves with normal crossings. For $\beta \in \Gamma$ and for each holomorphic function $h : (Z,z) \to (\mathbb{C},0)$ let $\nu_\beta(h)$ denote the vanishing order of $\overline{h} = h \circ \pi : X \to \mathbb{C}$ along the irreducible exceptional curve E_β (ν_β is just the divisorial valuation defined by E_β). If $\eta = \{h=0\}$ is the curve defined by h we use also the notation $\nu(\eta)$ instead of $\nu(h)$. In particular, this notation could be used for any curve germ if $(Z,z) = (\mathbb{C}^2,0)$ and π is a modification of it.

The lifting $\overline{\psi} = \psi \circ \pi$ of the meromorphic function $\psi = g/f$ is a meromorphic function defined in a suitable neighbourhood of the exceptional divisor E but in a finite number of points. The irreducible component E_β of E is called a *dicritical* component of the pencil $\Phi = \langle f,g\rangle$ (or of the meromorphic function ψ) if the restriction of $\overline{\psi} : E_\beta \to \mathbb{CP}^1$ is defined everywhere and not constant. If $P \in E_\beta$ (E_β a dicritical component) is a smooth point of E we say that P is a critical point of Φ if it is a critical point of $\overline{\psi} : E_\beta \to \mathbb{CP}^1$ (this does not depend on the pair of functions of Φ used to define the meromorphic function ψ).

A good resolution of the pencil Φ is a good resolution $\pi : (X,E) \to (Z,z)$ of (Z,z) such that $\overline{\psi}$ is everywhere defined. The zero locus $\{h=0\}$ of the elements $h \in \Phi$ are equisingular and called the *generic* fibres of Φ, excepted a finite number of them called the *special* fibres. Moreover from [5], we know that the strict transforms of the generic fibres by π intersect the exceptional divisor smoothly and transversally at smooth points of E, lying on the dicritical components. In fact, a good resolution of (Z,z) is a good resolution of Φ if and only if it is a resolution of all the fibres of Φ, excepted possibly some of the special fibres, and such that the strict transforms of any pair of generic fibres do not intersect each others.

Lemma 1.4 *With the above notations. Let $h : (Z, z) \to (\mathbb{C}, 0)$ be an analytic function, $\eta = \{h = 0\}$ and let $(\gamma, z) \subset (Z, z)$ be a branch such that its strict transform $\widetilde{\gamma}$ by π, intersects $E_\alpha \subset E$ at a smooth point P of E. Then if we denote by $\tilde{\eta}$ the strict transform of η by π, we have :*

$$I_z(h, \gamma) = [\widetilde{\eta}, \widetilde{\gamma}]_P + [E_\alpha, \widetilde{\gamma}]_P \; \nu_\alpha(h) \; .$$

In particular, if $\tilde{\eta} \cap \tilde{\gamma} = \emptyset$, then $I_z(h, \gamma) = [E_\alpha, \tilde{\gamma}]_P \; \nu_\alpha(h)$.

Proof The divisor $(\overline{h})$ defined by $\overline{h} = h \circ \pi$ on X may be written as

$$(\overline{h}) = (\widetilde{h}) + \sum_{\beta \in \Gamma} \nu_\beta(h) E_\beta \; ,$$

where the local part $(\widetilde{h}) = \widetilde{\eta}$ is the strict transform of the germ $\{h = 0\}$ by π. For each $\sigma \in \Gamma$ one has the well-known Mumford formula (see [11]):

$$(\overline{h}) \cdot E_\sigma = (\widetilde{h}) \cdot E_\sigma + \sum_{\beta} \nu_\beta(h)(E_\beta \cdot E_\sigma) = 0. \tag{2}$$

(Here "$\cdot$" stands for the intersection form on the smooth surface X).

Applying the Eq. (2) to the total transforms of γ and η we have:

$$I_z(h, \gamma) = (\overline{\eta}) \cdot (\overline{\gamma}) = (\widetilde{h}) \cdot \widetilde{\gamma} + \sum_{\beta \in \Gamma} \nu_\beta(h)(E_\beta \cdot \widetilde{\gamma}) = [\widetilde{\eta}, \widetilde{\gamma}]_P + \nu_\alpha(h)[E_\alpha, \widetilde{\gamma}]_P \; .$$

□

The above result will also be used in the paper in the case of $(Z, z) = (\mathbb{C}^2, 0)$. Note that, in such a case, every germ of curve is defined by a function (i.e., is a principal divisor).

1.5 (**Iterated pencils**) We denote $\Phi = \langle f, g \rangle = \{ag - bf \, : \, (a : b) \in \mathbb{CP}^1\}$ the pencil defined by f and g. Let us assume that $I_z(g, \gamma) \geq I_z(f, \gamma)$. If $I_z(g, \gamma) = \infty$ then $\varphi(\gamma)$ is the x axis $\{y = 0\}$ and the procedure stops here. Otherwise, let $q/p = I_z(g, \gamma)/I_z(f, \gamma) \geq 1$ with $\gcd(p, q) = 1$. Let Φ_1 be the pencil generated by g^p and f^q. By Proposition 1 of [5] the intersection multiplicity $I_z(ag^p - bf^q, \gamma)$ is constant for all $w = (a : b) \in \mathbb{CP}^1$, except for one value $w_0 = (a_0 : b_0)$ in $\mathbb{CP}^1$. With the hypothesis that $q/p \geq 1$, we have $I_z(g^p, \gamma) = I_z(f^q, \gamma)$ which shows that $a_0 \neq 0 \neq b_0$. Therefore we can write $w_0 = (1 : a_1)$ where $a_1 \in \mathbb{C}^*$. We denote

$$g_1 = g^p - a_1 f^q \; .$$

Let $f_1 := f^q$, and $\varphi_1 = (f_1, g_1) : (Z, z) \longrightarrow (\mathbb{C}^2, 0)$. If $I_z(g_1, \gamma) = \infty$ then γ is a branch of $\{g_1 = 0\}$ and in this case one has $\varphi_1(\gamma) = L$, $L = \{y_1 = 0\}$ for $y_1 = y^p - a_1 x^q$, so $\varphi(\gamma) = \{y^p - a_1 x^q = 0\}$ and we stop here.

Else, we consider $\Phi_1 = \langle g_1, f_1 \rangle$ and we can define a new rational number $q_1/p_1 = I_z(g_1, \gamma)/I_z(f_1, \gamma) > 1$. In this case proceed to construct $\Phi_2 = \langle g_2, f_2 \rangle$, $g_2 \in \Phi_2$, and q_2/p_2 as we did for $\Phi_1, g_1, q_1/p_1$.

Recursively, we define a sequence $\mathcal{P}(f, g, \gamma) = \{(\Phi_i, g_i, q_i/p_i) \, : \, i \geq 0\}$ where, $(\Phi_0, g_0, q_0/p_0) = (\Phi, g, q/p)$ and, for $i \geq 1$, $\Phi_i = \langle g_i, f_i \rangle = \langle g_{i-1}^{p_{i-1}}, f_{i-1}^{q_{i-1}} \rangle$ is a pencil of functions on (Z, z), $f_i = f_{i-1}^{q_{i-1}}$, $g_i = g_{i-1}^{p_{i-1}} - a_i f_{i-1}^{q_{i-1}}$ is the unique function of Φ_i (except for product with non-zero constants) such that $I_z(g_i, \gamma) > I_z(h, \gamma)$ ($h \in \Phi_i$ a generic function) and $q_i/p_i = I_z(g_i, \gamma)/I_z(f_i, \gamma) > 1$, with $\gcd(q_i, p_i) = 1$. We also denote by ξ_i the curve on (Z, z) defined by $g_i = 0$ and for the sake of completeness we take $g_{-1} = f$. For $i \geq 1$, note that, $f_i = f^{q_0 \cdots q_{i-1}}$ and also that the sequence is infinite unless there exists $l \in \mathbb{N}$ such that $q_l/p_l = \infty$, in which case the sequence is finite and ends at l.

Definition 1.6 The *iterated sequence of pencils* of the branch γ (with respect to $\varphi = (f, g)$) is the sequence $\mathcal{P}(f, g, \gamma)$ defined above.

Remark 1.7 Along the paper we use frequently the case $(Z, z) = (\mathbb{C}^2, 0)$ and $\varphi = (x, y)$, the identity map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$. In this case we denote by $\{(\Lambda_i, y_i, q_i/p_i) : i \geq 0\}$ the iterated pencil of a branch $\delta \subset (\mathbb{C}^2, 0)$. We will also denote λ_i the curve defined by $y_i = 0$.

Theorem 1.8 *Let k be the degree of the restriction map $\varphi_{|\gamma} : \gamma \longrightarrow \delta$ and let $\varphi_* \gamma = k \cdot \delta$ be the direct image of γ. The sequence $\{I_z(g_i, \gamma) : i = -1, 0, \ldots\}$ and the integer k determine the topological type of δ.*

An easy consequence is:

Corollary 1.9 *The sequence of rational numbers $\{q_i/p_i, i \geq 0\}$ determines the topological type of δ.*

Proof The proofs of Theorem 1.8 and Corollary 1.9 repeat the scheme used in the proof of Theorem 1 (see also Corollary 1) of [4]. The first step consists in the proof that the topological type (say e.g. the semigroup of values) of δ can be determined with the described procedure in the particular case in which we take the identity map $(x, y) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$. For the sake of completeness we will describe briefly the procedure, the complete proof could be read in [4].

Assume that δ is not a coordinate axis and $I_0(x, \delta) \leq I_0(y, \delta)$. Let us consider the iterated sequence of pencils $\mathcal{P}(x, y, \delta) = \{(\Lambda_i, y_i, q_i/p_i) : i \geq 0\}$ of δ with respect to the identity.

Let $m_{-1}, m_0, \ldots$ be the sequence of positive integers defined recursively as follows: $m_{-1} = I_0(x, \delta)$, $m_0 = I_0(y, \delta)$ and, if we assume that $m_1, \ldots, m_i$ have been defined, then

$$m_{i+1} := d_i m_i + I_0(y_{i+1}, \delta) - p_i I_0(y_i, \delta) \ , \tag{3}$$

where $d_i = \gcd(m_{-1}, \ldots, m_{i-1}) / \gcd(m_{-1}, \ldots, m_i)$.

Theorem 2 in [4] says that the sequence $m_{-1}, m_0, \ldots$ generates the semigroup of values of δ (see also the Remark below).

Let $\varphi^* : \mathcal{O}_{\mathbb{C}^2,0} \to \mathcal{O}_{Z,z}$ be the ring homomorphism associated to φ. Obviously $\varphi^* \Lambda = \Phi$ and, by the projection formula 1.1, $p/q = I_z(g, \gamma)/I_z(f, \gamma)$; so $\Phi_1 = \varphi^* \Lambda_1$. Again, the projection formula implies that

$$I_z(\varphi^* y_1, \gamma) = k I_0(y_1, \delta) > k I_0(h, \delta) = I_z(\varphi^* h, \gamma)$$

for any $h \in \Lambda_1$, $h \neq y_1$. Now, the unicity of $g_1 \in \Phi_1$ implies that $\varphi^* y_1 = g_1$. Recursively, for $i \geq 1$ one has $\varphi^*(y_i) = g_i$ and, on the other hand, the definition of $\Lambda_i = \{a y_i - b x_i : (a : b) \in \mathbb{CP}^1\}$ implies that

$$\varphi^* \Lambda_i = \{\varphi^*(a y_i - b x_i) : (a : b) \in \mathbb{CP}^1\} = \{a g_i - b f_i : (a : b) \in \mathbb{CP}^1\} = \Phi_i \ .$$

Thus, one can recover by pullback the iterated sequence of pencils $\mathcal{P}(f, g, \gamma)$ on (Z, z): that means $\varphi^* \Lambda_i = \Phi_i$ and $\varphi^* y_i = g_i$ for $i \geq 0$.

The projection formula 1.1 applied to the Eq. (3) gives:

$$m_{-1} = I_0(x, \delta) = I_z(f, \gamma)/k, \quad m_0 = I_0(y, \delta) = I_z(g, \gamma)/k$$

and for $i \geq 1$:

$$m_{i+1} = d_i m_i + I_0(y_{i+1}, \delta) - p_i I_0(y_i, \delta) = d_i m_i + \frac{I_z(g_{i+1}, \gamma)}{k} - p_i \frac{I_z(g_i, \gamma)}{k} \ .$$

So, we recover the semigroup of values of the branch δ from the sequence $\{m_i : i \geq -1\}$.

The proof of the Corollary is the same as the one of Corollary 1 in [4], however, it is also included in the proof of Theorem 3.5. □

Remark 1.10 In the proof of Theorem 1.8 we recover the semigroup of values, in the same way one can recover the characteristic exponents of δ. Let $\mu_i = I_z(g_i, \gamma)$ for $i \geq -1$. Let $m_{-1} = \widetilde{m}_{-1} = I_z(f, \gamma)/k$, $m_0 = \widetilde{m}_0 = I_z(g_0, \gamma)/k$ and for $i \geq 1$,

$$m_{i+1} = d_i m_i + \frac{\mu_{i+1}}{k} - p_i \frac{\mu_i}{k} = d_i m_i + \frac{\mu_{i+1}}{k} - q_i \frac{I_z(f_i, \gamma)}{k}$$
$$\widetilde{m}_{i+1} = \widetilde{m}_i + \frac{\mu_{i+1}}{k} - p_i \frac{\mu_i}{k} = \widetilde{m}_i + \frac{\mu_{i+1}}{k} - q_i \frac{I_z(f_i, \gamma)}{k}$$

where, $d_i = \gcd(m_{-1}, \ldots, m_{i-1})/\gcd(m_{-1}, \ldots, m_i)$. Note that

$$e_i := \gcd(m_{-1}, \ldots, m_i) = \gcd(\widetilde{m}_{-1}, \ldots, \widetilde{m}_i) = \gcd(\mu_{-1}, \ldots, \mu_i)/k\ .$$

As a consequence, denoting $\epsilon_i := \gcd(\mu_{-1}, \ldots, \mu_i)$, one has also $d_i = \epsilon_{i-1}/\epsilon_i$.

From [4] one knows that the minimal set of generators $\{\overline{\beta}_0, \ldots, \overline{\beta}_g\}$ of the semigroup of the plane germ δ is a subset of $\{m_i \mid i \geq -1\}$. More specifically $\{\overline{\beta}_0, \ldots, \overline{\beta}_g\} = \{m_{-1}\} \cup \{m_i : d_i > 1\}$. In the same way, the set of characteristic exponents $\{\beta_0, \ldots, \beta_g\}$ of δ is a subset of $\{\widetilde{m}_i \mid i \geq -1\}$, namely the set $\{\beta_0, \ldots, \beta_g\} = \{\widetilde{m}_{-1}\} \cup \{\widetilde{m}_i : d_i > 1\}$.

In particular, the computation of the sequences stops in a finite number of steps, just when $e_i = 1$ or equivalently $\gcd(\mu_{-1}, \ldots, \mu_i) = k$. Notice that $k = \gcd\{\mu_i : i \geq -1\}$, i.e. the set $\mu = \{\mu_i : i \geq -1\}$ generates a finitely generated subsemigroup of $k\mathbb{Z}$. So theoretically one can recover the integer k and so the topological type of δ from μ, however, if k is unknown, we do not have a criterion to decide when to stop the computation of the sequence $\{\mu_i : i \geq -1\}$.

2 Case of several branches

In this section, we treat the case in which $\gamma \subset (Z, z)$ is a reduced but not irreducible curve in (Z, z). We denote $\gamma = \bigcup_{i=1}^r \gamma_i$, with γ_i a branch in (Z, z), and its image by φ is $\delta = \bigcup_{i=1}^r \delta_i$ in such a way that $\varphi(\gamma_i) = \delta_i$, $1 \leq i \leq r$. Pay attention that we could have $\delta_i = \delta_j$ for some $i \neq j$.

To describe the topological type of δ, it suffices to describe the topological type of each branch δ_i, $1 \leq i \leq r$, and give the intersection multiplicities $[\delta_i, \delta_j]$ between pairs of different branches δ_i, δ_j. Thus, it is enough to resolve the case of two branches.

Along this section $\gamma, \gamma' \subset (Z, z)$ are two irreducible curves in (Z, z) and we assume that $\varphi(\gamma) = \delta$ is not equal to $\varphi(\gamma') = \delta'$.

2.1 Case of $(\mathbb{C}^2, 0)$

We start, as in the irreducible case, describing the intersection multiplicity $[\delta, \delta']$ by the use of the iterated pencils, starting with the pencil $\Lambda_0 = \langle x, y \rangle$ on $\mathbb{C}^2$ (which corresponds to the identity map from $(\mathbb{C}^2, 0)$ to $(\mathbb{C}^2, 0)$). So, we proceed by means of an iterative process.

2.1.1 First step

Let us assume that $I_0(\delta, y) \geq I_0(\delta, x)$, which means that $\{x = 0\}$ is transversal to δ. Let $I_0(\delta, y)/I_0(\delta, x) = q/p \geq 1$, with $\gcd(p, q) = 1$. We have to consider several situations according to the value of $I_0(\delta', y)/I_0(\delta', x)$:

(a) If $I_0(\delta', y) < I_0(\delta', x)$ then $\{x = 0\}$ is tangent to δ'. As it is transversal to δ, δ and δ' are transversal and we have $[\delta, \delta'] = I_0(\delta, x)I_0(\delta', y)$.

Otherwise we have $I_0(\delta', y)/I_0(\delta', x) = q'/p' \geq 1$, with $\gcd(p', q') = 1$.

(b) Let us suppose that $q'/p' \neq q/p$. Assume that $q'/p' > q/p \geq 1$.
Note that if $q/p = 1$, then δ and δ' are transversal, therefore

$$[\delta, \delta'] = I_0(\delta, x)I_0(\delta', x) = I_0(\delta, y)I_0(\delta', x) < I_0(\delta', y)I_0(\delta, x)\ .$$

Else we have $q'/p' > q/p > 1$ and in this case $\{y = 0\}$ is tangent to δ and δ'. Then, an easy computation (see also the proof of Proposition 2.3) allows to show that

$$[\delta, \delta'] = \min\{I_0(\delta, x)I_0(\delta', y), I_0(\delta, y)I_0(\delta', x)\} = I_0(\delta, y)I_0(\delta', x)\ .$$

(c) Finally, let us assume that $q/p = q'/p' \geq 1$. Let $\Lambda_1 = \langle y^p, x^q\rangle$ be the first iterated pencil. Notice that $\Lambda_1 = \Lambda_1'$. Taking into account the construction of (1.5), let $y_1 = y^p - ax^q$ with $a \neq 0$ defining the only fibre of Λ_1 such that $I_0(\delta, y_1) > I_0(\delta, y^p)(= I_0(\delta, x^q))$ (resp. $y_1' = y^p - a'x^q \in \Lambda_1$, $a' \neq 0$, the corresponding one for δ'). We have to distinguish two cases according to $a \neq a'$ or $a = a'$.

(c-1) *Case $a \neq a'$.*

Lemma 2.1 *If $a \neq a'$ then* $[\delta, \delta'] = I_0(\delta, x)I_0(\delta', y) = I_0(\delta, y)I_0(\delta', x)$.

Proof Let $\sigma : (X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of Λ_1. Notice that the fibres y_1, y_1' are generic for $\Lambda_1 = \langle y^p, x^q\rangle$, in particular they are equisingular. The fact that $I_0(\delta, y_1) > I_0(\delta, y^p)$ implies that the strict transform of δ and y_1 by σ intersect in a point $P \in E_\alpha \subset E$. As y_1 is generic for Λ_1 it implies that E_α is a dicritical divisor for Λ_1 (it is the unique one in this particular case, see [2]) and P is a smooth point of E_α which is not a critical point of Λ_1. (In this case, none smooth point of E_α is critical). In the same way one has that the strict transform of δ' and y_1' by σ intersect in a point $P' \in E_\alpha \subset E$. As $a \neq a'$ we have $P \neq P'$ and it is known that $[\delta, \delta'] = I_0(\delta, x)I_0(\delta', y) = I_0(\delta, y)I_0(\delta', x)$. □

(c-2) *Case $a = a'$.*

Following the notations of the previous Lemma, as $\gcd(p, q) = 1$, the fibres $\{y_1 = 0\}$ and $\{y_1' = 0\}$ are irreducible, so the case $a = a'$ is equivalent to the two following conditions $P = P'$ and $[\delta, \delta'] > I_0(\delta, x)I_0(\delta', y)$. In this case one has to iterate the process.

Remark 2.2 In all the described cases but in case c-2) we have proved that

$$[\delta, \delta'] = \min\{I_0(\delta, x)I_0(\delta', y), I_0(\delta, y)I_0(\delta', x)\}\ .$$

On the other hand, in the case c-2) for the first iterated pencils $(\Lambda_1, y_1, q_1/p_1)$ and $(\Lambda_1', y_1', q_1'/p_1')$ of, respectively, δ and δ', we have $\Lambda_1 = \Lambda_1'$, $y_1 = y_1'$ and $P = P'$. Moreover, for the same reasons as the ones in **c-1**, the divisor E_α is a dicritical divisor of Λ_1 and $P \in E_\alpha$ is not a critical point of Λ_1.

2.1.2 General recursive step

For an index $i \geq 1$, let $(\Lambda_j, y_j, q_j/p_j)$ and $(\Lambda'_j, y'_j, q'_j/p'_j)$, $j \leq i$, be the sequence of iterated pencils for δ and δ'. Let us assume that, for every $j \leq i$ one has $\Lambda_j = \Lambda'_j$ and $y_j = y'_j$ and moreover $q_j/p_j = q'_j/p'_j$ for $j < i$.

Let $\sigma : (X, E) \to (\mathbb{C}^2, 0)$ be the minimal resolution of the pencil $\Lambda_i = \langle x_i, y_i \rangle$ (and so σ is a resolution of Λ_j for all $j < i$). Let $\widetilde{\delta}$ (resp. $\widetilde{\delta}'$) be the strict transform of δ (resp. δ') by π. We suppose that $\widetilde{\delta} \cap E = \widetilde{\delta}' \cap E$ and we denote this point Q.

Following the idea of the proof of Theorem 2 in [4], we also assume that the following facts are true:

1. $Q = \widetilde{\delta} \cap E = \widetilde{\delta}' \cap E$ is a smooth point of the exceptional divisor E. So, there exists a (unique) irreducible component $E_\alpha \subset E$ such that $Q = \widetilde{\delta} \cap E_\alpha = \widetilde{\delta}' \cap E_\alpha$.
2. The irreducible component E_α is dicritical for the pencil Λ_i.
3. There exists a unique branch ζ_i of $\lambda_i = \{y_i = 0\}$ which is a curvette at the point Q (i.e., its strict transform by π is smooth and transversal to E_α at the point Q).

Let $\sigma' : (X', E') \to (\mathbb{C}^2, 0)$ be the composition of σ with the minimal modification of (X, Q) until the strict transforms of δ and δ' by σ' meet the exceptional locus E' at smooth points, P, P', and also such that the strict transforms of ζ_i by σ' do not contain P nor P'.

Proposition 2.3 *Following the above notations we obtain :*

1. *If $P \neq P'$ one has :*
$$[\delta, \delta'] = \min\{[E_\alpha, \widetilde{\delta}]_Q\ [\delta', \zeta_i], [E_\alpha, \widetilde{\delta}']_Q\ [\delta, \zeta_i]\}\ .$$
2. *If $P = P'$, then $q_i/p_i = q'_i/p'_i$, $\Lambda_{i+1} = \Lambda'_{i+1}$, $y_{i+1} = y'_{i+1}$ and we proceed with a new iteration. Moreover, one has $P = P'$ if and only if $[\delta, \delta'] > [E_\alpha, \widetilde{\delta}]_Q\ [\delta', \zeta_i] = [E_\alpha, \widetilde{\delta}']_Q\ [\delta, \zeta_i]$.*

Remark 2.4 In [4] the next property is also included in the above list, as fourth property, for the recursive step:

"Let $m_i = [\delta, \zeta_i]$, the set $\{m_{-1}, \ldots, m_{i-1}\}$ contains all the maximal contact values of δ smaller than m_i."

In our case one can add the corresponding one for δ', $m'_i = [\delta', \zeta_i]$. (See also the proof of Theorem 1.8 for the definition of the integers m_i). Here we do not need it, because the semigroup of each branch is already computed in Theorem 1.8. However it is interesting to notice that (see [4]) :
$[E_\alpha, \widetilde{\delta}]_Q = \gcd(m_{-1}, m_0, \ldots, m_{i-1}) := e_{i-1}$ and so, for $P \neq P'$:
$$[\delta, \delta'] = \min\{e_{i-1}m'_i, e'_{i-1}m_i\}$$

The next lemma will be useful for the proof of Proposition 2.3.

Lemma 2.5 *With the above notations, we have the following equivalence:*
$$\frac{q'_i}{p'_i} > \frac{q_i}{p_i} \iff \frac{[\widetilde{\delta}', \widetilde{\zeta}_i]_Q}{[E_\alpha, \widetilde{\delta}']_Q} > \frac{[\widetilde{\delta}, \widetilde{\zeta}_i]_Q}{[E_\alpha, \widetilde{\delta}]_Q}.$$

Proof Hypothesis 3 above implies that the strict transform of an irreducible component ζ of λ_i, different from ζ_i, does not contain the point Q. In particular the strict transform of ζ does not intersect the strict transforms $\widetilde{\delta}$ and $\widetilde{\delta'}$ of δ and δ' and we can apply Lemma 1.4 to obtain $[\delta, \zeta] = [E_\alpha, \widetilde{\delta}]_Q \nu_\alpha(\zeta)$ and $[\delta', \zeta] = [E_\alpha, \widetilde{\delta'}]_Q \nu_\alpha(\zeta)$. So

$$[\delta, \zeta]/[E_\alpha, \widetilde{\delta}]_Q = [\delta', \zeta]/[E_\alpha, \widetilde{\delta'}]_Q.$$

In the same way we have $I_0(x_i, \delta) = [E_\alpha, \widetilde{\delta}]_Q \nu_\alpha(x_i)$ and $I_0(x_i, \delta') = [E_\alpha, \widetilde{\delta'}]_Q \nu_\alpha(x_i)$, which implies:

$$\frac{q_i}{p_i} = \frac{[\delta, \lambda_i]}{I_0(\delta, x_i)} = \frac{[\delta, \zeta_i]}{I_0(\delta, x_i)} + \sum_{\zeta \in \lambda_i,\ \zeta \neq \zeta_i} \frac{[\delta, \zeta]}{[E_\alpha, \widetilde{\delta}]_Q\ \nu_\alpha(x_i)}$$

and

$$\frac{q'_i}{p'_i} = \frac{[\delta', \lambda_i]}{I_0(\delta', x_i)} = \frac{[\delta', \zeta_i]}{I_0(\delta', x_i)} + \sum_{\zeta \in \lambda_i,\ \zeta \neq \zeta_i} \frac{[\delta', \zeta]}{[E_\alpha, \widetilde{\delta'}]_Q\ \nu_\alpha(x_i)}\ .$$

Therefore

$$\begin{aligned}\frac{q'_i}{p'_i} > \frac{q_i}{p_i} &\iff \frac{[\delta', \zeta_i]}{I_0(\delta', x_i)} > \frac{[\delta, \zeta_i]}{I_0(\delta, x_i)} \iff \frac{[\delta', \zeta_i]}{q_0 \ldots q_{i-1} I_0(\delta', x)} > \frac{[\delta, \zeta_i]}{q_0 \ldots q_{i-1} I_0(\delta, x)} \\ &\iff \frac{[\widetilde{\delta'}, \widetilde{\zeta_i}]_Q + [E_\alpha, \widetilde{\delta'}]_Q\ \nu_\alpha(\zeta_i)}{[E_\alpha, \widetilde{\delta'}]_Q} > \frac{[\widetilde{\delta}, \widetilde{\zeta_i}]_Q + [E_\alpha, \widetilde{\delta}]_Q\ \nu_\alpha(\zeta_i)}{[E_\alpha, \widetilde{\delta}]_Q} \\ &\iff \frac{[\widetilde{\delta'}, \widetilde{\zeta_i}]_Q}{[E_\alpha, \widetilde{\delta'}]_Q} > \frac{[\widetilde{\delta}, \widetilde{\zeta_i}]_Q}{[E_\alpha, \widetilde{\delta}]_Q}\ .\end{aligned}$$

□

Proof of Proposition 2.3 **(1)** Let us assume $P \neq P'$. We start by computing $[\widetilde{\delta}, \widetilde{\delta'}]_Q$. Taking into account that $\widetilde{\zeta_i}$ and E_α are both smooth and intersect transversally at Q, we can choose a pair of local coordinates (u, v) at (X, Q) in such a way that $u = 0$ and $v = 0$ are the (local) equations of E_α and $\widetilde{\zeta_i}$ respectively. Thus one has locally, at $Q \in X$, exactly the same situation as in the First step 2.1.1 for the pencil $\langle u, v\rangle$ instead of $\Lambda = \langle x, y\rangle$. Thus, we have:

$$P \neq P' \iff [\widetilde{\delta}, \widetilde{\delta'}]_Q = \min\{[E_\alpha, \widetilde{\delta}]_Q[\widetilde{\delta'}, \widetilde{\zeta_i}]_Q, [E_\alpha, \widetilde{\delta'}]_Q[\widetilde{\delta}, \widetilde{\zeta_i}]_Q\}\,.$$

Now one has to compute $[\delta, \delta']$. By Lemma 1.4 we have :

$$[\delta, \delta'] = [\widetilde{\delta}, \widetilde{\delta'}]_Q + [E_\alpha, \widetilde{\delta'}]_Q\ \nu_\alpha(\delta) = [\widetilde{\delta}, \widetilde{\delta'}]_Q + [E_\alpha, \widetilde{\delta}]_Q\ \nu_\alpha(\delta').$$

Let us assume that $[E_\alpha, \widetilde{\delta'}]_Q[\widetilde{\delta}, \widetilde{\zeta_i}]_Q < [E_\alpha, \widetilde{\delta}]_Q[\widetilde{\delta'}, \widetilde{\zeta_i}]_Q$, (from Lemma 2.5 this is equivalent to $q'_i/p'_i > q_i/p_i$) then

$$\begin{aligned}[\delta, \delta'] &= [E_\alpha, \widetilde{\delta'}]_Q[\widetilde{\delta}, \widetilde{\zeta_i}]_Q + [E_\alpha, \widetilde{\delta'}]_Q \nu_\alpha(\delta) \\ &= [E_\alpha, \widetilde{\delta'}]_Q \left([\widetilde{\delta}, \widetilde{\zeta_i}]_Q + \nu_\alpha(\delta)\right) \\ &= [E_\alpha, \widetilde{\delta'}]_Q[\delta, \zeta_i] < [E_\alpha, \widetilde{\delta}]_Q[\delta', \zeta_i].\end{aligned}$$

Notice that if $q_i/p_i = q'_i/p'_i$ one has $[\delta, \delta'] = [E_\alpha, \widetilde{\delta'}]_Q[\delta, \zeta_i] = [E_\alpha, \widetilde{\delta}]_Q[\delta', \zeta_i]$.

(2) Let $P = P' \in E_\beta \subset E'$. The strict transform of ζ_i by σ' does not intersect E_β at P and obviously the same is true for all the branches of $\lambda_i = \{y_i = 0\}$. So by Lemma 1.4 we have $[\lambda_i, \delta] = [E_\beta, \widetilde{\delta}]_P\ \nu_\beta(y_i)$ and $[\lambda_i, \delta'] = [E_\beta, \widetilde{\delta'}]_P\ \nu_\beta(y_i)$. The same equalities are also true for x_i instead of y_i. Then

$$\frac{q_i}{p_i} = \frac{[\lambda_i, \delta]}{I_0(x_i, \delta)} = \frac{\nu_\beta(y_i)}{\nu_\beta(x_i)} = \frac{[\lambda_i, \delta']}{I_0(x_i, \delta')} = \frac{q'_i}{p'_i}$$

and as a consequence $\Lambda_{i+1} = \Lambda'_{i+1}$. As $\nu_\beta(x_i^{q_i}) = \nu_\beta(y_i^{p_i})$, the lifting of the meromorphic function $\psi = y_i^{p_i}/x_i^{q_i}$ is not equal to zero along E_β and has a zero at the point P, thus E_β is a dicritical divisor of Λ_{i+1}. As in the proof of Lemma 1 of [4], one can see that Q is not a critical point of Λ_{i+1} and so one has also that there exists an unique fibre $y_{i+1} = y_i^{p_i} - ax_i^{q_i} \in \Lambda_{i+1}$ ($a = \widetilde{\psi}(P) \in \mathbb{C}^*$) such that $I_0(y_{i+1}, \delta) > I_0(y_i^{p_i}, \delta)$ and $I_0(y_{i+1}, \delta') > I_0(y_i^{p_i}, \delta')$. The fact that E_β is dicritical and $P \in E_\beta$ is not a critical point of Λ_i implies that there exists an unique branch ζ_{i+1} of $\lambda_{i+1} = \{y_{i+1} = 0\}$ such that its strict transform is a curvette at the point P.

Notice that if σ' is not the minimal resolution of Λ_{i+1}, then the new blowing-ups needed to have a resolution of Λ_{i+1} do not affect the described situation on $P \in E_\beta$, because new blowing-ups at P are not needed.

Notice also that in this case we have $[\delta, \delta'] > [E_\alpha, \widetilde{\delta}']_Q[\delta, \zeta_i] = [E_\alpha, \widetilde{\delta}]_Q[\delta', \zeta_i]$. □

Remark 2.6 Let us summarize the case when $P \neq P'$ and $q_i/p_i = q'_i/p'_i$. Note that at this step, $q_i/p_i = q'_i/p'_i$ is equivalent to say $\Lambda_{i+1} = \Lambda'_{i+1}$. In this case P and P' belong to the same dicritical component $E_\beta \subset E'$ of the minimal resolution of Λ_{i+1}. Let $y_{i+1} = y_i^{p_i} - ax_i^{q_i}$, $\lambda_{i+1} = \{y_{i+1} = 0\}$ (resp. $y'_{i+1} = y_i^{p_i} - a'x_i^{q_i}$, $\lambda'_{i+1} = \{y'_{i+1} = 0\}$) be the unique fibre such that $[\delta, \lambda_{i+1}] > [\delta, \lambda]$ (resp. $[\delta', \lambda'_{i+1}] > [\delta', \lambda]$), for any other fibre λ of Λ_{i+1}. Thus we have two possibilities:

Either $a \neq a'$, and in this case $y_{i+1} \neq y'_{i+1}$ and $\Lambda_{i+2} \neq \Lambda'_{i+2}$.

Otherwise $a = a'$. In this case there exist ζ, ζ' irreducible components of the same fibre λ_{i+1} such that the strict transform $\widetilde{\zeta}$ of ζ (resp. $\widetilde{\zeta}'$ of ζ') meets E_β at P (resp. at P'). By using Lemma 1.4 it is easy to check that this is equivalent to

$$\frac{[\delta, \zeta]}{[E_\beta, \widetilde{\delta}]} > \frac{[\delta', \zeta]}{[E_\beta, \widetilde{\delta}']} \left(\text{resp. } \frac{[\delta', \zeta']}{[E_\beta, \widetilde{\delta}']} > \frac{[\delta, \zeta']}{[E_\beta, \widetilde{\delta}]} \right) . \qquad (*)$$

As a consequence, again using Lemma 1.4 we have $[\delta, \delta'] = \nu_\beta(\delta)[E_\beta, \tilde{\delta}']$. As $\tilde{\delta} \cap \tilde{\zeta}' = \emptyset$ and ζ' is transversal to E_β we have $\nu_\beta(\delta) = [\delta, \zeta']$ and we obtain

$$[\delta, \delta'] = [E_\beta, \widetilde{\delta}'][\delta, \zeta'] .$$

In the same way we have $[\delta, \delta'] = \nu_\beta(\delta')[E_\beta, \tilde{\delta}] = [E_\beta, \widetilde{\delta}][\delta', \zeta]$.

From Proposition 2.3 and its proof, it is obvious that the maximal integer such that the requirements of the General recursive step are satisfied can be determined in a more easy way: let $\mathcal{P}(x, y, \delta) = \{(\Lambda_i, y_i, q_i/p_i) : i \geq 0\}$ (resp. $\mathcal{P}(x, y, \delta') = \{(\Lambda'_i, y'_i, q'_i/p'_i) : i \geq 0\}$) be the sequence of iterated pencils for δ (resp. for δ'). Then one has:

Statement 2.7 *Let κ be the largest integer such that, if $\sigma : (X, E) \to (\mathbb{C}^2, 0)$ is the minimal resolution of the pencil Λ_κ, then the strict transforms of δ and δ' intersect E at the same point $Q \in E_\alpha \subset E$. Then E_α is a dicritical component for Λ_κ and Q is not a critical point of Λ_κ.*

Moreover, if ζ is the unique branch of $\lambda_\kappa = \{y_\kappa = 0\}$ such that its strict transform by π intersects E at the point Q one has:

$$[\delta, \delta'] = \min\{[E_\alpha, \widetilde{\delta}]_Q\ [\delta', \zeta], [E_\alpha, \widetilde{\delta}']_Q\ [\delta, \zeta]\} .$$

Proof We proceed by induction on κ, the case $\kappa = 0$ is just the first step 2.1.1 and the inductive step is the General recursive step 2.1.2. □

Although the above statement completely solves the problem of determining the intersection multiplicity of δ and δ', the computation of κ requires the resolution of the iterated pencils. Let us show that this computation can be set up in another way.

Let $\{m_i, i \geq -1\}$ and $\{m'_i, i \geq -1\}$ be the sequences defined in the proof of Theorem 1.8 and Remark 1.10 for δ and δ'. For $i \geq -1$, let $e_i = \gcd(m_{-1}, \ldots, m_i)$ and $e'_i = \gcd(m'_{-1}, \ldots, m'_i)$.

Theorem 2.8 *Let ℓ be the smallest integer such that there exists a branch ζ of the fibre $\lambda_\ell = \{y_\ell = 0\}$ of Λ_ℓ such that $[\delta, \zeta]/I_0(\delta, x) \neq [\delta', \zeta]/I_0(\delta', x)$. Then:*

$$[\delta, \delta'] = \min\{e_{\ell-1}[\delta', \zeta], e'_{\ell-1}[\delta, \zeta]\} .$$

Proof Let κ be the integer defined in Statement 2.7, then one has that $[\delta, \zeta]/e_{i-1} = [\delta', \zeta]/e'_{i-1}$ for all branches ζ of the fibres $\{y_i = 0\}$ for $i < \kappa$. Let ℓ be equal to $\kappa + 1$ if we are in the case $a = a'$ described in Remark 2.6 and let $\ell = \kappa$ otherwise. Proposition 2.3 and Remark 2.6 imply that there exists a branch ζ of $\lambda_\ell = \{y_\ell = 0\}$ such that $[\delta, \zeta]/e_{i-1} \neq [\delta', \zeta]/e'_{i-1}$. Then, the integer ℓ can be defined as the smallest positive integer such that $[\delta, \zeta]/e_{\ell-1} \neq [\delta', \zeta]/e'_{\ell-1}$ for some branch ζ of λ_ℓ. The same result implies the stated equality for $[\delta, \delta']$.

Now, it is well known that, in our conditions, $d_i = e_{i-1}/e_i = e'_{i-1}/e'_i = d'_i$ for $i \leq \ell - 1$ (see e.g. [3]). Moreover, $e_{-1} = m_{-1} = I_0(\delta, x)$ and $e'_{-1} = m'_{-1} = I_0(\delta', x)$. Then for the integer ℓ, defined just above, one has that also ℓ is the smallest integer such that $[\delta, \zeta]/I_0(\delta, x) \neq [\delta', \zeta]/I_0(\delta', x)$. □

Remark 2.9 Notice that the formula for $[\delta, \delta']$ in the above Theorem is not the same as in Remark 2.4. Let us assume that we are in the case $\ell = \kappa + 1$, then one has that $[\delta, \delta'] = e_{\kappa-1}m'_\kappa = e'_{\kappa-1}m_\kappa$ and also $[\delta, \delta'] = \min\{e_{\ell-1}[\delta', \zeta], e'_{\ell-1}[\delta, \zeta]\}$. However $m'_\ell \neq [\delta', \zeta]$ because ζ does not contain P'. In fact it could happen even that $m'_\ell/e'_{\ell-1} = m_\ell/e_{\ell-1}$ (see the Example below). Thus in order to detect the integer ℓ (and so κ) it does not suffice to know the sequences $\{m_i\}$ and $\{m'_i\}$.

Remark 2.10 If there exists an integer r such that $\Lambda_r \neq \Lambda'_r$ then there exist ζ and ζ' branches of $\lambda_r = \{y_r = 0\}$ and $\lambda'_r = \{y'_r = 0\}$ such that $\widetilde{\delta} \cap \widetilde{\zeta} \neq \emptyset$, and respectively, $\widetilde{\delta'} \cap \widetilde{\zeta'} \neq \emptyset$. Then, one has:

$$[\delta, \delta'] = e'_{r-1}[\delta, \zeta'] = e_{r-1}[\delta', \zeta] .$$

Note that the only condition is that $\Lambda_r \neq \Lambda'_r$, so in this case it is not necessary to know the factorization of the fibres of the pencils to compute the intersection multiplicity of δ and δ'. In particular the branches δ, δ' could be separated before the step r.

Example 2.11 The following example from [4] shows that the use we made of the branches of the fibres and not only of the fibres themselves is required. Let $\delta = \{4y^2 - 4x^3 - 4yx^3 + x^6 = 0\}$, $\delta' = \{4y^2 - 4x^3 + 4yx^3 + x^6 = 0\}$. Their Puiseux expansions are $(x = t^2, y = t^3 + (1/2)t^6)$ and $(x = t^2, y = t^3 - (1/2)t^6)$ respectively. Their intersection multiplicity $[\delta, \delta']$ is equal to 9. As one can prove in an easy way, the sequence of pencils and fibres for both are the same (as well as the sequences $\{m_i\}$ and $\{m'_i\}$). For the first steps one finds:

$$\begin{aligned}
y_1 &= y^2 - x^3 \\
y_2 &= (y^2 - x^3)^2 - x^9 \\
y_3 &= ((y^2 - x^3)^2 - x^9)^6 - (1/64)x^{63} \\
y_4 &= (((y^2 - x^3)^2 - x^9)^6 - (1/64)x^{63})^{42} - (3/256)^{42}(x^{63})^{43}.
\end{aligned}$$

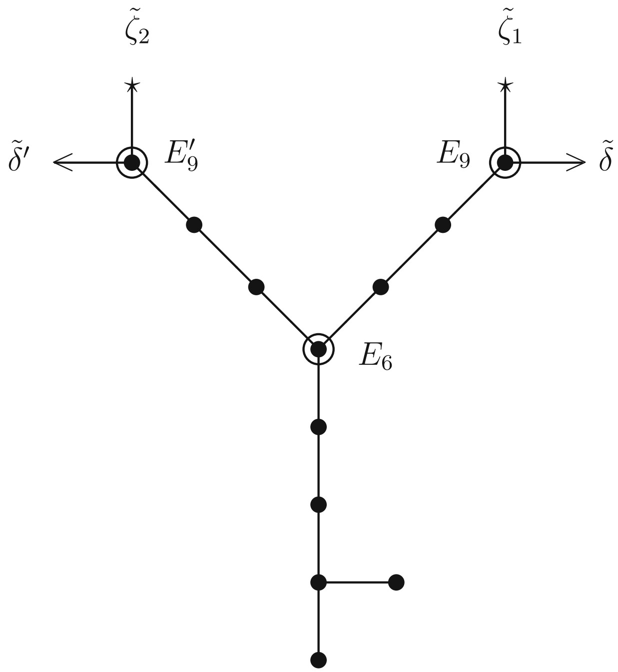

Fig. 1 Graph of the minimal resolution of $\delta \cup \delta' \cup \{y_2 = 0\}$

and the sequences of $m_i = m'_i$ is ($m_{-1} = 2, m_0 = 3, m_1 = 9, m_2 = 12, m_3 = 15$) and ($q_0/p_0 = q'_0/p'_0 = 3/2, q_1/p_1 = q'_1/p'_1 = 3/2, q_2/p_2 = q_2/p'_2 = 7/6, \ldots$). Note that $m_{-1} = 2$ and $m_0 = 3$ are enough to compute the semigroup of δ (see Corollary 1.9): the one generated by 2, 3. The same is true for δ'.

The fibre y_1 is irreducible. But at the second step, the fibre y_2 has two branches ζ_1, ζ_2 such that $[\delta, \zeta_1] = 12$ and $[\delta, \zeta_2] = 9$ and inversely $[\delta', \zeta_1] = 9$ and $[\delta', \zeta_2] = 12$. Thus following Theorem 2.8 we have that $\ell = 2$ and

$$[\delta, \delta'] = \min\{e_1[\delta', \zeta], e'_1[\delta, \zeta']\} = \min\{1 \cdot 9, 1 \cdot 9\} = 9\ .$$

As moreover $[\delta, \delta'] = 9$, the strict transforms of δ and δ' separate at the sixth blow-up (the corresponding divisor E_6 is the dicritical one of the pencil Λ_2, see Fig. 1), as do the ones of ζ_1 and ζ_2, while the strict transforms of δ and ζ_1, resp. of δ' and ζ_2, separate three blow-ups later (the corresponding divisors E_9 and E'_9 are dicritical for the pencil Λ_3).

With the above notations it means that we have $P \neq P'$ at E_6 although the pencils of δ and δ' are the same.

The resolution graph of the minimal resolution of $\delta \cup \delta' \cup \{y_2 = 0\}$ is depicted in Fig. 1.

2.2 Case of a normal singularity (Z, z)

Let γ, γ' be two branches in (Z, z) and $\varphi(\gamma) = \delta$ and $\varphi(\gamma') = \delta'$ their image in the plane $\mathbb{C}^2$. We assume that $\delta \neq \delta'$. Recall that $\varphi_* \gamma = k\delta$ and $\varphi_* \gamma' = k'\delta'$ for some positive integers k, k'.

Let $\mathcal{P}(f,g,\gamma)=\{(\Phi_i,g_i,q_i/p_i): i\geq 0\}$ and $\mathcal{P}(f,g,\gamma')=\{(\Phi'_i,g'_i,q'_i/p'_i): i\geq 0\}$ be the iterated sequences of pencils of the branches γ and γ' w.r.t. $\varphi=(f,g)$.

Theorem 2.12 *Let ℓ be the smallest integer such that there exists a branch ρ of the fibre $\xi_\ell=\{g_\ell=0\}$ of Φ_ℓ such that*

$$\frac{I_z(\gamma,\varphi^*(\varphi(\rho)))}{I_z(\gamma,f)}\neq\frac{I_z(\gamma',\varphi^*(\varphi(\rho)))}{I_z(\gamma',f)}\;.$$

Then:

$$[\delta,\delta']=min\left\{\frac{e'_{\ell-1}I_z(\gamma,\varphi^*(\varphi(\rho)))}{k},\frac{e_{\ell-1}I_z(\gamma',\varphi^*(\varphi(\rho)))}{k'}\right\}\;.$$

Proof Let $\mathcal{P}(x,y,\delta)=\{(\Lambda_i,y_i,q_i/p_i): i\geq 0\}$ (resp. $\mathcal{P}(x,y,\delta')=\{(\Lambda'_i,y'_i,q'_i/p'_i): i\geq 0\}$) be the sequence of iterated pencils for δ (resp. for δ'). As a consequence of the projection formula (Proposition 1.1), for $j\geq 0$, $(\varphi^*\Lambda_j,\varphi^*y_j,q_j/p_j)=(\Phi_j,g_j,q_j/p_j)$.

Let $\zeta\subset(\mathbb{C}^2,0)$ be a plane branch. By the projection formula 1.1 one has $[\delta,\zeta]=I_z(\gamma,\varphi^*\zeta)/k$ and so

$$\frac{[\delta,\zeta]}{I_0(\delta,x)}=\frac{I_z(\gamma,\varphi^*\zeta)}{I_z(\gamma,f)}\;.$$

These equalities applied to the branches ζ of $\lambda_i=\{y_i=0\}$ and both pairs (γ,δ), (γ',δ') imply that the integer ℓ defined in Theorem 2.8 is characterized as the one described in the statement.

Now, the equality for the intersection multiplicity of δ and δ' given in Theorem 2.8 is (by using the projection formula) the one described in the statement of the Theorem. □

Remark 2.13 Using Remark 1.10, the expression of the formula of Theorem 2.12 in terms of intersection multiplicities is the following one :

$$[\delta,\delta']=\frac{1}{kk'}\min\left\{\epsilon'_{\ell-1}I_z(\gamma,\varphi^*(\varphi(\rho))),\epsilon_{\ell-1}I_z(\gamma',\varphi^*(\varphi(\rho)))\right\}.$$

3 The discriminant case

The results proved in [5] about the behaviour of the critical locus of the map φ and its relation with the special fibres of the pencil $\Phi=\langle f,g\rangle$ (see also [2] for the plane case) allows to determine the topology of the irreducible components of the discriminant curve, $D(\varphi)$, of the morphism φ. We recall that the discriminant curve is the image by φ of the critical locus of the map φ, denoted $C(\varphi)$ (we consider only the one-dimensional components of the critical locus, so $C(\varphi)$ is the topological closure of the vanishing of the restriction of the jacobian determinant of (f,g) to $(Z,z)\setminus\{z\}$). Moreover, the sequence of rational numbers q_i/p_i, $i\geq 0$, for each branch of $C(\varphi)$ can be computed directly from the pencils without the knowledge of the concrete branches of $C(\varphi)$ (in [5] see part 2 of Theorem 3). We will show here how to obtain the whole topological type of the discriminant curve using only the construction of the pencils.

To make this paragraph self-contained, we recall here the construction of $\mathcal{S}$ (the set of sequences $(B_i)_{i\geq 0}$ associated to each branch of the discriminant curve) in a similar way to the one given in section 4 of [4].

3.1 Let $\pi:(X,E)\to(Z,z)$ be the minimal good resolution of (Z,z) which is also a resolution of the pencil $\langle g,f\rangle$ and of the curve $\{fg=0\}$ (i.e. the minimal good resolution of the singularity (Z, z) such that $(g/f) \circ \pi$ is a morphism and the strict transform of $\{fg = 0\}$ is smooth and transversal to the exceptional locus, see [5]).

An irreducible component E_α of the exceptional divisor E is called a *nodal* component (called rupture component in [4]) if either E_α is non-rational or the number of connected components of $\bar{C}_{red} \backslash E_\alpha$ is at least three, where $\bar{C}_{red}$ stands for the total transform of $C = \{fg = 0\}$ by π with reduced structure. A *nodal zone* $R \subset E$ is a maximal connected union of irreducible exceptional components containing at least one nodal component and such that the ratio $\frac{\nu_\alpha(g)}{\nu_\alpha(f)}$ is constant for $E_\alpha \subset R$. We denote this ratio by $Q(R)$. The term r-nodal zone is used to indicate that $Q(R) = r$.

Let $\mathcal{RZ}$ be the set of all nodal zones R such that $Q(R) \neq 1$. Let $E^{(1)} = \cup_\alpha E_\alpha \subset E$ be the union of the irreducible components E_α of E such that $\frac{\nu_\alpha(g)}{\nu_\alpha(f)} = 1$. Notice that the union $\mathcal{D}$ of dicritical components of E is included in $E^{(1)}$. Let $\mathcal{A}$ be the set of connected components of $\overline{E^{(1)} \backslash \mathcal{D}}$ which contain a nodal component. For $A \in \mathcal{A}$ we define $Q(A) := \nu_\alpha(g)/\nu_\alpha(f) = 1$, $E_\alpha \subset A$. Moreover let $\mathcal{P_C}$ be the set of points $P \in \mathcal{D}$ such that either P is a singular point of $\mathcal{D}$ or P is a smooth point of $\bar{C}_{red}$ which is a critical point of the map $(g/f) \circ \pi_{|\mathcal{D}} : \mathcal{D} \longrightarrow \mathbb{CP}^1$. For $P \in \mathcal{P_C}$ we put $Q(P) := 1$.

We denote $\mathcal{B}^0 = \mathcal{RZ} \cup \mathcal{A} \cup \mathcal{P_C}$ and fix $B \in \mathcal{B}^0$. We write $Q(B) = \frac{q}{p} \in \mathbb{Q}$ with $\gcd(p, q) = 1$ and consider the pencil $\Phi_B = \langle g^p, f^q \rangle$. Notice that π is also an embedded resolution of $\{f^q g^p = 0\}$. Thus by theorem 3 of [5] there exists a unique special fibre $\xi_B = \{g^p - af^q = 0\}$ of Φ_B whose strict transform $\tilde{\xi}_B$ by π intersects B.

Let us denote $\varphi_1 = (f_1, g_1) = (f^q, g^p - af^q)$ and let σ_B be the minimal sequence of blowing-ups of points such that $\pi_B := \sigma_B \circ \pi$ is an embedded resolution of $\{f_1 g_1 = 0\}$.

One can construct the set $\mathcal{B}^0(\varphi_1)$ for φ_1 in the same way we have defined $\mathcal{B}^0$ for (f, g) and for each $B^1 \in \mathcal{B}^0(\varphi_1)$ we define $Q_1(B_1)$ as $Q(B_1)$ with respect to g_1, f_1. Moreover we add the symbol B_∞ when there exists a branch ζ of $\{g_1 = 0\}$ such that $r\zeta$ is a component of $\{g_1 = 0\}$ for some $r > 1$ and whose strict transform $\widetilde{r\zeta}$ by π_B intersects $\sigma_B^{-1}(B)$. When B_∞ exists we put $Q_1(B_\infty) := \infty$.

Let us denote :

$$\mathcal{B}^1_B = \{B_1 \in \mathcal{B}^0(\varphi_1) \ / \ \sigma_B(B^1) \subset B\} \cup \{B_\infty\}$$

and we set $\mathcal{B}^1 = \cup_{B \in \mathcal{B}^0} \mathcal{B}^1_B$. Moreover one has the map $\psi_1 : \mathcal{B}^1 \longrightarrow \mathcal{B}^0$ defined by $\psi_1(B_1) = B$ if and only if $B_1 \in \mathcal{B}^1_B$. Thus we can follow the same process for each element $B_1 \in \mathcal{B}^1$ with $B_1 \neq B_\infty$ for any $B \in \mathcal{B}^0$ and so we inductively construct the collection of sets $\{\mathcal{B}^i\}_{i \geq 0}$ together with maps $\psi_i : \mathcal{B}^i \longrightarrow \mathcal{B}^{i-1}$ and $Q_i : \mathcal{B}^i \longrightarrow \mathbb{Q} \cup \{\infty\}$ in such a way that the definition of $\psi_i^{-1}(B)$ for $B \in \mathcal{B}^{i-1}$ (and the map $Q_{i}{}_{|\psi_i^{-1}(B)}$) follows the same process described above for $B \in \mathcal{B}^0$.

In a such a way we obtain

$$\mathcal{S} := \{(B_i)_{i \geq 0} : B_i \in \mathcal{B}^i \text{ and } \psi_i(B_i) = B_{i-1} \text{ for } i \geq 1\} .$$

We assume that if $B_i = (B_{i-1})_\infty$ then the sequence is finite and ends at B_i. For our purposes it is better to encode this information in the form of a weighted oriented graph in the following way:

Definition 3.2 The graph of φ is the oriented graph $\mathcal{T}$ whose vertices are the elements $B \in \mathcal{B}^i$, $(i \geq 0)$, with weight equal to $Q_i(B)$, and the set of oriented edges is the set of pairs $(B_{i-1}, B_i) \in \mathcal{B}^{i-1} \times \mathcal{B}^i$ such that $\psi_i(B_i) = B_{i-1}$.

Note that the graph $\mathcal{T}$ is the union of $\#\mathcal{B}^0$ trees, each one having as root the corresponding element (vertex) of $\mathcal{B}^0$. In this way, the set $\mathcal{S}$ corresponds to the maximal completely ordered (and weighted) subtrees of $\mathcal{T}$.

3.3 [**The branches of the discriminant**] Let $\mathcal{C}(\varphi)$ be the set of branches of $C(\varphi)$ which are not branches of $fg = 0$. Let us also denote by $\Delta(\varphi)$ the set of branches of the discriminant locus which are the image of a branch of $\mathcal{C}(\varphi)$, so the map φ gives a surjective map $\varphi : \mathcal{C}(\varphi) \to \Delta(\varphi)$.

As a consequence of the results of [5, 9, 10] one has that the set $\mathcal{C}(\varphi)$ can be decomposed in $\bigcup_{B\in\mathcal{B}^0} \mathcal{J}_B$, $\mathcal{J}_B \neq \emptyset$ for all $B \in \mathcal{B}^0$, and such that $\gamma \in \mathcal{J}_B$ if and only if its strict transform by π intersects B. Moreover, for $\gamma \in \mathcal{J}_B$, one has $I_z(g,\gamma)/I_z(f,\gamma) = Q(B)$. Note that $Q(B) = q/p$ (the rational number defined in 1.5).

Remark 3.4 The decomposition described in [5] relating the special values and the branches of the critical locus $\mathcal{C}(\varphi)$ is not the same as the one described here: in fact it could happen that a special component (i.e. a connected component of $E \setminus \mathcal{D}$) can be decomposed in several different r-nodal zones. On the other hand, the points of a divisorial component in $\mathcal{P}_{\mathcal{C}}$ offer a more precise decomposition than the one given by the corresponding 1-nodal zone described in [10]. In this sense, the decomposition from the elements of $\mathcal{B}$ is a mixture of both decompositions.

Now, by Theorem 4 and Corollary 3 in [5], the special fibre $\xi_B = \{g^p - af^q = 0\}$ of $\Phi = \langle g, f\rangle$ defined above coincides with the fibre g_1 defined in 1.5. Therefore, the pencil $\Phi_1 = \langle f_1, g_1\rangle$ is exactly the one corresponding to the map $\varphi_1 = (g_1, f_1)$. Moreover, the critical locus $\mathcal{C}(\varphi_1)$ is the same as $\mathcal{C}(\varphi)$, the one of the map φ (this is a straightforward computation, see also the proof of Theorem 4 in [4] or lemma 6.4 in [7]). That means that we can repeat the same process for the new map and so consider $B_1 \in \mathcal{B}^0(\varphi_1)$ such that the strict transform of γ by the corresponding resolution map intersects B_1.

The above construction makes possible to define the map $F : \mathcal{C}(\varphi) \to \mathcal{S}$ such that for $\gamma \in \mathcal{C}(\varphi)$, $F(\gamma) \in \mathcal{S}$ is the unique sequence $(B_i)_{i\geq 0}$ such that the corresponding strict transform of γ intersects B_i for each i. Note that the fact that $\mathcal{J}_{B_i} \neq \emptyset$ for all i implies that the map F is surjective.

Theorem 3.5 *Let $\gamma \in \mathcal{C}(\varphi)$, $\delta = \varphi(\gamma) \in \Delta(\varphi)$ and let $F(\gamma) = (B_i)_{i\geq 0}$. Then the sequence $(Q_i(B_i))_{i\geq 0}$ together with the integer k such that $\varphi_*\gamma = k\delta$ determines the semigroup (and therefore the topological type) of the branch δ.*

Proof One needs to compute the sequence $\{m_i : i \geq -1\}$ described in Theorem 1.8. For $i \geq 0$ let $r_i = q_i/p_i = Q_i(B_i)$, $\gcd(q_i, p_i) = 1$. As $r_i = q_i/p_i = I_z(g_i,\gamma)/I_z(f_i,\gamma)$ and $f_i = f_{i-1}^{q_{i-1}} = f^{q_0\cdots q_{i-1}}$ we deduce:

$$I_z(g_i,\gamma) = r_i I_z(f_i,\gamma) = r_i q_{i-1} I_z(f_{i-1},\gamma) = (\prod_{k=0}^{i-1} q_k) r_i I_z(f,\gamma)\ .$$

Moreover, using notations of Rmark 1.10, $\mu_{-1} = I_z(f,\gamma)$ and for $i \geq 0$:

$$\mu_i = I_z(g_i,\gamma) = I_z(\varphi^* y_i,\gamma) = I_0(y_i,\varphi_*\gamma) = kI_0(y_i,\delta)\ .$$

Thus

$$\mu_i/k = I_0(y_i,\delta) = I_z(g_i,\gamma)/k = \left(\prod_{k=0}^{i-1} q_k\right) r_i I_z(f,\gamma)/k = \left(\prod_{k=0}^{i-1} q_k\right) r_i \mu_{-1}/k\ .$$

So for the elements $m_i, i \geq -1$ of the semigroup of δ we obtain:

$$m_{i+1} = d_i m_i + \frac{\mu_{i+1}}{k} - p_i \frac{\mu_i}{k} = d_i m_i + \left(\prod_{k=0}^{i} q_k\right) r_{i+1} \frac{\mu_{-1}}{k} - p_i \left(\prod_{k=0}^{i-1} q_k\right) r_i \frac{\mu_{-1}}{k}$$
$$= d_i m_i + (r_{i+1} - 1) \left(\prod_{k=0}^{i} q_k\right) \frac{\mu_{-1}}{k} .$$

(As in Remark 1.10 $\epsilon_i = \gcd(\mu_{-1}, \ldots, \mu_i)$ and $d_i = \epsilon_{i-1}/\epsilon_i$.) □

Remark 3.6 Let $\mathcal{S}$ be the set of maximal completely ordered subtrees of $\mathcal{T}$ (see Definition 3.2). Then $\mathcal{C}(\varphi) = \bigcup_{S \in \mathcal{S}} F^{-1}(S)$ is a partition of the set of branches $\mathcal{C}(\varphi)$. It is clear that, if $\gamma, \gamma' \in F^{-1}(S)$ for some $S \in \mathcal{S}$, then $\varphi(\gamma)$ and $\varphi(\gamma')$ are equisingular: both have the same semigroup. Notice that $F^{-1}(S) \neq \emptyset$ for any $S \in \mathcal{S}$, however it could happen that $\# F^{-1}(S) > 1$ for some S.

Moreover, if $F(\gamma) = F(\gamma')$, then the iterated sequences of pencils $\mathcal{P}(f, g, \gamma) = \{(\Phi_i, g_i, q_i/p_i) : i \geq 0\}$ and $\mathcal{P}(f, g, \gamma') = \{(\Phi'_i, g'_i, q'_i/p'_i) : i \geq 0\}$ are the same but the converse is false, i.e. it could happen that $\mathcal{P}(f, g, \gamma) = \mathcal{P}(f, g, \gamma')$ but $F(\gamma) \neq F(\gamma')$. When $\gamma, \gamma' \in \mathcal{C}(\varphi)$, a key point is that the iterated sequences of pencils can be determined without the previous knowledge of the branches γ and γ'.

For each $S = (B_i)_{i \geq 0} \in \mathcal{S}$, let consider $Q(S) = (Q_i(B_i))_{i \geq 0}$. Theorem 3.5 above implies that, theoretically, the set of sequences $\{Q(S) : S \in \mathcal{S}\}$ permits to recover the set of topological types (the semigroups) of the branches of $\Delta(\varphi)$ (see the end of Remark 1.10). So in this sense, both sets are equivalent.

3.7 The intersection multiplicity

Let $\gamma, \gamma' \in \mathcal{C}(\varphi)$ and let $\varphi_*(\gamma) = k\delta$, $\varphi_*(\gamma') = k'\delta'$. Assume that $\delta \neq \delta'$. The computation of the intersection multiplicity $[\delta, \delta']$ can be made as in Theorem 2.12. So it depends on a more detailed knowledge of the branches of the fibres of the iterated pencils of γ and γ'.

Theorem 3.8 *Let ℓ be the smallest integer such that there exists a branch ρ of the fibre $\xi_\ell = \{g_\ell = 0\}$ of Φ_ℓ such that*

$$\frac{I_z(\gamma, \varphi^*(\varphi(\rho)))}{I_z(\gamma, f)} \neq \frac{I_z(\gamma', \varphi^*(\varphi(\rho)))}{I_z(\gamma', f)} .$$

Then:

$$[\delta, \delta'] = min \left\{ \frac{e'_{\ell-1} I_z(\gamma, \varphi^*(\varphi(\rho)))}{k}, \frac{e_{\ell-1} I_z(\gamma', \varphi^*(\varphi(\rho)))}{k'} \right\} .$$

Remark 3.9 Let $\gamma, \gamma' \in \mathcal{C}(\varphi)$ be such that $F(\gamma) = (B_i)_{i \geq 0} \neq F(\gamma') = (B'_i)_{i \geq 0}$ and let ℓ be such that $B_i = B'_i$ for $i < \ell$ and $B_\ell \neq B'_\ell$. In this case there exists a branch ρ of $\xi_\ell = \{g_\ell = 0\}$ such that its strict transform intersects B_ℓ and not B'_ℓ. For such a branch one has that

$$\frac{I_z(\gamma, \varphi^*(\varphi(\rho)))}{I_z(\gamma, f)} > \frac{I_z(\gamma', \varphi^*(\varphi(\rho)))}{I_z(\gamma', f)} .$$

So, in this case one has that

$$[\delta, \delta'] = \frac{e_{\ell-1} I_z(\gamma', \varphi^*(\varphi(\rho)))}{k'} .$$

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