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The Tjurina number of a plane curve with two branches and high intersection multiplicity

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Abstract

In this paper we provide a family of reduced plane curves with two branches that have a constant Tjurina number in their equisingularity class, along with a closed formula for it in terms of topological data.

Keywords Tjurina number · Plane curve singularities

Mathematics Subject Classification 14H20 · 14H50 · 32S05

1 Introduction

Let $(C_f, \mathbf{0})$ be a germ of a reduced plane curve singularity with $r \geq 1$ branches. One of its main analytic invariants is the Tjurina number, which can be computed as

$$\tau(C_f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x, y\}}{(f, f_x, f_y)},$$

where $f = 0$ is a defining equation for $(C_f, \mathbf{0})$. Despite being a well-studied invariant, several important problems remain open. Among these, determining a closed formula for the minimal Tjurina number in a fixed topological class in terms of topological invariants stands

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out as one of the most challenging. A closely related and even more difficult problem is the characterization of plane curves with a constant Tjurina number within an equisingularity class.

While determining a closed formula for the minimal Tjurina number in a fixed equisingularity class is completely solved for $r = 1$ (see [2, 23]), it remains open for $r > 1$. In fact, for $r > 1$ a recursive formula was proven by Briançon et al. [8] for positive weight deformations of a semi-quasihomogeneous curve $x^a + y^b$, i.e. plane curves with $r = \gcd(a, b)$ branches that are all topologically equivalent and have a single Puiseux pair. Recently, Genzmer [21, 22] has provided an algorithm that allows the computation of the minimal Tjurina number in a fixed topological class. However, a closed formula, or even a manageable recursive one in terms of topological data, is still far beyond reach at this moment.

The search for curves with a constant Tjurina number in the equisingularity class is even less developed. Except for cases where the equisingularity class contains only one analytic representative, there are few known irreducible cases: irreducible plane curve singularities with value semigroup $\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ satisfying $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{g-1}) = 2$. The case $g = 2$ was first described in 1990 by Luengo and Pfister [27], and the general case for $g \geq 2$ was provided by Abreu and the second-named author [11] in 2022. Up to best of the authors' knowledge, there are no cases in the literature of plane curves with $r > 1$ branches in which all different analytical representatives in a class of equisingularity have the same Tjurina number.

The main goal of this paper is to present a family of plane curve singularities with two branches that have a constant Tjurina number in their equisingularity class but whose branches can achieve distinct Tjurina numbers, along with a closed formula for this number in terms of topological data.

To provide such a family, we have used the ideas of [11]. There, a computation of the Tjurina number in terms of the set of values of Kähler differentials plays a crucial role. For an irreducible plane curve, Berger [7] shows that $\mu(C) - \tau(C) = \sharp(\Lambda_f \setminus S)$ where $\mu(C)$ is the Milnor number, S is the semigroup of values of C and Λ_C is the set of values of the Kähler differentials of C. Our first main result, Theorem 2.10, presents a generalization of this result for any reduced plane curve. More precisely, we show that for a reduced plane curve C with $r \geq 1$ branches we always have

$$\mu(C) - \tau(C) = d(\overline{\Lambda_C}, S)$$

where $d(\overline{\Lambda_f}, S)$ indicates the distance between the values set $\overline{\Lambda_f} := \Lambda_f \cup \{0\}$ and S as presented in Sect. 2. This result has its own interest as it is provided for any $r \geq 1$ and presents an alternative method to compute the Tjurina number of a reduced plane curve to the ones presented in [6, 24].

The next step is a careful study of the set of values of the Kähler differentials Λ_C . In [6, 24] some relations are presented that allow us to express $\tau(C)$ in terms of the Tjurina number of irreducible components $\{C_i;\ i = 1, \ldots, r\}$ of C, the intersection multiplicity of $[C_i, C_j]_0$ and an analytical invariant Θ_i for $2 \leq i \leq r$ related with the values of Kähler differentials of C. In Sect 3, we study the set Λ_C for $C = C_1 \cup C_2$ considering C_1 and C_2 equisingular plane branches with value semigroup $\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and $I := [C_1, C_2]_0 > n_g\overline{\beta_g}$ where $n_g = gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{g-1})$. In this case, considering logarithmic differentials, we describe the infinite fibers of Λ_C (see Sect 3.1) that allow us to compute Θ_2 and to show that the conductor of Λ_C depends only on $[C_1, C_2]_0$ and the multiplicity of C_i (see Theorem 3.14). Therefore, we show that in this case the conductor of Λ_C only depends on topological data and thus it is independent of the analytic type of the curve. This result also has its own interest as for the family of irreducible plane curve singularities with value semigroup $\langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$, where $\gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{g-1}) = 2$ provided in [11] the conductor of Λ may change but the Tjurina number is still constant. Then, our family with two branches has even a stronger behavior in this sense.

These two ingredients, the generalization of Berger's result and the description of the infinite fibers of Λ_C, are enough to achieve our main goal. In Sect. 4, considering $C = C_1 \cup C_2$ such that C_1 and C_2 sharing the same value semigroup $\langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ and $I > n_g \overline{\beta}_g$, we prove (Theorem 4.1)

$$\tau(C) = 2I + \mu(C_1) - 1$$

that is, the Tjurina number is given by only topological data, i.e. the intersection multiplicity and the Milnor number $\mu(C_1) = \mu(C_2)$ of the branches. This result gives us new examples of topological classes where the Tjurina number is constant. Moreover, in contrast to the irreducible case, the number of possible plane curves with two branches belonging to this family is quite large. In addition, our formula allows to provide a new proof in this particular case of the inequality $\mu(C)/\tau(C) < 4/3$ provided by the first named author [3] for any plane curve.

A deeper analysis of our results reveals that achieving similar results in more general cases presents an interesting yet challenging problem. To conclude the paper, we discuss some of the key difficulties in these cases and propose conjectures that could guide future investigations.

2 Value sets of fractional ideals

Let C_f be a (reduced) germ of complex plane curve singularity with equation $f = \prod_{i=1}^{r} f_i = 0$ where $f_i \in \mathbb{C}\{x, y\}$ is irreducible. Each f_i defines a branch C_i and its analytical type is characterized by the local ring $\mathscr{O}_i := \mathbb{C}\{x, y\}/(f_i)$ up to $\mathbb{C}$-algebra isomorphism. The field of fraction $\mathscr{K}_i$ of $\mathscr{O}_i$ is isomorphic to $\mathbb{C}(t_i)$ and associated to it we have a canonical discrete valuation $v_i : \mathscr{K}_i \to \overline{\mathbb{Z}} := \mathbb{Z} \cup \{\infty\}$. The isomorphic image of $\mathscr{O}_i$ in $\mathbb{C}(t_i) \cong \mathscr{K}_i$ can be given by $\mathbb{C}\{t_i^n, \sum_{j \geq n} a_j t_i^j\}$ where n is the multiplicity of f_i, the set $\{n, j;\ a_j \neq 0\}$ does not admit a nontrivial common divisor and

$$\varphi_i(t_i) = \left(t_i^n, \sum_{j \geq n} a_j t_i^j \right) \tag{2.1}$$

is a Newton–Puiseux parametrization of C_i. Notice that we have $f_i \left(t_i^n, \sum_{j \geq n} a_j t_i^j \right) = 0$, or equivalently $f_i \left(x, \sum_{j \geq n} a_j x^{\frac{j}{n}} \right) = 0$. We call $s_i(x) = \sum_{j \geq n} a_j x^{\frac{j}{n}}$ a Puiseux series of the branch C_i. If $h \in \mathbb{C}\{x, y\}$ and $\varphi_i(t_i) = (x_i, y_i) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ is a parametrization of C_i, we denote $\varphi_i^*(h) := h(x_i, y_i) \in \mathbb{C}\{t_i\}$. Thus, if $\overline{h} \in \mathscr{O}_i$ with $h \in \mathbb{C}\{x, y\}$, then we get $v_i(\overline{h}) = \operatorname{ord}_{t_i} \varphi_i^*(h) = [f_i, h]_0$, where $[f_i, h]_0 := [C_i, C_h]_0$ denotes the intersection multiplicity of C_i and C_h at the origin.

The image of v_i of $\mathscr{O}_i \setminus \{0\}$, that is

$$S(C_i) := \{v_i(\overline{h}) \in \mathbb{N} \ :\ \overline{h} \in \mathscr{O}_i,\ \overline{h} \neq 0\}$$

is the semigroup of values of C_i and it is a numerical semigroup whose minimal generating set can be computed from a Newton–Puiseux parametrization (cf. [31]) as follows.

If $f \in \mathbb{C}\{x, y\}$ is irreducible with Newton–Puiseux parametrization $(t^n, \sum_{j\geq n} a_j t^j)$, let $(z) \subseteq \mathbb{Z}$ be the set of multiples of z, $\beta_0 = n$ and set

$$e_{i-1} = \gcd\{\beta_0, \ldots, \beta_{i-1}\},\ \beta_i = \min\{j : a_j \neq 0,\ j \notin (e_{i-1})\}\ \text{ for } i = 1, \ldots, g \text{ where } e_g = 1.$$

Let us define $n_0 = 1$,

$$\overline{\beta}_0 = \beta_0,\ \overline{\beta}_1 = \beta_1 \quad \text{and} \quad \overline{\beta}_{i+1} = n_i\overline{\beta}_i + \beta_{i+1} - \beta_i \quad \text{where },\ n_i = e_{i-1}/e_i \text{ for } 1 \leq i < g. \tag{2.2}$$

It follows (cf. [31]) that the values semigroup $S(C_f)$ is minimally generated by the elements $\overline{\beta}_0, \ldots, \overline{\beta}_g$, i.e.

$$S(C_f) = \langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle = \left\{\gamma \in \mathbb{N} : \gamma = m_0\overline{\beta}_0 + \cdots + m_g\overline{\beta}_g \text{ with } m_i \in \mathbb{N},\ \text{ for } i = 0, \ldots, g\right\}.$$

Since $S(C_f)$ is a numerical semigroup it admits a conductor $\mathbf{c}(S(C_f))$, that is the minimun element in $S(C_f)$ such that $\mathbf{c}(S(C_f)) + \mathbb{N} \subset S(C_f)$ and it can be computed (cf. [31]) by

$$\mathbf{c}(S(C_f)) = \sum_{i=1}^{g}(n_i - 1)\overline{\beta}_i - \overline{\beta}_0 + 1 = v(f_y) - \overline{\beta}_0 + 1. \tag{2.3}$$

The topological (equisingularity) class of the branch C_f is totally determined by its values semigroup.

For $f = \prod_{i=1}^{r} f_i$ with $r > 1$ the topological class of C_f is also characterized by a semigroup as we describe in the sequel.

The total ring of fraction $\mathcal{K}$ of the local ring $\mathcal{O} = \mathbb{C}\{x, y\}/(f)$ is isomorphic to $\prod_{i=1}^{r} \mathbb{C}(t_i)$. If $\pi_i : \mathcal{K} \to \mathcal{K}_i$ denotes the natural projection then we can consider $\underline{v} : \mathcal{K} \to \overline{\mathbb{Z}}^r$ defined by $\underline{v}(q) = (v_1(q), \ldots, v_r(q))$ where $q \in \mathcal{K}$ and $v_i(q)$ stands for $v_i(\pi_i(q))$.

In what follows we set $\mathtt{I} := \{1, \ldots, r\}$ and we consider the product order on $\mathbb{Z}^r$, that is, given $\alpha = (\alpha_1, \ldots, \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$,

$$\alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i \text{ for all } i \in \mathtt{I}.$$

The values semigroup of C_f is the additive submonoid of $\mathbb{N}^r$ defined by

$$S = S(C_f) := \{\underline{v}(h) = (v_1(h), \ldots, v_r(h)) \in \mathbb{N}^r\ :\ h \in \mathcal{O},\ h \text{ is not a divisor of } 0\}.$$

For $\mathtt{J} = \{j_1, \ldots, j_k\} \subset \mathtt{I}$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{Z}^r$ we put $pr_J(\alpha) = (\alpha_{j_1}, \ldots, \alpha_{j_k}) \in \mathbb{N}^k$. Notice that $S \subsetneq S(C_1) \times \cdots \times S(C_r)$ and $pr_J(S) = S(C_{f_\mathtt{J}})$ where $f_J = \prod_{j\in\mathtt{J}} f_j$.

Some elementary properties of the semigroup of values S are the following (see [17]):

(1) If $\alpha, \beta \in S$, then
$$\min\{\alpha, \beta\} := (\min\{\alpha_i, \beta_i\})_{i\in\mathtt{I}} \in S.$$

(2) If $\alpha, \beta \in S$ and $j \in \mathtt{I}$ with $\alpha_j = \beta_j$, then there exists $\epsilon \in S$ such that $\epsilon_j > \alpha_j = \beta_j$ and $\epsilon_i \geq \min\{\alpha_i, \beta_i\}$ for all $i \in \mathtt{I}\backslash\{j\}$, with equality if $\alpha_i \neq \beta_i$.

(3) The semigroup S has a conductor $\mathbf{c}_S := \mathbf{c}(S)$, which is defined to be the minimal element of S such that $\gamma \in S$ whenever $\gamma \geq \mathbf{c}_S$. Moreover, we get (see [17])
$$\mathbf{c}_S = (c_1 + \sum_{i\in\mathtt{I}\backslash\{1\}} I_{1,i}, \ldots, c_r + \sum_{i\in\mathtt{I}\backslash\{r\}} I_{r,i}), \tag{2.4}$$
where $c_i := \mathbf{c}(S(C_i))$ and $I_{i,j} = [f_i, f_j]_0$.

Remark 2.1 The conductor of the semigroup of a plane curve C defined by f is closely related to its Milnor number $\mu(C) = \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f_x, f_y)$. In fact, if f is irreducible then $\mu(C) = \mathbf{c}_S$ and for $f = \prod_{i=1}^r f_i$ we get

$$\mu(C) = \sum_{i=1}^{r} pr_i(\mathbf{c}_S) - r + 1 = \sum_{i=1}^{r} c_i + 2 \sum_{1 \leq i < j \leq r} I_{i,j} - r + 1. \tag{2.5}$$

For $r > 1$, the semigroup S is no longer finitely generated, but it is finitely determined (see [9, 17]). However, if we allow $\underline{v}(h) \in S \subset \overline{\mathbb{Z}}^r$ for *any* $h \in \mathscr{O}$, that is, $v_i(0) = \infty$, then $(S, \min, +)$ is a finite generated semiring (see [14]).

Now, for a given $A \subseteq \overline{\mathbb{Z}}^r$, $\alpha \in \mathbb{Z}^r$ and an index subset $\emptyset \neq \mathtt{J} \subset \mathtt{I}$, we define

$$F_{\mathtt{J}}(A, \alpha) = \left\{\beta \in A \ : \ \beta_j = \alpha_j \quad \forall j \in \mathtt{J} \quad \text{and} \quad \beta_k > \alpha_k \quad \forall k \notin \mathtt{J}\right\},$$

$$\overline{F}_{\mathtt{J}}(A, \alpha) = \left\{\beta \in A \ : \ \beta_j = \alpha_j \quad \forall j \in \mathtt{J} \quad \text{and} \quad \beta_k \geq \alpha_k \quad \forall k \notin \mathtt{J}\right\}.$$

The fiber of α in A is defined as $F(A, \alpha) = \cup_{i=1}^r F_i(A, \alpha)$. We will abuse of notation and denote $F_i(A, \alpha) := F_{\{i\}}(A, \alpha)$.

The fibers $F(S, \alpha)$ are important in order to determine S in terms of its projections (see [17]).

An element $\gamma \in S$ is called a maximal element of S if $F(S, \gamma) = \emptyset$. If, moreover, $F_{\mathtt{J}}(S, \gamma) = \emptyset$ for all $\mathtt{J} \subset \mathtt{I}$ such that $\emptyset \neq \mathtt{J} \neq \mathtt{I}$, then γ is said to be absolute maximal. On the other hand, if γ is a maximal and if $F_{\mathtt{J}}(S, \alpha) \neq \emptyset$ for all $\mathtt{J} \subset \mathtt{I}$ such that $\sharp\mathtt{J} \geq 2$, then γ will be called relative maximal. It is easily checked that the set of maximal elements of S is finite.

Definition 2.2 Let $A \subset \overline{\mathbb{Z}}^r$ be a set satisfying the properties (1), (2) and (3). We will say that α has infinite fiber in A with respect to $\mathtt{J} \subset \mathtt{I}$, writing $F_{\mathtt{J}}^A(\alpha) = \infty$, if there exists $\beta \in \overline{F}_{\mathtt{J}}(A, \alpha)$ such that $pr_{\mathtt{I}\setminus\mathtt{J}}(\beta) \geq pr_{\mathtt{I}\setminus\mathtt{J}}(\mathbf{c}_A)$.

Observe that α has infinite fiber in A with respect to $\mathtt{J} \subset \mathtt{I}$ is the same as saying that

$$\{\delta \in \mathbb{N}^r \mid pr_{\mathtt{J}}(\delta) = pr_{\mathtt{J}}(\alpha), pr_{\mathtt{I}\setminus\mathtt{J}}(\delta) \geq pr_{\mathtt{I}\setminus\mathtt{J}}(\mathbf{c}_A)\} \subseteq A,$$

or equivalently, there exists $\beta \in F_{\mathtt{J}}(A, \alpha)$ such that $\beta + \overline{F}_{\mathtt{J}}(\mathbb{N}^r, 0) \subset \overline{F}_{\mathtt{J}}(A, \alpha)$. There are several other characterizations of infinite fibers see [17, Proposition 2.4] and [18, Lemma 1.8] for further details.

Many of the above properties hold for fractional ideals of $\mathscr{O}$ and its set of values. For the convenience of the reader, we present them in the next section.

2.1 Basic properties of value set of fractional ideals

A fractional ideal $J \subset \mathscr{K}$ of $\mathscr{O}$ is a $\mathscr{O}$-module such that there exists a regular element $h \in \mathscr{O}$ satisfying $hJ \subset \mathscr{O}$. It follows that for any fractional ideal J of $\mathscr{O}$ its value set $E := \underline{v}(J)$ is a relative ideal of S, i.e. $S + E \subseteq E$ and there exists $\gamma \in S$ such that $\gamma + E \subseteq S$.

By [4, 10], for any two fractional ideals $J_2 \subset J_1$ of $\mathscr{O}$ we can compute the length $l(J_1/J_2)$ (as $\mathscr{O}$-modules) by comparing their value sets by means a saturated chain as follows.

For a fractional ideal $J \subset \mathscr{K}$, a saturated chain in $E = \underline{v}(J)$ is a sequence

$$\alpha^0 < \alpha^1 < \cdots < \alpha^n$$

of elements in E such that for every element $\epsilon \in \mathbb{Z}^r$ such that $\alpha^i < \epsilon < \alpha^{i+1}$ one has $\epsilon \notin E$. Such a chain is said to have length n. According to [10, Proposition 2.3], any saturated chain in E between α^0 and α^n has the same length. This property allows us to define a distance function between two elements in E: if $\alpha^0, \alpha^n \in E$ with $\alpha^0 < \alpha^n$ then its distance in E, denoted by $d_E(\alpha^0, \alpha^n)$, is the length of any saturated chain in E with α^0 as initial element and α^n as final element.

Example 2.3 Let $f = (y^2 - 2x^2y - x^3 + x^4)(y^2 + x^3)$. According to [12, Example 3.11] we get

$$S = S(C_f) = \{(0,0), (k,k);\ 2 \leq k \leq 7\} \cup \{(6, 6+k_1), (6+k_2, 6),\ k_1, k_2 \in \mathbb{N}\} \cup \{(8,8)+\mathbb{N}^2\}.$$

Thus, $\mathbf{c}_S = (8,8)$. We have $d_S((0,0),(8,8)) = 8$ as we can see with the following two different saturated chains of length 8 in S :

$$\alpha^0 = (0,0) < \alpha^1 = (2,2) < \alpha^2 = (3,3) < \alpha^3 = (4,4) < \alpha^4 = (5,5) < \alpha^5 = (6,6) < \alpha^6 = (7,6) < \alpha^7 = (7,7) < \alpha^8 = (8,8),$$

$$\alpha^0 = (0,0) < \alpha^1 = (2,2) < \alpha^2 = (3,3) < \alpha^3 = (4,4) < \alpha^4 = (5,5) < \alpha^5 = (6,6) < \alpha^6 = (6,7) < \alpha^7 = (6,8) < \alpha^8 = (8,8).$$

As for S, the value set of a fractional ideal also satisfies properties (1), (2) and (3) (see [10]). So, if J is a fractional ideal of $\mathscr{O}$, then its value set E has always a minimum $m_E := \min\{\alpha \in E\}$ and a conductor $\mathbf{c}_E := \min\{\gamma \in E\ :\ \gamma + \mathbb{N}^r \subseteq E\}$.

In this way, the colength of fractional ideals can be computed according the following

Theorem 2.4 [10, Section 2] *Let $J_2 \subset J_1$ be fractional ideals of $\mathscr{O}$ with $E_i = \underline{v}(J_i)$ for $i = 1, 2$. Then,*

$$l(J_1/J_2) = d(E_1 \backslash E_2) := d_{E_1}(m_{E_1}, \boldsymbol{c}_{E_2}) - d_{E_2}(m_{E_2}, \boldsymbol{c}_{E_2}).$$

Remark 2.5 In general, for any two subsets $E_1, E_2 \subseteq \mathbb{Z}^r$, $E_2 \subsetneq E_1$ satisfying properties (1), (2), (3) we define its distance as $d(E_1 \backslash E_2) = d_{E_1}(m_{E_1}, \mathbf{c}_{E_2}) - d_{E_2}(m_{E_2}, \mathbf{c}_{E_2})$.

Also, it is obvious that for any $\gamma \geq \mathbf{c}_{E_2}$ we have

$$d(E_1 \backslash E_2) = d_{E_1}(m_{E_1}, \gamma) - d_{E_2}(m_{E_2}, \gamma).$$

This method has the disadvantage that we need a lot of information about the value set E. In order to avoid the use of a saturated chain in E, Guzmán and Hefez [15] provided an alternative method to compute colengths just by using the set of relative maximal points of the value set and its projections.

Remark 2.6 The notion of maximal, relative maximal and absolute maximal for a value set is defined in an analogous way as in the semigroup case. For any value set E, we will denote by $M(E)$, $RM(E)$, $AM(E)$ the sets of maximal, relative maximals and absolute maximals of E.

Let us briefly explain the Guzmán and Hefez's method to compute colengths without the use of a saturated chain, we refer to [15] for further details.

For any fractional ideal J of $\mathscr{O}$, there is a canonical filtration indexed by $\alpha \in \mathbb{Z}^r$ defined as

$$J(\alpha) = \{h \in J\ ;\ \underline{v}(h) \geq \alpha\}.$$

Therefore, given $J_2 \subseteq J_1$ two fractional ideals with value sets $E_i = \underline{v}(J_i)$, we have that for any $\gamma \geq \mathbf{c}_{E_2}$ the colength (and hence its distance) is

$$l(J_1/J_2) = l(J_1/J_1(\gamma)) - l(J_2/J_2(\gamma)). \tag{2.6}$$

Recall that $J_1(\gamma) = J_2(\gamma)$. In this way, we have

Theorem 2.7 [15, Cor. 11] *Let J be a fractional ideal of $\mathscr{O}$ with value set $E = \underline{v}(J)$ and $m_E = \alpha^0$. If $\gamma \geq \mathbf{c}_E$, then*

$$l\left(\frac{J}{J(\gamma)}\right) = \sum_{i=1}^{r}(\gamma_i - \alpha_i^0 - \sharp((\mathbb{N}+\alpha_i^0)\setminus pr_i(E)) - \Theta_i),$$

where Θ_i is defined as

$$\Theta_1 = 0, \quad \Theta_i = \sharp \bigcup_{\{i\}\subsetneq\mathcal{J}\subseteq\{1,\ldots,i\}} pr_*(RM(E_{\mathcal{J}})), \quad \text{for } 2 \leq i \leq r,$$

with $pr_*(\alpha_{j_1},\ldots,\alpha_{j_s}) = \alpha_{j_s}$ and $E_{\mathcal{J}} = pr_{\mathcal{J}}(E)$ if $\mathcal{J} = \{j_1,\ldots,j_s\}$.

As an application of Theorem 2.7 we will provide an alternative way to compute the delta invariant $\delta(C)$ of a reduced plane curve C.

Example 2.8 Let $C = \cup_{i\in\mathtt{I}} C_i$ be a reduced plane curve such that each branch C_i is defined by f_i. If $\mathscr{O}$ denotes its local ring, then its normalization $\overline{\mathscr{O}}$ is isomorphic to $\mathbb{C}\{t_1\}\times\cdots\times\mathbb{C}\{t_r\}$. Since $\mathscr{O}$ is a Gorenstein ring (see [18]), the conductor ideal $\mathscr{C} := \{z \in \overline{\mathscr{O}} : z\overline{\mathscr{O}} \subset \mathscr{O}\}$ is such that

$$\delta(C) = l\left(\frac{\overline{\mathscr{O}}}{\mathscr{O}}\right) = l\left(\frac{\mathscr{O}}{\mathscr{C}}\right) = \frac{1}{2}l\left(\frac{\overline{\mathscr{O}}}{\mathscr{C}}\right).$$

Notice that $\underline{v}(\overline{\mathscr{O}}) = \mathbb{N}^r$ and $\underline{v}(\mathscr{C}) = \mathbf{c}_S + \mathbb{N}^r$, in particular, these values sets do not have maximal points, that is, $\Theta_i(\underline{v}(\overline{\mathscr{O}})) = \Theta_i(\underline{v}(\mathscr{C})) = 0$ for all $i \in \mathtt{I}$. In addition, we get

$$m_{\underline{v}(\overline{\mathscr{O}})} = \mathbf{c}_{\underline{v}(\overline{\mathscr{O}})} = (0,\ldots,0) \in \mathbb{N}^r \quad\text{and}\quad m_{\underline{v}(\mathscr{C})} = \mathbf{c}_{\underline{v}(\mathscr{C})} = \mathbf{c}_S.$$

According to (2.6) we get $l(\frac{\overline{\mathscr{O}}}{\mathscr{C}}) = l(\frac{\overline{\mathscr{O}}}{\overline{\mathscr{O}}(\mathbf{c}_S)}) - l(\frac{\mathscr{C}}{\mathscr{C}(\mathbf{c}_S)})$. Notice that $\mathscr{C}(\mathbf{c}_S) = \mathscr{C}$, that is $l(\frac{\mathscr{C}}{\mathscr{C}(\mathbf{c}_S)}) = 0$ and, by Theorem 2.7 we have

$$l\left(\frac{\overline{\mathscr{O}}}{\overline{\mathscr{O}}(\mathbf{c}_S)}\right) = \sum_{i\in\mathtt{I}}(pr_i(\mathbf{c}_S) - 0 - \sharp(\mathbb{N}+0)\setminus pr_i(\mathbb{N}^r)) - 0)) = \sum_{i\in\mathtt{I}} pr_i(\mathbf{c}_S) = \sum_{i\in\mathtt{I}}(c_i + \sum_{j\neq i} I_{i,j}),$$

where the last equality follows by (2.4).

Since $c_i = 2\delta_i$ where $\delta_i = \delta(C_i)$ is the delta invariant of the branch we get

$$\delta(C) = \frac{1}{2}l\left(\frac{\overline{\mathscr{O}}}{\mathscr{C}}\right) = \frac{1}{2}l\left(\frac{\overline{\mathscr{O}}}{\overline{\mathscr{O}}(\mathbf{c}_S)}\right) = \frac{1}{2}\sum_{i\in\mathtt{I}}(c_i + \sum_{j\neq i} I_{i,j}) = \sum_{i\in\mathtt{I}}(\delta_i + \sum_{j<i} I_{i,j}).$$

In this way, we recover the well-known formula for the delta invariant of the curve C and, as an immediate consequence of Proposition 2.5 we also obtain that

$$\delta(C) = \frac{1}{2}l\left(\frac{\overline{\mathscr{O}}}{\mathscr{C}}\right) = l\left(\frac{\mathscr{O}}{\mathscr{C}}\right).$$

In [24] is presented a way to compute the data Θ_i given in Theorem 2.7 without knowing the relative maximal of the values set of a fractional ideal or any information of the values set of E. Let us present this result.

Given $J \subset \mathscr{K}$ a fractional ideal of $\mathscr{O}$ with $E := \underline{v}(J)$ and $\pi : \mathscr{K} \to \mathscr{K}_i$ the natural projection we put

$$\mathscr{N}_i(J) := J \cap \ker \pi_i \quad\text{and}\quad \mathscr{N}_{\mathtt{J}}(J) := \cap_{j\in\mathtt{J}}\mathscr{N}_j(J).$$

In this way, setting $[1, i) = \{1, \ldots, i-1\}$ for $1 < i \leq r$ we get (cf. [24, Cor. 2.8])

$$\Theta_i = \sharp(\mathrm{pr}_i(E) \setminus v_i(\mathcal{N}_{[1,i)}(J))). \tag{2.7}$$

We consider $\mathcal{N}_{[1,1)}(J) = pr_i(E)$ to obtain $\Theta_1 = 0$ as in Theorem 2.7. In this way, we get the following

Proposition 2.9 [24, Cor. 2.9] *Let $J_2 \subset J_1$ be fractional ideals of $\mathcal{O}$ with $E_i = \underline{v}(J_i)$ for $i = 1, 2$. Then,*

$$l\left(\frac{J_1}{J_2}\right) = \sum_{i \in \mathcal{I}} \left(\sharp(pr_i(E_1)\backslash pr_i(E_2)) - \sharp(pr_i(E_1)\backslash v_i(\mathcal{N}_{[1,i)}(J_1))) + \sharp(pr_i(E_2)\backslash v_i(\mathcal{N}_{[1,i)}(J_2)))\right).$$

Notice that Theorems 2.5, 2.7 (and (2.6)) and Proposition 2.9 provide alternative ways to compute colength of fractional ideals according to the data available in each situation.

In this paper, we are interested in computing the codimension of a particular fractional ideal that gives us an important analytic invariant of a plane curve: the Tjurina number.

2.2 Tjurina number

Let C be a plane curve defined by $f = \prod_{i \in \mathcal{I}} f_i$ and C_i be the branch given by $f_i = 0$. The Tjurina number of C is $\tau = \tau(f) := \dim_{\mathbb{C}} \mathbb{C}\{x, y\}/(f, f_x, f_y)$. Denoting by $\overline{h}$ the class of $h \in \mathbb{C}\{x, y\}$ in $\mathcal{O}$ and considering the ideal $\mathbf{J} := \mathcal{O}\overline{f_x} + \mathcal{O}\overline{f_y}$ we have that $\tau = l(\frac{\mathcal{O}}{\mathbf{J}})$.

An alternative, but equivalent, approach to compute the Tjurina number is by using the module of Kähler differentials of the curve. Let $\Omega^1 = \mathbb{C}\{x, y\}\mathrm{d}x + \mathbb{C}\{x, y\}\mathrm{d}y$ be the $\mathbb{C}\{x, y\}$-module of holomorphic forms on $\mathbb{C}^2$ and consider the submodule $\mathscr{F}(f) := \mathbb{C}\{x, y\}\mathrm{d}f + f\Omega^1$. The module of Kähler differentials of C is

$$\Omega_f := \frac{\Omega^1}{\mathscr{F}(f)},$$

that is the $\mathcal{O}$-module $\mathcal{O}\mathrm{d}x + \mathcal{O}\mathrm{d}y$ module the relation $\mathrm{d}f = 0$.

If $\varphi_i = (x_i, y_i) \in \mathbb{C}\{t_i\} \times \mathbb{C}\{t_i\}$ is a parameterization (non necessarily a Newton–Puiseux parameterization) of the branch C_i and $h(x, y) \in \mathcal{O}$ then, as before, we denote $\varphi_i^*(h) := h(x_i, y_i) \in \mathbb{C}\{t_i\}$. In addition, given $\omega = A(x, y)\mathrm{d}x + B(x, y)\mathrm{d}y \in \Omega_f$, we define

$$\varphi_i^*(\omega) \colon = t_i(\varphi_i^*(A) \cdot x_i' + \varphi_i^*(B) \cdot y_i') \in \mathbb{C}\{t_i\},$$

where x_i', y_i' denote, respectively, the derivative of $x_i, y_i \in \mathbb{C}\{t_i\}$ with respect to t_i. We put

$$\varphi^*(\Omega_f) = \{(\varphi_1^*(\omega), \ldots, \varphi_r^*(\omega)) : \omega \in \Omega_f\} \subset \mathcal{K}.$$

By [6, Theorem 3], if $Tor(\Omega_f)$ denotes the torsion submodule of Ω_f then we have $\ker(\varphi^*) = Tor(\Omega_f)$ and the following $\mathcal{O}$-module isomorphism:

$$\varphi^*(\Omega_f) \cong \frac{\Omega_f}{Tor(\Omega_f)}. \tag{2.8}$$

In this way, $\varphi^*(\Omega_f)$ is a fractional ideal of $\mathcal{O}$ and, considering $v_i(\omega) := v_i(\varphi_i^*(\omega))$, its value set is given by

$$\Lambda_f = \underline{v}(\varphi^*(\Omega_f)) = \{\underline{v}(\omega) \colon = (v_1(\omega), \ldots, v_r(\omega)); \quad \omega \in \Omega_f\}.$$

Recall that $S \setminus \{0\} \subset \Lambda_f$ so, $\mathbf{c}_{\Lambda_f} \leq \mathbf{c}_S$ and $m_{\Lambda_f} = (\beta_0^1, \ldots, \beta_0^r)$ where $\beta_0^i = \min\{\alpha \in S(C_i) \setminus \{0\}\}$ is the multiplicity of the branch C_i.

The set Λ_f is an important analytic invariant of the curve which was used for the analytical classification of plane curves (see [25]) and, according to Pol [29, Proposition 3.31], the Jacobian ideal $\mathbf{J}$ is isomorphic to $\varphi^*(\Omega_f)$. Moreover, she shows that

$$\underline{v}(\overline{Af_x + Bf_y}) = \underline{v}(\overline{A}dy - \overline{B}dx) + \mathbf{c}_S - \underline{1}, \quad \text{consequently} \quad \underline{v}(\mathbf{J}) = \Lambda_f + \mathbf{c}_S - \underline{1} \tag{2.9}$$

where $\underline{1} = (1, \ldots, 1) \in \mathbb{N}^r$.

In the case $r = 1$, Berger [7] proved that the Milnor number and the Tjurina number of C_f are related by $\tau = \mu - \sharp(\Lambda_f \setminus S)$. Since the distance function is the natural generalization for the difference of values set of fractional ideals in the irreducible case, it is natural to ask for a extension of Berger's expression for the case $r > 1$. From the identity (2.9) and the previous results, we obtain the following generalization of Berger's result.

Theorem 2.10 *Let C be a reduced plane curve. With the previous notation, let us denote by $\overline{\Lambda} = \Lambda_f \cup \{0\}$. Then, we have*

$$\tau = \mu - d(\overline{\Lambda} \setminus S). \tag{2.10}$$

Proof Consider $C = \cup_{i=1}^r C_i$. Since $\mathbf{J} \subset \mathscr{O}$ we have $E := \underline{v}(\mathbf{J}) \subset S$ and $\mathbf{c}_E \leq m_E + \mathbf{c}_S$. As we have remarked, $m_\Lambda = (\beta_0^1, \ldots, \beta_0^r)$ and by (2.9) we get $m_E = \mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r) - \underline{1}$. Thus, considering the relation (2.6) with $\gamma = 2\mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r) - \underline{1}$ we have

$$\tau = l\left(\frac{\mathscr{O}}{\mathbf{J}}\right) = l\left(\frac{\mathscr{O}}{\mathscr{O}(\gamma)}\right) - l\left(\frac{\mathbf{J}}{\mathbf{J}(\gamma)}\right). \tag{2.11}$$

Notice that $l(\mathscr{O}/\mathscr{O}(\gamma)) = l(\mathscr{O}/\mathscr{O}(\mathbf{c}_S)) + l(\mathscr{O}(\mathbf{c}_S)/\mathscr{O}(\gamma))$ with $\underline{v}(\mathscr{O}(\mathbf{c}_S)) = \mathbf{c}_S + \mathbb{N}^r$ and $\underline{v}(\mathscr{O}(\gamma)) = \gamma + \mathbb{N}^r$. In this way, by Theorems 2.5 and 2.7, we have

$$\begin{aligned} l\left(\tfrac{\mathscr{O}}{\mathscr{O}(\gamma)}\right) = l\left(\tfrac{\mathscr{O}}{\mathscr{O}(\mathbf{c}_S)}\right) + l\left(\tfrac{\mathscr{O}(\mathbf{c}_S)}{\mathscr{O}(\gamma)}\right) &= d_S(0, \mathbf{c}_S) + \textstyle\sum_{i=1}^r (pr_i(\mathbf{c}_S) + \beta_0^i - 1) \\ &= d_S(0, \mathbf{c}_S) + \textstyle\sum_{i=1}^r \beta_0^i + \mu - \underline{1} \end{aligned} \tag{2.12}$$

where the last equality follows by Remark 2.5.

On the other hand, the relation (2.9) between $\underline{v}(\mathbf{J})$ and Λ_f gives

$$l\left(\frac{\mathbf{J}}{\mathbf{J}(\gamma)}\right) = l\left(\frac{\varphi^*(\Omega_f)}{\varphi^*(\Omega)(\mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r))}\right) = l\left(\frac{\varphi^*(\Omega_f)}{\varphi^*(\Omega)(\mathbf{c}_S)}\right) + l\left(\frac{\varphi^*(\Omega_f)(\mathbf{c}_S)}{\varphi^*(\Omega)(\mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r))}\right).$$

Since $\mathbf{c}_{\Lambda_f} \leq \mathbf{c}_S$, we get $\underline{v}(\varphi^*(\Omega_f)(\mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r)) = \mathbf{c}_S + (\beta_0^1, \ldots, \beta_0^r) + \mathbb{N}^r$ and $\underline{v}(\varphi^*(\Omega_f)(\mathbf{c}_S)) = \mathbf{c}_S + \mathbb{N}^r$, that is, the second above summand is $\sum_{i=1}^r \beta_0^i$. In addition, denoting $\overline{\Lambda} = \Lambda_f \cup \{0\}$ we have

$$l\left(\frac{\varphi^*(\Omega_f)}{\varphi^*(\Omega)(\mathbf{c}_S)}\right) = d_{\Lambda_f}((\beta_0^1, \ldots, \beta_0^r), \mathbf{c}_S) = d_{\overline{\Lambda}}(0, \mathbf{c}_S) - \underline{1}.$$

In this way, we get

$$l\left(\frac{\mathbf{J}}{\mathbf{J}(\gamma)}\right) = d_{\overline{\Lambda}}(0, \mathbf{c}_S) - \underline{1} + \sum_{i=1}^r \beta_0^i. \tag{2.13}$$

Therefore, by (2.11), (2.12) and (2.13), we obtain

$$\tau = \mu + d_S(0, \mathbf{c}_S) - d_{\overline{\Lambda}}(0, \mathbf{c}_S) = \mu - d(\overline{\Lambda} \setminus S).$$

□

Example 2.11 Let $f = f_1 f_2$ where $f_1 = y^2 - 2x^2y - x^3 + x^4$ and $f_2 = y^2 + x^3$. Since $\mu(C_1) = \mu(C_2) = c_1 = c_2 = 2$ and $I_{1,2} = 6$ it follows, by (2.5), that $\mu = \mu(C_f) = 15$. By Example 2.3, we have $\mathbf{c}_S = (8, 8)$ and $d_S((0, 0), (8, 8)) = 8$. According to [12, Example 3.11] we get

$$\overline{\Lambda} = \Lambda_f \cup \{(0, 0)\} = \{(0, 0), (k, k);\ 2 \le k \le 5\} \cup \{(6, 6) + \mathbb{N}^2\}.$$

A saturated chain in $\overline{\Lambda}$ connecting $(0, 0)$ to $(8, 8)$ is

$$\begin{aligned}\alpha^0 = (0, 0) < \alpha^1 = (2, 2) < \alpha^2 = (3, 3) < \alpha^3 = (4, 4) < \alpha^4 = (5, 5)\\ < \alpha^5 = (6, 6) < \alpha^6 = (7, 6) < \alpha^7 = (7, 7) < \alpha^8 = (8, 7) < \alpha^9 = (8, 8),\end{aligned}$$

that is, $d_{\overline{\Lambda}}((0, 0), (8, 8)) = 9$. Hence, by Theorem 2.10, we get

$$\tau = \mu - d(\overline{\Lambda}\backslash S) = \mu + d_S((0, 0), (8, 8)) - d_{\overline{\Lambda}}((0, 0), (8, 8)) = \mu + 8 - 9 = \mu - 1 = 14,$$

that coincides with calculation with Singular [16].

3 Logarithmic differentials

As before, let C be a reduced plane curve defined by f. According to (2.9), the values set $\underline{v}(\mathbf{J})$ determines and it is determined by the set Λ. In [29], Pol shows that such analytical invariants are related to the values set of residues of logarithmic differentials or equivalently to the values set of the Saito module. Let us recall these objects and some results concerning to them.

According to Saito [30], a meromorphic differential $W \in (1/f)\Omega^1$ is a logarithmic form along C if there exist $\eta \in \Omega^1$, $p, q \in \mathbb{C}\{x, y\}$ with $gcd(q, f) = 1$ such that $qW = (p\mathrm{d}f + f\eta)/f$ or equivalently, $\frac{W\wedge \mathrm{d}f}{\mathrm{d}x\wedge\mathrm{d}y} \in \mathbb{C}\{x, y\}$ and he denotes the set of logarithmic forms along C by $\Omega^1(\log\ C)$.

Since $W = \omega/f \in \Omega^1(\log\ C)$ is equivalent to get $q\omega \in \mathscr{F}(f) = \mathbb{C}\{x, y\}\mathrm{d}f + f\Omega^1$ for some $q \in \mathbb{C}\{x, y\}$ coprime with f or $\frac{\omega\wedge\mathrm{d}f}{\mathrm{d}x\wedge\mathrm{d}y} \in (f)$ we can consider the $\mathbb{C}\{x, y\}$-module

$$\begin{aligned} f \cdot \Omega^1(\log\ C) &= \left\{\omega \in \Omega^1;\ \exists\, q \in \mathbb{C}\{x, y\}, gcd(q, f) = 1 \text{ such that } q\omega \in \mathscr{F}(f)\right\}\\ &= \left\{\omega \in \Omega^1;\ \tfrac{\omega\wedge\mathrm{d}f}{\mathrm{d}x\wedge\mathrm{d}x} \in (f)\right\}, \end{aligned} \tag{3.1}$$

called the Saito module associated to C.

We have that $f \cdot \Omega^1(\log\ C)$ is generated by two elements. Moreover,

Saito's criterion: $\{\omega_1, \omega_2\}$ is a set of generators for $f\cdot\Omega^1(\log C)$ if and only if $\frac{\omega_1\wedge\omega_2}{\mathrm{d}x\wedge\mathrm{d}y} = uf$ where $u \in \mathbb{C}\{x, y\}$ is a unit.

Notice that $\mathscr{F}(f) \subset f \cdot \Omega^1(\log\ C)$, by [29, Proposition 3.22], we have that $Tor(\Omega_f)$ is isomorphic (as $\mathscr{O}$-module) to $f \cdot \Omega^1(\log\ C)/\mathscr{F}(f)$. In particular, by [29, Proposition 3.31] and (2.8), we get

$$\mathbf{J} \cong \varphi^*(\Omega_f) \cong \frac{\Omega^1}{f \cdot \Omega^1(\log\ C)}.$$

Given $\omega \in f \cdot \Omega^1(\log\ C)$, such that $q\omega = p\mathrm{d}f + f\eta$ where $\eta \in \Omega^1$, $p, q \in \mathbb{C}\{x, y\}$ with $gcd(q, f) = 1$ the residue of ω is $res(\omega) = \overline{p}/\overline{q} \in \mathscr{K}$ where $\overline{h}$ denotes the class of $h \in \mathbb{C}\{x, y\}$ in $\mathscr{O}$. The $\mathscr{O}$-module of logarithmic residues along C is then defined as

$$\mathscr{R}_C := \{res(\omega);\ \omega \in f \cdot \Omega^1(\log\ C)\} \subset \mathscr{K}.$$

We have that $\mathscr{R}_C$ is a fractional ideal of $\mathscr{O}$ and its values set $\Delta_f := \underline{v}(\mathscr{R}_C)$ satisfies (cf. [29, Cor. 3.32])

$$\alpha \in \Delta_f \Leftrightarrow F(-\alpha, \Lambda_f) = \emptyset. \tag{3.2}$$

According to Pol [29] we have that $\mathbf{c}(\Delta_f)$ is $-(\overline{\beta}_0^1, \dots, \overline{\beta}_0^r) + (1, \dots, 1)$.

Remark 3.1 If $\mathtt{I} = \{1, \dots, r\}$ with $r > 1$ and $C = \cup_{i \in \mathtt{I}} C_i$ is a plane curve where each branch C_i is defined by f_i then for $\emptyset \neq \mathtt{J} \subsetneq \mathtt{I}$ we denote $f_{\mathtt{J}} = \prod_{j \in \mathtt{J}} f_j$. In this case, given $\omega \in f_{\mathtt{J}} \cdot \Omega^1(\log\ C_{\mathtt{J}})$ whose class module $\mathscr{F}(f)$ is $\overline{\omega}$, we get $v_j(\overline{\omega}) = \infty$ for all $j \in \mathtt{J}$. Moreover,

$$\omega \in f_{\mathtt{J}} \cdot \Omega^1(\log\ C_{\mathtt{J}}) \backslash f_{\mathtt{I} \backslash \mathtt{J}} \cdot \Omega^1(\log\ C_{\mathtt{I} \backslash \mathtt{J}}) \quad \text{if and only if} \quad F^{\Lambda}_{\mathtt{I} \backslash \mathtt{J}}(\underline{v}(\overline{\omega})) = \infty,$$

that is $\underline{v}(\overline{\omega})$ has infinite fiber in $\Lambda = \Lambda_f$ (see Definition 2.2).

3.1 Infinite fibers of diagonal curve with two branches

In what follows, we will use the values set of logarithmic residues along C to obtain information on the set Λ, as the infinite fibers and its conductor, when C is defined by $f = f_1 f_2$ and it has two equisingular branches with values semigroup $S(C_1) = S(C_2) = \langle \overline{\beta}_0, \dots, \overline{\beta}_g \rangle$ and intersection multiplicity $I = [f_1, f_2]_0 > n_g \overline{\beta}_g$ that we call *diagonal curve with two branches*.

Remark 3.2 A complete description of the infinite fibers of a semigroup with two equisingular branches was given by Bayer [5].The semigroup of values of a diagonal curve with any number of branches was completely described by Delgado de la Mata [19], as well as provided the name *diagonal* to this family.

Remark 3.3 Diagonal curves with two branches can be described in the following equivalent way: it is a curve $C : f = f_1 f_2 = 0$ where both branches have the same multiplicity at the origin and $I = [f_1, f_2]_0 > n_g \overline{\beta}_g$ where $\langle \bar{\beta}_0, \dots, \bar{\beta}_g \rangle$ is the semigroup of values of one of them. Clearly, if $I \geq n_g \overline{\beta}_g$ and both branches are equisingular then they have same semigroup of values and consequently, they have the same multiplicity. On the other hand, assume f_1, f_2 are two branches with the same multiplicity say $\overline{\beta}_0$, one of them with semigroup of values $\langle \overline{\beta}_0, \dots, \overline{\beta}_g \rangle$ and $I = [f_1, f_2]_0 \geq n_g \overline{\beta}_g$. Let us first assume that f_1, f_2 are transversal, i.e. they don't have the same tangent. In this case, we have $I = \overline{\beta}_0^2$. Now, by definition of the generators of the semigroup (see Eq. (2.2)) we have $\overline{\beta}_{i+1} = n_i \overline{\beta}_i + (\beta_{i+1} - \beta_i)$, where β_i denotes the Puiseux characteristic exponents. A straightforward induction on this formula together with the fact that $\overline{\beta}_1 > \overline{\beta}_0$ leads to prove $n_g \overline{\beta}_g > \overline{\beta}_0^2$ against the hypothesis. Thus, the two branches must have the same tangent. If both branches have the same tangent with the required condition on the intersection multiplicity then a straightforward argument with Noether's formula shows that both have the same value semigroup.

Before that, let us recall some important fact about irreducible curves. Let C be an irreducible plane curve with values semigroup $S = \langle \overline{\beta}_0, \dots, \overline{\beta}_g \rangle$. Without loss of generality, we can assume that C is defined by a Weierstrass polynomial $f \in \mathbb{C}\{x\}[y]$ with $\deg_y(f) = \beta_0$ and Newton–Puiseux parametrization φ as (2.1).

Given $n \in \mathbb{N} \setminus \{0\}$ we consider the $\mathbb{C}$-vector space

$$P_n = \{h \in \mathbb{C}\{x\}[y];\ \deg_y(h) < n\}. \tag{3.3}$$

Since $f \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial with $\deg_y(f) = \beta_0$, any $H \in \mathbb{C}\{x, y\}$ can be uniquely expressed (by the Weierstrass division theorem) as $H = qf + h$ with $q \in \mathbb{C}\{x, y\}$ and $h \in P_{\beta_0}$. So, $\mathbb{C}\{x, y\} = P_{\beta_0} \oplus f \cdot \mathbb{C}\{x, y\}$ and the classes of H and h in $\mathscr{O} = \frac{\mathbb{C}\{x,y\}}{(f)}$ are equal.

If v_f indicates the discrete valuation associated to C then we put $v_f(H) := v_f(\overline{H})$ where $\overline{H}$ denotes the class of $H \in \mathbb{C}\{x, y\}$ in $\mathscr{O}$. In this way, $S = v_f(\mathscr{O}\setminus\{0\}) = v_f(\mathbb{C}\{x, y\}\setminus(f)) = v_f(P_{\beta_0} \setminus \{0\})$. Following [11, Sec. 3], we introduce the following $\mathbb{C}$-vector spaces:

$$\mathscr{E}(f) = P_{\beta_0}\,\mathrm{d}x + P_{\beta_0-1}\,\mathrm{d}y, \quad \mathscr{G}(f) = \mathbb{C}\{x, y\}\,\mathrm{d}f + \mathbb{C}\{x, y\}\,f\,\mathrm{d}x,$$

where P_{β_0} is given in (3.3). With the above notation, we have [11, Lemma 3.3]

$$\Omega^1 = \mathscr{E}(f) \oplus \mathscr{G}(f).$$

Since $\mathscr{G}(f) \subset \mathscr{F}(f)$, if $\omega = \omega_0 + \omega_1 \in \Omega^1$ with $\omega_0 \in \mathscr{E}(f)$ and $\omega_1 \in \mathscr{G}(f)$ then the classes $\overline{\omega}$ and $\overline{\omega_0}$ of ω, respectively of ω_0, in Ω_f are equal and $v_f(\overline{\omega}) = v_f(\overline{\omega_0})$. In this way, we get $\Lambda_f = \{v_f(\overline{\omega});\ \omega \in \mathscr{E}(f)\}$. In what follows, to simplify the notation, given $\omega \in \Omega^1$ when we put $v_f(\omega)$ we understand $v_f(\overline{\omega})$ where $\overline{\omega}$ indicates the class of ω module $\mathscr{F}(f)$.

Let us now consider $\omega = A\,\mathrm{d}x - B\,\mathrm{d}y \in f \cdot \Omega(\log\ C_f)$. According to (3.1) we get $Af_y + Bf_x = Mf$ for some $M \in \mathbb{C}\{x, y\}$. In particular, the relations

$$f_x\omega = A\,\mathrm{d}f - fM\,\mathrm{d}y \quad \text{and} \quad f_y\omega = -B\,\mathrm{d}f + fM\,\mathrm{d}x, \tag{3.4}$$

allow us to compute the residue of ω as $\mathrm{res}(\omega) = \frac{\overline{A}}{\overline{f_x}} = \frac{\overline{B}}{\overline{f_y}}$. Thus, the value set of logarithmic residues along C_f can be done by

$$\begin{aligned} \Delta_f &= \{v_f(res(\omega)) = v_f(B) - v_f(f_y);\ \omega = Adx + Bdy \in f \cdot \Omega(\log\ C_f)\} \\ &= \{v_f(B) - (\mu + \overline{\beta}_0 - 1);\ Adx + Bdy \in f \cdot \Omega(\log\ C_f)\}, \end{aligned} \tag{3.5}$$

where $v_f(f_y) = \mu + \overline{\beta}_0 - 1$ (see (2.3)). In particular, as we are considering the irreducible case, the property (3.2) translates to the fact that

$$\lambda \in \Lambda_f \quad \Leftrightarrow \quad -\lambda \notin \Delta_f. \tag{3.6}$$

Proposition 3.4 [11, Proposition 3.7] *Let $\omega = A\,\mathrm{d}x - B\,\mathrm{d}y \in \mathscr{E}(f) \cap f \cdot \Omega^1(\log\ C_f)$, i.e. $Bf_x + Af_y = Mf$. Under the previous notation, we have*

$$\varphi^*(M) = \frac{e_k\overline{\beta}_{k+1}}{\overline{\beta}_0} b\,t^{v_f(M)} + (h.o.t.), \quad \varphi^*(B) = b\,t^{v_f(B)} + (h.o.t.),$$

with $b \in \mathbb{C}^*$ and $v_f(B) = v_f(M) + \overline{\beta}_0$, where $k = \max_{0\le i<g}\{i;\ e_i \nmid v_f(res(\omega))\}$.

In what follows, let $C_f = C_1 \cup C_2$ where C_1 and C_2 are two equisingular plane branches defined by Weierstrass polynomials $f_1, f_2 \in \mathbb{C}\{x\}[y]$ with value semigroup $S_1 = S_2 = \langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and intersection multiplicity $I := [f_1, f_2]_0 \ge n_g\overline{\beta}_g$, where $n_g = e_{g-1} = \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_{g-1})$. In particular, there exists a Newton–Puiseux parametrization $\varphi_i(t_i) = (t_i^{\overline{\beta}_0}, \sum_{k\ge\overline{\beta}_0} a_k^{(i)} t_i^k)$ for C_i and $i = 1, 2$ such that

$$\begin{aligned} a_k^{(1)} &= a_k^{(2)} \text{ for every } k \le \beta_g \text{ if } I > n_g\overline{\beta}_g \\ a_k^{(1)} &= a_k^{(2)} \text{ for every } k < \beta_g \text{ if } I = n_g\overline{\beta}_g. \end{aligned} \tag{3.7}$$

We would like to use Proposition 3.4 in the context of a plane curve with two equisingular branches with $I \ge n_g\overline{\beta}_g$. To do so, we need the following technical lemmas to generalize

Proposition 3.4 in this context. Let us recall a few facts regarding the maximal contact curves associated to a branch. For a fixed $i \in \{1, 2\}$, let say $i = 1$, any $h \in \mathbb{C}\{x, y\}$ satisfying $[f_1, h]_0 = \overline{\beta}_q$ will be called a maximal contact curve of genus $q - 1$ with f_1. In [31], Zariski considers a particular set of maximal contact curves D_q for $1 \leq q \leq g$ defined by a truncation of the Puiseux series of the curve C_1 as follows:

$$s_q^{(1)}(x) = \sum_{\substack{j \in (\beta_0) \\ \beta_0 \leq j < \beta_1}} a_j^{(1)} x^{j/\beta_0} + \cdots + \sum_{\substack{j \in (e_{q-1}) \\ \beta_{q-1} \leq j < \beta_q}} a_j^{(1)} x^{j/\beta_0}.$$

The minimal polynomial $\mathfrak{g}_q(x, y) \in \mathbb{C}[x, y]$ of $s_q^{(1)}(x)$ is given by

$$\mathfrak{g}_q(x, y) = \prod_{\epsilon^{\beta_0/e_{q-1}}=1} (y - s_q^{(1)}(\epsilon x)) \tag{3.8}$$

and the plane branch D_q given by $\mathfrak{g}_q$ has maximal contact with C_1. In addition, $\mathfrak{g}_q$ is monic with $\deg_y(\mathfrak{g}_q) = \frac{\overline{\beta_0}}{e_{q-1}}$. In what follows, we put $\mathfrak{g}_0 = x$. The branches D_q are called semiroots of C_1 and any element $h \in \mathbb{C}\{x\}[y]$ with $\deg_y(h) < \deg_y(f_1)$, that is, $h \in P_{\beta_0}$, admits a unique G–adic expansion (cf. [1, Chap. 1 and 3]) in terms of $G = \{\mathfrak{g}_0, \mathfrak{g}_1, \ldots, \mathfrak{g}_g\}$, that is,

$$h = \sum_{\alpha=(\alpha_0,\ldots,\alpha_g)\in\mathbb{N}^{g+1}} b_\alpha \mathfrak{g}_0^{\alpha_0} \mathfrak{g}_1^{\alpha_1} \cdots \mathfrak{g}_{g-1}^{\alpha_{g-1}} \mathfrak{g}_g^{\alpha_g}, \tag{3.9}$$

with $b_\alpha \in \mathbb{C}$ and

(1) $0 \leq \alpha_k < n_k = \frac{e_{k-1}}{e_k}$ for $1 \leq k \leq g$.
(2) If we denote $h_\alpha = \mathfrak{g}_0^{\alpha_0} \mathfrak{g}_1^{\alpha_1} \cdots \mathfrak{g}_{g-1}^{\alpha_{g-1}} \mathfrak{g}_g^{\alpha_g}$, then $v_1(h_\alpha) \neq v_1(h_\gamma)$ for $\alpha \neq \gamma$.

Remark 3.5 Notice that if $h \in \mathbb{C}\{x\}[y]$ is given as (3.9), that is

$$h = \sum_{\gamma=(\gamma_0,\ldots,\gamma_g)\in\mathbb{N}^{g+1}} b_\gamma \mathfrak{g}_0^{\gamma_0} \mathfrak{g}_1^{\gamma_1} \cdots \mathfrak{g}_{g-1}^{\gamma_{g-1}} \mathfrak{g}_g^{\gamma_g}$$

and $v_1(h) = \alpha_0\overline{\beta}_0 + \cdots + \alpha_q\overline{\beta}_q$ with $\alpha_q \neq 0$ for $q \leq g$ and $0 \leq \alpha_k < n_k$ for $1 \leq k \leq q$, that is, $e_q = \max_{1\leq i\leq g}\{e_i : e_i \mid v_1(B)\}$, then

$$\varphi_1^*(h) = \varphi_1^*(b_\alpha \mathfrak{g}_0^{\alpha_0} \cdots \mathfrak{g}_q^{\alpha_q}) + \text{(h.o.t.)} = Q^h_{v_1(h)}(a^{(1)}) t^{v_f(h)} + \text{(h.o.t.)},$$

where $Q^h_{v_1(h)}(a^{(1)})$ is a polynomial in the coefficients of the Puiseux series of C_1. The polynomial $Q^h_{v_1(h)}(a^{(1)})$ can be explicitly computed and satisfies some useful properties related to the coefficients of the Puiseux series. We refer to [28, Sec. 1] for further details.

Remark 3.6 Notice that, by (3.7), C_1 and C_2 share the same maximal contact curves. In particular, given $H \in \mathbb{C}\{x\}[y]$ with $\deg_y(H) < \beta_0$ we get $v_1(H) = v_2(H)$. Moreover, if $H \in (x, y)$ then $v_1(\mathrm{d}H) = v_2(\mathrm{d}H) = v_1(H) = v_2(H)$ so $(\alpha, \alpha) \in \Lambda_{f_1 f_2}$ for every $0 \neq \alpha \in S_1 = S_2$.

Now we are ready to provide the generalization of Proposition 3.4.

Lemma 3.7 *Let C_1 and C_2 be two equisingular plane branches with semigroup $\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and intersection multiplicity $I \geq n_g\overline{\beta}_g$. If $\omega = A\,dx - B\,dy \in \mathscr{E}(f_1) \cap f_1 \cdot \Omega^1(\log C_1)$ with $B(f_1)_x + A(f_1)_y = Mf_1$, then we have*

$$\varphi_i^*(M) = \frac{e_k\overline{\beta}_{k+1}}{\overline{\beta}_0} b_i\, t_i^{v_1(B)-\overline{\beta}_0} + (h.o.t.), \quad \varphi_i^*(B) = b_i\, t_i^{v_1(B)} + (h.o.t.) \quad \textit{for} \quad i = 1, 2$$

where $k=\max\{l\,:\, e_l\nmid(v_1(B)-v_1((f_1)_y))\}$*. Moreover,* $b_1=b_2$ *if* $I>n_g\overline{\beta}_g$ *or* $e_q\mid v_1(B)$ *for some* $q<g$*. Otherwise, each* b_i *depends on the coefficients* $a_l^{(i)}$ *with* $l\leq\beta_g$*.*

Proof From Proposition 3.4, we have the required expression for $i=1$.

Since $M,B\in\mathbb{C}\{x\}[y]$ with $\deg_y(M),\deg_y(B)<\beta_0$, by Remark 3.6, we get $v_2(B)=v_1(B)=\sum_{l=0}^{q}\alpha_l\overline{\beta}_l$ with $\alpha_q\neq 0$ and $v_2(M)=v_1(M)=v_1(B)-\overline{\beta}_0$. So, considering the G–expansion of M and B, according to Remark 3.5, for $i=1,2$ we have

$$\varphi_i^*(M)=\varphi_i^*(b_\alpha^M\mathfrak{g}_0^{\alpha_0-1}\mathfrak{g}_1^{\alpha_1}\cdots\mathfrak{g}_q^{\alpha_q})+(\text{h.o.t.})=Q^M_{v_1(M)}(a^{(i)})t_i^{v_1(B)-\overline{\beta}_0}+(\text{h.o.t.}),$$

$$\varphi_i^*(B)=\varphi_i^*(b_\alpha^B\mathfrak{g}_0^{\alpha_0}\mathfrak{g}_1^{\alpha_1}\cdots\mathfrak{g}_q^{\alpha_q})+(\text{h.o.t.})=b_it_i^{v_1(B)}+(\text{h.o.t.}).$$

As $\varphi_i^*(\mathfrak{g}_0)=t_i^{\overline{\beta}_0}$, independently of i, the coefficient of $t_i^{v_1(B)-\overline{\beta}_0}$ in $\varphi_i^*(\mathfrak{g}_0^{\alpha_0-1}\mathfrak{g}_1^{\alpha_1}\cdots\mathfrak{g}_q^{\alpha_q})$ and the coefficient of $t_i^{v_1(B)}$ in $\varphi_i^*(\mathfrak{g}_0^{\alpha_0}\mathfrak{g}_1^{\alpha_1}\cdots\mathfrak{g}_q^{\alpha_q})$ are the same, so

$$\frac{b_i}{b_\alpha^B}=\frac{Q^M_{v_1(M)}(a^{(i)})}{b_\alpha^M},\quad\text{that is}\quad Q^M_{v_1(M)}(a^{(i)})=\frac{b_\alpha^M}{b_\alpha^B}b_i=\frac{e_k\overline{\beta}_{k+1}}{\overline{\beta}_0}b_i,\tag{3.10}$$

where the last equality follows using $i=1$.

By [28, Lemma 1.7], $Q^M_{v_j(M)}(a^{(i)})$ and $b_i=Q^B_{v_j(B)}(a^{(i)})$ are non-zero homogeneous polynomials in the coefficients $a_k^{(i)}$ of $\varphi_i(t_i)$ with $k\leq\beta_q$. In this way, if $q\neq g$ or $I>n_g\overline{\beta}_g$, by (3.7), we get

$$b_2=b_1\ \text{ and }\ Q^M_{v_1(M)}(a^{(2)})=Q^M_{v_1(M)}(a^{(1)})=\frac{e_k\overline{\beta}_{k+1}}{\overline{\beta}_0}b_1,$$

that is, the coefficients are independent of the branch C_i. If $q=g$ and $I=n_g\overline{\beta}_g$ then $Q^M_{v_1(M)}(a^{(i)})$ and $Q^B_{v_1(B)}(a^{(i)})$ satisfy (3.10) and they depend on $a_l^{(i)}$ with $l\leq\beta_g$. □

By Remark 3.1, given $\alpha\in\Lambda_f$ with $f=f_1f_2$, to characterize the infinite fiber $F_i^\Lambda(\alpha)=\infty$ it is enough to characterize the values of differential forms in $f_j\cdot\Omega^1(\log C_j)$ with respect to f_i. Firstly, we will analyze the differential forms in $\mathscr{E}(f_j)\cap f_j\cdot\Omega^1(\log\ C_j)$. To do so, we will apply similar ideas to the ones in [11, Sec. 4].

Proposition 3.8 *Let* C_1 *and* C_2 *be two equisingular plane branches with semigroup* $\langle\overline{\beta}_0,\ldots,\overline{\beta}_g\rangle$ *and intersection multiplicity* I*. Let* $\omega=A\,\mathrm{d}x-B\,\mathrm{d}y\in\mathscr{E}(f_1)\cap f_1\cdot\Omega^1(\log\ C_1)$*. If*

(a) $I>n_g\overline{\beta}_g$ *or*
(b) $I=n_g\overline{\beta}_g$ *and* $e_{g-1}\mid\ (v_1(B)-(c+\overline{\beta}_0-1))$

then

$$\nu_2(\omega)=\nu_1(B)+I-(c+\overline{\beta}_0-1)=\nu_2(B)+I-(c+\overline{\beta}_0-1),\tag{3.11}$$

where c *denotes the conductor of* $\langle\overline{\beta}_0,\ldots,\overline{\beta}_g\rangle$*. In particular,* $\omega\notin f_2\cdot\Omega^1(\log\ C_2)$*.*

Proof Since $\omega=A\,\mathrm{d}x-B\,\mathrm{d}y\in\mathscr{E}(f_1)\cap f_1\cdot\Omega(\log\ C_1)$ there exists $M\in\mathbb{C}\{x\}[y]$ satisfying $B(f_1)_x+A(f_1)_y=Mf_1$ and, by (3.4),

$$(f_1)_y\omega=Mf_1\mathrm{d}x-B\mathrm{d}f_1.\tag{3.12}$$

According to Proposition 3.4 and Lemma 3.7, we have

$$\varphi_2^*(M) = \frac{e_k\overline{\beta}_{k+1}}{\overline{\beta}_0} b_2\, t_2^{v_1(B)-\overline{\beta}_0} + \text{(h.o.t.)}, \quad \varphi_2^*(B) = b_2\, t_2^{v_1(B)} + \text{(h.o.t.)},$$

where $k = \max\{l :\ e_l \nmid (v_1(B) - v_1((f_1)_y))\}$. In addition, we get

$$\varphi_2^*(f_1) = at_2^I + \text{(h.o.t.)}.$$

Thus,

$$\begin{aligned}\varphi_2^*((f_1)_y\omega) &= \varphi_2^*(Mf_1\mathrm{d}x - B\mathrm{d}f_1)\\ &= (ae_k\overline{\beta}_{k+1}b_2t_2^{v_1(B)+I-1} + \text{h.o.t.})(aIb_2t_2^{v_1(B)+I-1} + \text{h.o.t.}) \qquad (3.13)\\ &= (e_k\overline{\beta}_{k+1} - I)ab_2t_2^{v_1(B)+I-1} + \text{(h.o.t.)}.\end{aligned}$$

If $I > n_g\overline{\beta}_g = e_{g-1}\overline{\beta}_g$ then for all $0 \le k \le g-1$ we have $I \ne e_k\overline{\beta}_{k+1}$. If $I = n_g\overline{\beta}_g$ and $e_{g-1} \mid (\nu_1(B) - \nu_1((f_1)_y)$ then $k < g-1$ and thus $I = n_g\overline{\beta}_g = e_{g-1}\overline{\beta}_g \ne e_k\overline{\beta}_{k+1}$. So, for the condition *a*) or *b*) we get $v_2((f_1)_y\omega) = v_1(B) + I$.

Since we have $B, (f_1)_y \in \mathbb{C}\{x\}[y]$ with $\deg_y(B), \deg_y((f_1)_y) < \beta_0$, it follows by Remark 3.6 and (2.3) that

$v_2(B) = v_1(B)$ and $v_2((f_1)_y) = v_1((f_1)_y) = c + \overline{\beta}_0 - 1$. So, $v_2(\omega) = v_1(B) + I - (c - \overline{\beta}_0 - 1) = v_2(B) + I - (c - \overline{\beta}_0 - 1)$. □

Let us now focus on the case of a reduced plane curve singularity with two equisingular branches and $I = [f_1, f_2]_0 > n_g\overline{\beta}_g$. Recall that given $\omega = A\mathrm{d}x - B\mathrm{d}y \in \Omega^1$, its *Weierstrass 1-form* (see [20]) of ω with respect to f_1 is given as

$$\omega = \omega_0 + \omega_1 \qquad (3.14)$$

with $\omega_0 = Pf_1\mathrm{d}x - Q\mathrm{d}f_1 \in \mathcal{G}(f_1)$ and $\omega_1 = A_1\mathrm{d}x - B_1\mathrm{d}y \in \mathcal{E}(f_1)$.

Remark 3.9 For $i, j \in \{1, 2\}$, $i \ne j$, given $\omega \in \mathcal{G}(f_j) = \mathbb{C}\{x, y\}f_j\mathrm{d}x + \mathbb{C}\{x, y\}\mathrm{d}f_j \subset f_j \cdot \Omega^1(\log\ C_j)$ we get $v_i(\omega) \ge I = [f_i, f_j]_0 = v_i(f_j) = v_i(\mathrm{d}f_j)$. In addition we have that

$$I + S(C_i) \subset v_i(\mathcal{G}(f_j)) \subset v_i(f_j \cdot \Omega^1(\log\ C_j)).$$

In fact, given $\gamma \in S(C_i)$ we consider $h \in \mathbb{C}\{x, y\}$ such that $v_i(h) = \gamma$. Since $h\mathrm{d}f_j \in \mathcal{G}(f_j) \subset f_j \cdot \Omega^1(\log\ C_j)$, we get $v_i(\mathrm{h}df_j) = \gamma + I \in v_i(\mathcal{G}(f_j)) \subset v_i(f_j \cdot \Omega^1(\log\ C_j))$.

Notice that Remark 3.9 implies that in order to compute the infinite fibers for which we need to work a bit we only need to look at those $\omega \in f_j \cdot \Omega^1(\log\ C_j)$ for which $v_i(\omega) \le I + c$.

Remark 3.10 Given $\omega = A\mathrm{d}x - B\mathrm{d}y \in f_1 \cdot \Omega^1(\log\ C_1)$ we consider its Weierstrass 1-form with respect to f_1, that is $\omega = \omega_0 + \omega_1$ with $\omega_0 \in \mathcal{G}(f_1)$ and $\omega_1 \in \mathcal{E}(f_1)$. Since $res(\omega) = res(\omega_0) + res(\omega_1)$, we have $v_1(res(\omega)) \ge \min\{v_1(res(\omega_0)), v_1(res(\omega_1))\}$. In this way if $v_1(res(\omega)) < 0$ then, since $v_1(res(\omega_0)) \ge 0$ by construction, we get $v_1(res(\omega)) = v_1(res(\omega_1))$.

Lemma 3.11 *Suppose that $I > n_g\overline{\beta}_g$. Given $\omega = \omega_0 + \omega_1 \in f_1 \cdot \Omega^1(\log\ C_1)$ as before.*

If $v_1(res(\omega)) < 0$ then $v_2(\omega) = v_2(\omega_1) = I + v_2(res(\omega)) = I + v_2(res(\omega_1)) = I + v_1(res(\omega_1))$.

Proof Given $\omega = A\mathrm{d}x - B\mathrm{d}y \in f_1 \cdot \Omega^1(\log C_1)$ we consider its Weierstrass 1-form with respect to f_1, that is $\omega = \omega_0 + \omega_1$ with $\omega_0 \in \mathscr{G}(f_1)$ and $\omega_1 := A_1\mathrm{d}x - B_1\mathrm{d}y \in \mathscr{E}(f_1)$. By Remark 3.10 we get $v_1(res(\omega)) = v_1(res(\omega_1))$ where $res(\omega_1) = B_1/(f_1)_y$.

By hypothesis $I > n_g\overline{\beta}_g$, then since $deg_y(B_1), deg_y((f_1)_y) < \overline{\beta}_0$, by Remark 3.6, it follows that

$$v_2(res(\omega_1)) = v_2(B_1) - v_2((f_1)_y) = v_1(B_1) - v_1((f_1)_y) = v_1(res(\omega_1))$$

and by Proposition 3.8, we get $v_2(\omega_1) = I + v_2(res(\omega_1)) = I + v_1(res(\omega)) < I$.

On the other hand, $v_2(\omega) \geq \min\{v_2(\omega_0), v_2(\omega_1)\}$. By Remark 3.9, $v_2(\omega_0) \geq I$ and as $v_2(\omega_1) = I + v_2(res(\omega_1)) < I$ it follows that

$$v_2(\omega) = v_2(\omega_1) = I + v_2(res(\omega)) = I + v_2(res(\omega_1)) = I + v_1(res(\omega_1)).$$

□

In addition, we have the following

Lemma 3.12 *Suppose that* $I > n_g\overline{\beta}_g$*, then* $\mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1))) \leq I - \overline{\beta}_0 + 1$.

Proof It is sufficient to show that for any $k \in \mathbb{N}$ with $0 \leq k \leq \overline{\beta}_0 - 1$ we have

$$I - k + \overline{\beta}_0\mathbb{N} \subset v_2(f_1 \cdot \Omega^1(\log\ C_1)).$$

For $k = 0$ if we take $n \in \mathbb{N}$ then $x^n\mathrm{d}f_1 \in \mathscr{G}(f_1) \subset f_1 \cdot \Omega^1(\log\ C_1)$ and

$$I + n\overline{\beta}_0 = v_2(x^n\mathrm{d}f_1) \in v_2(f_1 \cdot \Omega^1(\log\ C_1)).$$

Now consider $k \in \mathbb{N}$ such that $0 < k \leq \overline{\beta}_0 - 1$. By [29, Prop. 3.21], we have that $\mathbf{c}(\Delta_{f_1}) = -\overline{\beta}_0 + 1$. Thus, we get $-k \in v_1(res(f_1 \cdot \Omega^1(\log\ C_1)))$ so, there exists $\omega \in f_1 \cdot \Omega^1(\log\ C_1)$ such that $v_1(res(\omega)) = -k < 0$. In this way, by Lemma 3.11, $v_2(\omega) = I + v_1(res(\omega)) = I - k$. Taking any $n \in \mathbb{N}$ we get

$$I - k + n\overline{\beta}_0 = v_2(x^n\omega) \in v_2(f_1 \cdot \Omega^1(\log\ C_1)).$$

Hence

$$I - \overline{\beta}_0 + 1 + \mathbb{N} = \bigcup_{k=0}^{\overline{\beta}_0 - 1}(I - k + \overline{\beta}_0\mathbb{N}) \subseteq v_2(f_1 \cdot \Omega^1(\log\ C_1))$$

and consequently, $\mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1))) \leq I - \overline{\beta}_0 + 1$. □

Now we are able to describe the infinite fibers of Λ_f where $f = f_1 f_2$ and $I > n_g\overline{\beta}_g$.

Theorem 3.13 *Suppose that* $I > n_g\overline{\beta}_g$*, then*

$$v_2(f_1 \cdot \Omega^1(\log\ C_1)) = I + \Delta_{f_1} \ \textit{and} \ \mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1))) = I - \overline{\beta}_0 + 1.$$

Proof Let $\omega \in f_1 \cdot \Omega^1(\log\ C_1)$ and $\omega = \omega_0 + \omega_1$ its Weierstrass 1-form with respect to f_1, that is $\omega_0 \in \mathscr{G}(f_1) \cap f_1 \cdot \Omega^1(\log\ C_1)$ and $\omega_1 = A_1\mathrm{d}x - B_1\mathrm{d}y \in \mathscr{E}(f_1) \cap f_1 \cdot \Omega^1(\log\ C_1)$.

We have that $v_2(\omega) \geq \min\{v_2(\omega_0), v_2(\omega_1)\}$.

If $v_2(\omega) \geq v_2(\omega_0)$, then by Remark 3.9 we get $v_2(\omega) \geq I$ and $v_2(\omega) \in I + \Delta_{f_1}$ because $\mathbf{c}(\Delta_{f_1}) = -\overline{\beta}_0 + 1$. If $v_2(\omega) = v_2(\omega_1)$, then by Proposition 3.8 we get $v_2(\omega_1) = I + v_1(B_1) - v_1((f_1)_y) \in I + \Delta_{f_1}$. Hence, $v_2(f_1 \cdot \Omega^1(\log\ C_1)) \subseteq I + \Delta_{f_1}$.

On the other hand, let us consider $I + \delta \in I + \Delta_{f_1}$, that is, $\delta \in \Delta_{f_1}$.

If $\delta \geq -\overline{\beta}_0 + 1$ then, by Lemma 3.12, there exists $\omega \in f_1 \cdot \Omega^1(\log\ C_1)$ such that $v_2(\omega) = I + \delta$.

If $-\overline{\beta}_0 + 1 > \delta \in \Delta_{f_1}$ then there exists $\omega \in f_1 \cdot \Omega^1(\log\ C_1)$ such that $\delta = v_1(res(\omega)) < 0$. So, by Lemma 3.11, we get

$$v_2(\omega) = v_2(\omega_1) = I + v_2(res(\omega_1)) = I + v_1(res(\omega_1)) = I + \delta.$$

In this way, $I + \Delta_{f_1} \subseteq v_2(f_1 \cdot \Omega^1(\log\ C_1))$. This conclude that $v_2(f_1 \cdot \Omega^1(\log\ C_1)) = I + \Delta_{f_1}$ and, since $\mathbf{c}(\Delta_{f_1}) = -\overline{\beta}_0 + 1$, it follows that $\mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1))) = I - \overline{\beta}_0 + 1$. □

As a consequence we obtain

Theorem 3.14 *Let $f = f_1 f_2$ such that C_1, C_2 are equisingular with values semigroup $\langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$, $I > n_g \overline{\beta}_g$ and Λ_f its value set of Kähler differentials. Then,*

$$\boldsymbol{c}(\Lambda_f) = (I - \overline{\beta}_0 + 1, I - \overline{\beta}_0 + 1) \tag{3.15}$$

In particular, $\boldsymbol{c}(\Lambda_f)$ is independent of the analytic type of each of the branches.

Proof Since $I > n_g \overline{\beta}_g$ it follows by (2.2) that $I - \overline{\beta}_0 > \mathbf{c}(S_1)$, by (2.3) and Remark 3.6 we get $(\delta, \delta) \in \Lambda_f$ for any $\delta \geq I - \overline{\beta}_0$. In addition, we have that $\overline{\beta}_0 = \min \Lambda_1 = \min \Lambda_2$, and Theorem 3.13 implies that $\mathbf{c}_{\Lambda_f} \leq (I - \overline{\beta}_0 + 1, I - \overline{\beta}_0 + 1)$. To prove the equality, it is enough to show that $(I - \overline{\beta}_0, I - \overline{\beta}_0) \in \Lambda_f$ is a maximal point (in fact an absolute maximal) of Λ_f.

By Theorem 3.13, we get $F_{\{i\}}^{\Lambda_f}(I - \overline{\beta}_0, I - \overline{\beta}_0) \neq \infty$ for $i \in \{1, 2\}$ (see Definition 2.2). Let us consider for example $i = 2$, as the other case follows similarly. As $F_{\{2\}}^{\Lambda_f}(I - \overline{\beta}_0, I - \overline{\beta}_0) \neq \infty$ then there exist $\alpha \in \overline{F}_{\{2\}}(\Lambda_f, (I - \overline{\beta}_0, I - \overline{\beta}_0))$ such that α is a maximal element of Λ_f. As the number of branches is $r = 2$ then the notion of maximal, relative maximal and absolute maximal agree. Hence $\alpha = (\alpha_1, I - \overline{\beta}_0)$ with $\alpha_1 \geq I - \overline{\beta}_0$ is an absolute maximal in Λ_f. Let us assume $\alpha \neq (I - \overline{\beta}_0, I - \overline{\beta}_0)$, then $\alpha = (I - \overline{\beta}_0 + n, I - \overline{\beta}_0)$ for some $n \in \mathbb{N} \setminus \{0\}$ is such that $F_{\{1\}}(\Lambda_f, \alpha) = F_{\{2\}}(\Lambda_f, \alpha) = \emptyset$ and $\alpha_1 \in I + \Delta_2$ since $\mathbf{c}(\Delta_2) = -\overline{\beta}_0 + 1$, but this is a contradiction with Theorem 3.13. □

Remark 3.15 A natural problem is to determine $\mathbf{c}(\Lambda_f)$ for $f = \prod_{i=1}^{r} f_i$ where the branches C_i are equisingular with semigroup $\langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ and satisfy $I_{i,j} > n_g \overline{\beta}_g$ for $1 \leq i < j \leq r$. A straightforward generalization of previous theorem would suggest that $\mathbf{c}(\Lambda_f) = (1 - \overline{\beta}_0 + \sum_{i \in \mathtt{I} \setminus \{1\}} I_{1,i}, \ldots, 1 - \overline{\beta}_0 + \sum_{i \in \mathtt{I} \setminus \{r\}} I_{r,i})$, but this turns out not to hold in general. Indeed, consider $f_1 = y$, $f_2 = y - x^n$, $f_3 = y - x^m$ with $1 < n < m$. In this case, $g = 0, n_0 = \overline{\beta}_0 = 1$ and $I_{1,3} = m > I_{1,2} = I_{2,3} = n > n_0 \overline{\beta}_0 = 1$. According to [13, Example 20], one has $\mathbf{c}(\Lambda_f) = (m+1, n+1, m+1) \neq (1 - \overline{\beta}_0 + I_{1,2} + I_{1,3}, 1 - \overline{\beta}_0 + I_{2,1} + I_{2,3}, 1 - \overline{\beta}_0 + I_{3,1} + I_{3,2})$.

Theorem 3.14 allows us to compute the invariant Θ_2 described in Sect 2 for the fractional ideal $J = \frac{\Omega_f}{Tor(\Omega_f)} \simeq \varphi^*(\Omega_f)$ (see 2.8) whose values set is $E = \Lambda_f$. In fact, according to Remark 3.1 we get $\omega \in f_1 \cdot \Omega^1(\log\ C_1)$ if and only if $v_1(\overline{\omega}) = \infty$. So, we have that $\mathscr{N}_1(J) = f_1 \cdot \Omega^1(\log\ C_1)$ and, by (2.7) we get $\Theta_2 = \sharp(\Lambda_2 \setminus v_2(f_1 \cdot \Omega^1(\log\ C_1)))$.

Corollary 3.16 *With the previous notation we have that*

$$\Theta_2 = I - \beta_0 + 1 - \sharp \mathbb{N} \setminus \Lambda_2 - \sharp\{\lambda > \beta_0 : \lambda \notin \Lambda_1\}.$$

Proof By Theorem 3.13 we have that $v_2(f_1 \cdot \Omega^1(\log\ C_1)) = I + \Delta_{f_1}$ and $\mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1))) = I - \beta_0 + 1$. Since $\mathbf{c}(\Lambda_2) \leq \mathbf{c}(v_2(f_1 \cdot \Omega^1(\log\ C_1)))$ we have

$$\begin{aligned}\Theta_2 &= \sharp(\Lambda_2 \setminus v_2(f_1 \cdot \Omega^1(\log\ C_1)))\\ &= \sharp\{\alpha \in \Lambda_2 :\ \alpha < I - \beta_0 + 1\} - \sharp\{\delta \in I + \Delta_{f_1} :\ \delta < I - \beta_0 + 1\}\\ &= I - \beta_0 + 1 - \sharp\mathbb{N} \setminus \Lambda_2 - \sharp\{\gamma \in \Delta_{f_1} : \gamma < -\beta_0 + 1\}.\end{aligned}$$

It follows, by (3.6), that $\Theta_2 = I - \beta_0 + 1 - \sharp\mathbb{N}\setminus\Lambda_2 - \sharp\{\lambda > \beta_0 :\ \lambda \notin \Lambda_1\}$. □

4 The Tjurina number for two branches

In Sect. 3 we have computed the conductor $\mathbf{c}(\Lambda)$ of the value set Λ of Kähler differentials for a plane curve $C = C_1 \cup C_2$ defined by $f = f_1 f_2$ such that C_1, C_2 are equisingular with values semigroup $\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and intersection multiplicity $I > n_g\overline{\beta}_g$. Also, in Theorem 2.10 we have shown that the Tjurina number of a reduced plane curve C with any number of branches can be computed in terms of the distance $d(\overline{\Lambda}\setminus S)$ between its values semigroup S and its set $\overline{\Lambda} = \Lambda \cup \{0\}$. In addition, in Sect. 2 we explained how to compute this distance if one knows the conductor of the values set. All this together allows us to provide explicit formulas for the Tjurina number in this case.

Theorem 4.1 *Let $C = C_1 \cup C_2$ be a plane curve such that C_1, C_2 are equisingular with values semigroup $S(C_i) = \langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$, conductor $\mathbf{c}(S(C_i)) = c$ and intersection multiplicity $I = [C_1, C_2]_0 > n_g\overline{\beta}_g$. Then, the Tjurina number of C is given by*

$$\tau = 2I + c - 1.$$

In particular, τ is constant in the equisingularity class of C.

Proof Let S be the values semigroup and Λ be the values set of Kähler differentials of C. By Theorem 2.10 and Remark 2.1 we get

$$\tau = \mu - d(\overline{\Lambda} \setminus S) = 2c + 2I - 1 - d(\overline{\Lambda} \setminus S). \tag{4.1}$$

Since the values semigroup S of C is such that $S \subseteq \overline{\Lambda} = \Lambda \cup \{(0,0)\}$ and $m_\Lambda = (\beta_0, \beta_0)$ we get

$$d(\overline{\Lambda} \setminus S) = d_{\overline{\Lambda}}((0,0), \mathbf{c}(S)) - d_S((0,0), \mathbf{c}(S)) = d_\Lambda((\beta_0, \beta_0), \mathbf{c}(S)) + 1 - (c + I), \tag{4.2}$$

where $d_S((0,0), \mathbf{c}(S)) = \delta(C) = c + I$ (see Example 2.8).

By Theorem 3.14 we have $\mathbf{c}(\Lambda) = (I - \beta_0 + 1, I - \beta_0 + 1) \leq (I + c, I + c) = \mathbf{c}(S)$, so we get

$$d_\Lambda((\beta_0, \beta_0), \mathbf{c}(S)) = d_\Lambda((\beta_0, \beta_0), \mathbf{c}(\Lambda)) + d_\Lambda(\mathbf{c}(\Lambda), \mathbf{c}(S)). \tag{4.3}$$

If $J = \varphi^*(\Omega_f)$ then, by Theorem 2.7, we have that

$$d_\Lambda((\beta_0, \beta_0), \mathbf{c}(\Lambda)) = l\left(\frac{J}{J(\mathbf{c}(\Lambda))}\right) = \sum_{i=1}^{2}(I - \beta_0 + 1 - \beta_0 - \sharp((\mathbb{N} + \beta_0)\setminus\Lambda_i) - \Theta_i),$$

where $\Theta_1 = 0$ and, by Corollary 3.16, $\Theta_2 = I - \beta_0 + 1 - \sharp\mathbb{N} \setminus \Lambda_2 - \sharp\{\lambda > \beta_0 :\ \lambda \notin \Lambda_1\}$.

Since, $\sharp\mathbb{N} \setminus \Lambda_2 = \beta_0 + \sharp((\mathbb{N} + \beta_0) \setminus \Lambda_2)$ and $(\mathbb{N} + \beta_0) \setminus \Lambda_1 = \{\lambda > \beta_0 :\ \lambda \notin \Lambda_1\}$ we get

$$d_\Lambda((\beta_0, \beta_0), \mathbf{c}(\Lambda)) = I - 2\beta_0 + 1. \tag{4.4}$$

Considering $J_c = \{\omega \in \varphi^*(\Omega_f) : \underline{v}(\omega) \geq \mathbf{c}(\Lambda)\}$ then $\underline{v}(J_c) = (I-\beta_0+1, I-\beta_1+1)+\mathbb{N}^2$ and, by Theorem 2.7, we have that

$$d_\Lambda(\mathbf{c}(\Lambda), \mathbf{c}(S)) = l\left(\frac{J_c}{J_c(\mathbf{c}(S))}\right) = 2(I + c - (I - \beta_0 + 1)) = 2(c + \beta_0 - 1). \qquad (4.5)$$

The expressions (4.4) and (4.5) give us, by (4.3), that $d_\Lambda((\beta_0, \beta_0), \mathbf{c}(S)) = I + 2c - 1$. So, by (4.2) we have $d(\overline{\Lambda} \setminus S) = c$ and, consequently by (4.1) we have $\tau = 2I + c - 1$. □

Notice that, by the previous result, we get $\mu - \tau = c = \mu_1 = \mu_2$ for a plane curve $C = C_1 \cup C_2$ with C_1 and C_2 equisingular with values semigroup $S(C_i) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ and intersection multiplicity $I > n_g\overline{\beta}_g$ where $\mu_i = c$ is the Milnor number of C_i.

Example 4.2 Let us consider

$$f = y^6 - 3x^3y^4 - 2x^5y^3 + 3x^6y^2 - 6x^8y - x^9 + x^{10}$$
$$g = y^6 - 3x^3y^4 + 4x^5y^3 + \left(3x^6 - \tfrac{3}{2}x^7\right)y^2 - \left(12x^8 + \tfrac{3}{8}x^9\right)y - x^9 + \tfrac{13}{2}x^{10} - \tfrac{1}{64}x^{11}.$$
$$\begin{aligned} h = {} & y^6 - 3y^4x^3 + \left(-\tfrac{6}{19}x^6 + 4x^5\right)y^3 + \left(-\tfrac{3}{27436}x^{10} + \tfrac{9}{76}x^8 - \tfrac{3}{2}x^7 + 3x^6\right)y^2 + \\ & + \left(-\tfrac{3}{1042568}x^{13} + \tfrac{3}{13718}x^{12} - \tfrac{9}{1444}x^{11} + \tfrac{3}{38}x^{10} - \tfrac{9}{152}x^9 - 12x^8\right)y - \tfrac{1}{3010936384}x^{17} \\ & + \tfrac{3}{79235168}x^{16} - \tfrac{15}{8340544}x^{15} + \tfrac{5}{109744}x^{14} - \tfrac{301}{438976}x^{13} + \tfrac{153}{11552}x^{12} - \tfrac{547}{1216}x^{11} + \tfrac{13}{2}x^{10} - x^9 \end{aligned}$$

We have that f, g and h are equisingular plane branches sharing the semigroup $S = \langle 6, 9, 19 \rangle$ with conductor $c = 42$ and $I(g,h) = 63 > I(f,h) = I(f,g) = 58 > n_2\overline{\beta}_2 = 57$.

In this case, using the SINGULAR software [16], we get $\tau(f) = 35$, $\tau(g) = 36$, $\tau(h) = 37$,

$$\tau(fg) = 2I(f,g) + c - 1 = 157 = 2I(f,h) + c - 1 = \tau(f,h) \text{ and } \tau(gh) = 169 = 2I(g,h) + c - 1.$$

that illustrate Theorem 4.1.

In [3], the inequality $\mu/\tau < 4/3$ was showed for any plane curve. The previous results allows to provide a new proof of that inequality in the case of a plane curve with two equisingular branches with values semigroup $S(C_i) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ and $I > n_g\overline{\beta}_g$.

Corollary 4.3 *Let $C = C_1 \cup C_2$ be a plane curve defined such that C_1, C_2 are equisingular with values semigroup $S(C_i) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$ and intersection multiplicity $I > n_g\overline{\beta}_g$. Then, $\mu/\tau < 4/3$.*

Proof From Theorem 4.1 we have $4\tau - 3\mu = 2I - 2c - 1$ where $c = \mu_i$ is the Milnor number of the branch C_i. By hypothesis $I > n_g\overline{\beta}_g > c$, then the results follows. □

4.1 Remarks on the minimal Tjurina number in more general cases

To conclude, we draw attention to some challenges concerning the Tjurina number in a more general setting.

First, Theorem 4.1 establishes that for a curve with two equisingular branches with values semigroup $S(C_i) = \langle \overline{\beta}_0, \ldots, \overline{\beta}_g \rangle$, the condition $I > n_g\overline{\beta}_g$ is sufficient to guarantee a constant Tjurina number within the equisingularity class. It is natural to ask up to what extent this holds in the case of a curve with more than two branches. The following example shows that if $C = \cup_{i=1}^r C_i$ is a curve with $r \geq 3$ branches with values semigroup $S(C_i) =$

$\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$, then condition $I_{i,j} := [C_i, C_j]_0 > n_g\overline{\beta}_g$ is not enough to have constant Tjurina number in the whole equisingularity class.

Example 4.4 Consider $f_1 = y^5 - x^8 + 2x^5y^2$, $f_2 = y^5 - x^8 + 3x^5y^2$, $f_3 = y^5 - x^8 + x^4y^3$ and $f_4 = y^5 - x^8 + 7x^5y^2$. We have that $f = f_1f_2f_3$ and $g = f_1f_2f_4$ define two equisingular curves each one with three equisingular branches with values semigroup $\langle 5, 8\rangle$, all of them have intersection multiplicity $I_{i,j} = 41 > 40 = n_g\overline{\beta}_g$. A computation with SINGULAR [16] shows $\tau(f) = 258 \neq 261 = \tau(g)$.

It would be certainly good to obtain sufficient topological conditions for a curve with several branches to have constant Tjurina number in the equisingularity class. In [11, 27] some families of plane branches with constant Tjurina number in the equisingularity class are shown. One could think that for a curve with two branches, to have constant Tjurina number in the equisingularity class of each of the branches could be a sufficient condition to have constant Tjurina number in the equisingularity class of the curve. The following example shows that this is not enough.

Example 4.5 Let us consider the branches

$$f_1 = (y^5 - x^7)^2 - x^{10}y^3, \quad f_2 = (y^5 - x^7)^2 - 5x^{10}y^3 \quad \text{and} \quad f_3 = (y^5 - x^7 + x^4y^3)^2 - 3x^{10}y^3.$$

All branches are equisingular with semigroup $\langle 10, 14, 71\rangle$ and, by [11], for any branch in this equisingularity class the Tjurina number is constant $\tau(f_i) = 94$. Let us denote $f = f_1f_2$ and $g = f_2f_3$. In both cases $[f_i, f_j]_0 = 142 = 2\cdot 71 = n_g\overline{\beta}_g$. A calculation with SINGULAR [16] shows that $\tau(f) = 402 \neq 406 = \tau(g)$.

Following with Example 4.5, we observe that it is quite close to the curves considered in Theorem 4.1. The difficulty here relies on the remaining cases of (b) of Proposition 3.8, i.e. to compute values, with respect to f_2, of those differentials $\omega \in \mathscr{E}(f_1) \cap f_1 \cdot \Omega^1(\log\ C_1)$ such that $e_{g-1} \nmid res(\omega)$. In that cases, one can check that the initial term in Eq. (3.13) cancels and one need to impose some open conditions in order to guarantee the value of ω. A careful analysis of this situation leads us to think that in the case of a plane curve with two equisingular branches with values semigroup $\langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and $I = n_g\overline{\beta}_g$ there should exists an open Zariski set for which $v_2(f_1 \cdot \Omega^1(\log\ C_1)) = I + 1 + \Delta_{f_1}$. This leads us to propose the following conjecture.

Conjecture 4.6 *Let $C = C_1 \cup C_2$ be a plane curve with two equisingular branches with semigroup $S_1 = S_2 = \langle\overline{\beta}_0, \ldots, \overline{\beta}_g\rangle$ and $I = [C_1, C_2]_0 = n_g\overline{\beta}_g$. Denote by c the conductor of S_i. Then, the minimal Tjurina number in the equisingularity class of C is*

$$\tau_{min} = 2I + c.$$

In fact, Conjecture 4.6 is actually true for $S = \langle\overline{\beta}_0, \overline{\beta}_1\rangle$ as showed in [8, Tableu 3, δ pair] (see also [26]).

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