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The convolution of two differentiable functions on the circle need not be differentiable

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Abstract
The convolution operator is well-known for preserving the best properties of its parent functions, and is often presented as a "smoothing" operator. In the present result, we construct two differentiable functions whose convolution is not differentiable.

Keywords Convolution · Non differentiable function · Lineability · Spaceability · Algebrability

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1 Introduction

Let us denote, as usual, by $L^p[-1,1]$ the space of 2-periodic functions so that $\int_{-1}^{1} |f(x)|^p \, \mathrm{d}x < \infty$ and by $C^k[-1,1]$ the space of 2-periodic functions that are k times differentiable and whose kth derivative is continuous. In particular, $C^0[-1,1]$, or $C[-1,1]$ for simplicity, will denote the space of continuous 2-periodic functions and $C^\infty[-1,1]$ will denote the space of 2-periodic functions which are infinitely many times differentiable. We will also denote by $D[-1,1]$ the space of 2-periodic differentiable functions.

If $f, g \in L^1[-1,1]$, we define the *convolution* of f and g as the function

$$(f * g)(x) = \int_{-1}^{1} f(s)g(x-s)\, \mathrm{d}s.$$

The functions f and g that take part in the convolution are sometimes referred to as "parent" functions. This operator consists basically on the "*limit of averaging of the area between f and translations of g*" [4]. Because of this "averaging", some of the properties of the parent functions translate to their convolution (see, e.g., [3]):

Lemma 1.1 *Let $f, g \in L^1[-1,1]$.*

(1) *If $f \in C^k[-1,1]$ for some $0 \leq k \leq \infty$, then $f * g \in C^k[-1,1]$ (without any other assumption on g). Furthermore, the kth derivative of $f * g$ can be calculated via*

$$\frac{d^k}{dx^k}(f * g)(x) = \left(\frac{d^k f}{dx^k} * g \right)(x).$$

(2) *If f is L-Lipschitz (that is, $|f(x) - f(y)| \leq L|x - y|$ for every x, y), then $f * g$ is L-Lipschitz (without any other assumption on g).*

In the present paper we show that this regularity is not maintained for the property of differentiability. To prove this claim, we provide an example of two differentiable functions whose convolution is not differentiable at 0. This breaks the pattern that is maintained for continuous functions, Lipschitz functions (which are sometimes regarded as a "middle way between continuity and differentiability", [1, p. 10]) and differentiable functions with continuous derivative. In particular, we provide a counter example to a theorem announced by Folland in [2, Theorem 7.2], where it is stated that $f * g$ is differentiable whenever f is differentiable and the convolutions $f * g$ and $f' * g$ are well defined.

2 A property which is not preserved by the convolution operator

Theorem 2.1 *There exist differentiable functions f and g so that $f * g$ is not differentiable at 0. On the other hand, such functions can be chosen so that $(\frac{df}{dx} * g)(x)$ is well-defined for every x.*

Proof Let us define the functions f and g as follows: for each $i \geq 1$, divide the interval $\left[\frac{1}{2^i}, \frac{1}{2^{i-1}}\right]$ into 2^{7i} sub-intervals of the same length ($\frac{1}{2^{8i}}$). For each $k = 0, 1, \dots, 2^{7i-1} - 1$, consider $\varphi_{i,k}$ and $\psi_{i,k}$ two C^∞—hat functions so that

$$\begin{aligned}
\text{supp } \varphi_{i,k} &\subseteq \left(\frac{1}{2^i} + \frac{2k+1}{2^{8i}}, \frac{1}{2^i} + \frac{2k+2}{2^{8i}}\right), \\
\text{supp } \psi_{i,k} &\subseteq \left(\frac{1}{2^i} + \frac{2k}{2^{8i}}, \frac{1}{2^i} + \frac{2k+1}{2^{8i}}\right), \\
\varphi_{i,k}(x) &= 1 \text{ for } \frac{1}{2^i} + \frac{8k+5}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}} \text{ and} \\
\psi_{i,k}(x) &= 1 \text{ for } \frac{1}{2^i} + \frac{8k+1}{2^{8i+2}} \leq x \leq \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}}.
\end{aligned}$$

For $0 \leq x \leq 1$ define

$$\begin{aligned}
f(x) &= x^2 \sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}(x), \\
g(x) &= x^2 \sum_{j=1}^{\infty} \sum_{l=0}^{2^{7j-1}-1} \psi_{j,l}(x)
\end{aligned}$$

(notice that f and g are well-defined, since for each x all but at most one of the terms that appear in the expression of $f(x)$ and $g(x)$, respectively, are 0) (Fig. 1).

Extend f and g to $[-1, 0]$ by $f(x) = f(-x)$, $g(x) = g(-x)$ for $-1 \leq x < 0$.

Let us show that f is differentiable over $[-1, 1]$ (for which it will be enough to show differentiability for $0 < x < 1$ and that $f'_+(0) = 0$). Indeed, assume first $x \in \left(\frac{1}{2^{i_0}}, \frac{1}{2^{i_0-1}}\right)$ for some $i_0 \geq 1$. Then, since $f(y) = y^2 \sum_{k=0}^{2^{7i_0-1}-1} \varphi_{i_0,k}(y)$ for every $y \in \left(\frac{1}{2^{i_0}}, \frac{1}{2^{i_0-1}}\right)$, we obtain that f is differentiable at x.

If now $x = \frac{1}{2^{i_0}}$ for some $i_0 \geq 1$, we can see that we can consider $f(y) = y^2\varphi_{i_0+1, 2^{7i_0+6}-1}(y)$ for

$$y \in \left(\frac{1}{2^{i_0+1}} + \frac{2(2^{7(i_0+1)-1} - 1) + 1}{2^{8(i_0+1)}}, \frac{1}{2^{i_0}} + \frac{1}{2^{8i_0}}\right).$$

Finally, for $x = 0$, notice that

$$\lim_{x \to 0^+} \frac{f(x)}{x} = \lim_{x \to 0} x \sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \varphi_{i,k}(x) = 0.$$

Note that g is differentiable as well, using a similar argument. Also, $(\text{supp } f) \cap (\text{supp } g) = \varnothing$, so $(f * g)(0) = 0$.

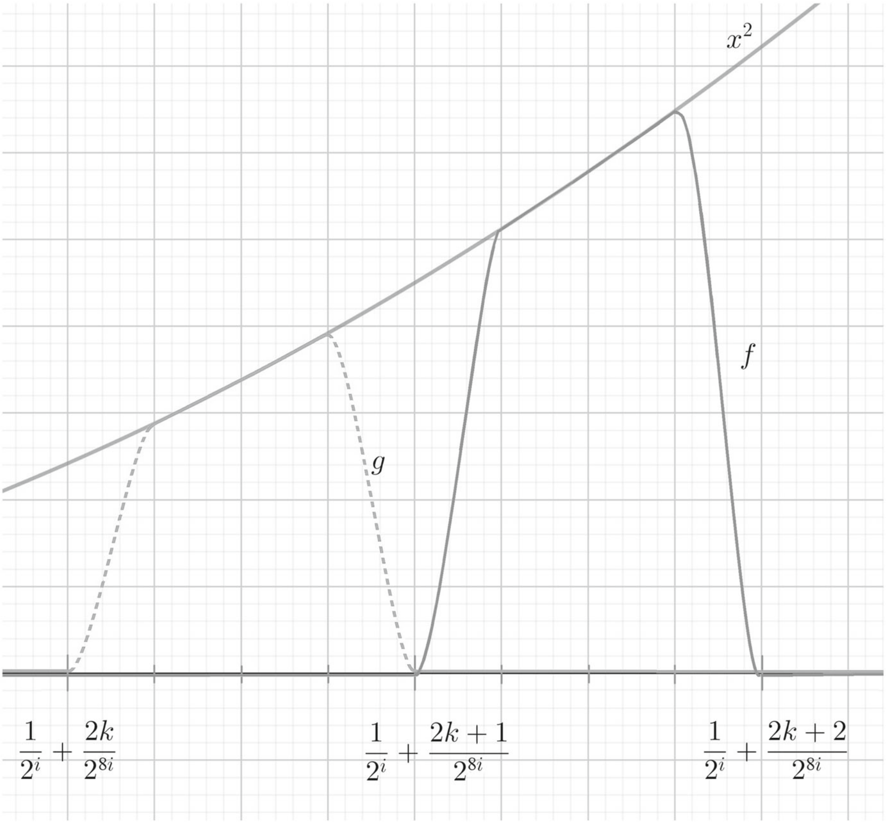

Fig. 1 Construction of the functions f and g

Furthermore, f and g are even functions, so $(f * g)(x)$ is an even function as well. Assume $f * g$ is differentiable on $[-1, 1]$. Then, $\frac{d}{dx}(f * g)$ is an odd function and therefore $\frac{d}{dx}(f * g)(0) = 0$.

On the other hand, for every $i \geq 1$,

$$
\begin{aligned}
(f * g)\left(\frac{1}{2^{8i}}\right) &= \int_{-1}^{1} f(s) g\left(\frac{1}{2^{8i}} - s\right) ds \\
&= \int_{-1}^{1} f(s) g\left(s - \frac{1}{2^{8i}}\right) ds \quad \text{(since } g \text{ is even)} \\
&\geq \int_{\frac{1}{2^i}}^{\frac{1}{2^{i-1}}} f(s) g\left(s - \frac{1}{2^{8i}}\right) ds \quad \text{(since } f, g \geq 0\text{)} \\
&\geq \sum_{k=0}^{2^{7i-1}-1} \int_{\frac{1}{2^i}+\frac{8k+5}{2^{8i+2}}}^{\frac{1}{2^i}+\frac{8k+7}{2^{8i+2}}} f(s) g\left(s - \frac{1}{2^{8i}}\right) ds.
\end{aligned}
$$

Now, notice that if

$$s \in \left[\frac{1}{2^i} + \frac{8k+5}{2^{8i+2}}, \frac{1}{2^i} + \frac{8k+7}{2^{8i+2}}\right],$$

we have that

$$s - \frac{1}{2^{8i}} \in \left[\frac{1}{2^i} + \frac{8k+1}{2^{8i+2}}, \frac{1}{2^i} + \frac{8k+3}{2^{8i+2}}\right],$$

so that

$$f(s) = s^2,$$
$$g\left(s - \frac{1}{2^{8i}}\right) = \left(s - \frac{1}{2^{8i}}\right)^2$$

and then

$$\begin{aligned}
\int_{\frac{1}{2^i}+\frac{8k+5}{2^{8i+2}}}^{\frac{1}{2^i}+\frac{8k+7}{2^{8i+2}}} f(s)g\left(s-\frac{1}{2^{8i}}\right) ds &= \int_{\frac{1}{2^i}+\frac{8k+5}{2^{8i+2}}}^{\frac{1}{2^i}+\frac{8k+7}{2^{8i+2}}} s^2 \left(s-\frac{1}{2^{8i}}\right)^2 ds \\
&\geq \int_{\frac{1}{2^i}+\frac{8k+5}{2^{4i+2}}}^{\frac{1}{2^i}+\frac{8k+7}{2^{4i+2}}} \left(s-\frac{1}{2^{8i}}\right)^4 ds \\
&= \frac{1}{5}\left[\left(\frac{1}{2^i}+\frac{8k+3}{2^{8i+2}}\right)^5 - \left(\frac{1}{2^i}+\frac{8k+1}{2^{8i+2}}\right)^5\right] \\
&= \frac{1}{5}\left[\left(\frac{1}{2^i}+\frac{8k+3}{2^{8i+2}}\right) - \left(\frac{1}{2^i}+\frac{8k+1}{2^{8i+2}}\right)\right] \\
&\quad \cdot \left[\left(\frac{1}{2^i}+\frac{8k+3}{2^{8i+2}}\right)^4 + \left(\frac{1}{2^i}+\frac{8k+3}{2^{8i+2}}\right)^3\left(\frac{1}{2^i}+\frac{8k+1}{2^{8i+2}}\right)\right. \\
&\quad \left. + \cdots + \left(\frac{1}{2^i}+\frac{8k+1}{2^{8i+2}}\right)^4\right] \\
&\geq \frac{1}{5}\cdot\frac{1}{2^{8i+1}}\cdot\frac{1}{2^{4i}}.
\end{aligned}$$

Therefore,

$$(f*g)\left(\frac{1}{2^{8i}}\right) \geq \sum_{k=0}^{2^{7i-1}-1} \frac{1}{5\cdot 2^{12i+1}} = \frac{1}{20\cdot 2^{5i}},$$

so that

$$\frac{(f*g)\left(\frac{1}{2^{8i}}\right)}{\frac{1}{2^{8i}}} \geq \frac{2^{3i}}{20} \xrightarrow[i\to\infty]{} \infty$$

and therefore $f*g$ can not be differentiable at 0.

Next, to simplify the notation, we shall write $a_{i,k} = \frac{1}{2^i} + \frac{2k+1}{2^{8i}},\ b_{i,k} = \frac{1}{2^i} + \frac{2k+2}{2^{8i}}$ for $i \geq 1,\ 0 \leq k \leq 2^{7i-1} - 1$ and $a^{j,l} = \frac{1}{2^j} + \frac{2l}{2^{8j}},\ b^{j,l} = \frac{1}{2^j} + \frac{2l+1}{2^{8j}}$ for $j \geq 1,\ 0 \leq l \leq 2^{7j-1} - 1$. Let $x \in [0,1]$.

If $x = 0$, we notice that supp $\frac{df}{dx} \cap$ supp $g = \varnothing$, so

$$\left(\frac{df}{dx} * g\right)(0) = \int_{-1}^{1} \frac{df}{dx}(s)g(s)\,ds = 0.$$

If $x \in (a_{i_0,k_0}, b_{i_0,k_0}]$ for some $i_0 \geq 1,\ 0 \leq k_0 \leq 2^{7i_0-1} - 1$, we can find $i_x \geq 1$ so that $x - s \in (a_{i_0,k_0}, b_{i_0,k_0}]$ for every $i \geq i_x,\ s \in \left(\frac{1}{2^i}, \frac{1}{2^{i-1}}\right)$. Therefore,

$$\begin{aligned}\left(\frac{df}{dx} * g\right)(x) &= \sum_{i=1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \int_{a_{i,k}}^{b_{i,k}} \frac{df}{dx}(s)g(x-s)\,ds \\ &= \sum_{i=1}^{i_x} \sum_{k=0}^{2^{7i-1}-1} \int_{a_{i,k}}^{b_{i,k}} \frac{df}{dx}(s)g(x-s)\,ds \\ &\quad + \sum_{i=i_x+1}^{\infty} \sum_{k=0}^{2^{7i-1}-1} \int_{a_{i,k}}^{b_{i,k}} \frac{df}{dx}(s)g(x-s)\,ds \\ &= \sum_{i=1}^{i_x} \sum_{k=0}^{2^{7i-1}-1} \int_{a_{i,k}}^{b_{i,k}} \frac{df}{dx}(s)g(x-s)\,ds.\end{aligned}$$

If now $x \in (a^{j,l}, b^{j,l}]$ for some $j \geq 1,\ 0 \leq l \leq 2^{7j-1} - 1$ we can find then $j_x \geq 1$ so that $x - s \in (a^{j,l}, b^{j,l}]$ for every $j \geq j_x,\ s \in \left(\frac{1}{2^j}, \frac{1}{2^{j-1}}\right)$. Therefore, using at this point the commutativity of the convolution,

$$\begin{aligned}\left(\frac{df}{dx} * g\right)(x) &= \sum_{j=1}^{\infty} \sum_{l=0}^{2^{7j-1}-1} \int_{a^{j,l}}^{b^{j,l}} \frac{df}{dx}(x-s)g(s)\,ds \\ &= \sum_{j=1}^{j_x} \sum_{l=0}^{2^{7j-1}-1} \int_{a^{j,l}}^{b^{j,l}} \frac{df}{dx}(x-s)g(s)\,ds\end{aligned}$$

$$+\sum_{j=j_x+1}^{\infty}\sum_{l=0}^{2^{7j-1}-1}\int_{a^{j,l}}^{b^{j,l}}\frac{df}{dx}(x-s)g(s)\,ds$$
$$=\sum_{j=1}^{j_x}\sum_{l=0}^{2^{7j-1}-1}\int_{a^{j,l}}^{b^{j,l}}\frac{df}{dx}(x-s)g(s)\,ds.$$

For $x\in[-1,0)$, we may just notice that $\frac{df}{dx}*g$ may be extended via $\left(\frac{df}{dx}*g\right)(x)=-\left(\frac{df}{dx}*g\right)(-x)$, since the convolution of an odd function and an even function is odd as well. □

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