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On the size of special families of linear operators

R.M. Aron [a,1], L. Bernal-González [b,2], P. Jiménez-Rodríguez [c,3], G.A. Muñoz-Fernández [d,3], J.B. Seoane-Sepúlveda [d,*,3]

[a] *Department of Mathematical Sciences, Kent State University, Kent, OH 44240, USA*
[b] *Departamento de Análisis Matemático, Universidad de Sevilla, Apdo. 1160, Avenida Reina Mercedes, Sevilla, 41080, Spain*
[c] *Department of Mathematical Sciences, Mathematics and Computer Science Building, Summit Street, Kent, OH 44242, USA*
[d] *Instituto de Matemática Interdisciplinar (IMI), Facultad de Ciencias Matemáticas, Plaza de Ciencias 3, Universidad Complutense de Madrid, Madrid, 28040, Spain*

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ABSTRACT

We continue the study, started in [12], of the search for algebraic structures one can find within the sets of injective linear functions. We shall focus on the cases when the operators are considered both on finite dimensional and infinite dimensional domains. We also study the set of continuous surjective linear operators.

* Corresponding author.
E-mail addresses: aron@math.kent.edu (R.M. Aron), lbernal@us.es (L. Bernal-González), pjimene1@kent.edu (P. Jiménez-Rodríguez), gustavo_fernandez@mat.ucm.es (G.A. Muñoz-Fernández), jseoane@ucm.es (J.B. Seoane-Sepúlveda).

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Lineability
Spaceability

1. Introduction and preliminaries

In [12], the authors were able to prove that it was not possible to find a vector space of dimension $n+1$ every nonzero element of which is an injective function $f : \mathbb{R}^n \to \mathbb{R}^n$, even though they could construct a vector space of dimension n of such functions. That generalized a result shown in [21] to arbitrary finite dimensions, and it carried out the problem giving some insights concerning infinite dimensions.

In this paper we will have a deeper study concerning the implications of the existence of a vector space contained in the set of injective linear functions between spaces of finite dimensions (plus the zero function). We will also address the case for infinite dimensions and we will introduce the analogous problem for the case of surjective linear functions.

The search of linear structures inside of sets has been a very fruitful field in Mathematics, and in the beginning of the 21st century, some terminology was introduced in an attempt to formalize this idea. More concretely we have the following definition (see, e.g., [1,2,5,21]):

If X is a vector space, μ is a cardinal number and $A \subset X$, then A is said to be *lineable* if there is an infinite dimensional vector space M such that $M \setminus \{0\} \subset A$, and μ*-lineable* if there exists a vector space M with $\dim(M) = \mu$ and $M \setminus \{0\} \subset A$. If, in addition, X is a topological vector space, then A is said to be *spaceable* in X whenever there is a closed infinite dimensional vector subspace M of X satisfying $M \setminus \{0\} \subset A$.

Up to now, a great number of cases have been studied (giving even optimal results, if we want to talk about maximal dimension or cardinality). The monographs [1,5] provide a broad range of the results obtained in this topic. Nevertheless, the amount of examples showing the impossibility of finding large structures where their elements fulfill a certain property (not necessarily pathological) is very scarce. One such example was presented in [12], and in Section 2 we will take further steps towards resolving some of the open questions that were posed in the mentioned article (see also [14]).

In Section 3 we will also consider the case for infinite dimensions. For that we will use the notion of Schauder and monotone basis of a Banach space, whose definition we review below (see, e.g., [3]):

Definition 1.1. Let X be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\{e_n\}_{n=1}^{\infty} \subset X$. We say that $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis of X if for every $x \in X$ one can find unique elements $\{a_i\}_{i=1}^{\infty} \subseteq \mathbb{K}$ so that

$$x = \sum_{i=1}^{\infty} a_i e_i.$$

If $\{e_n\}_{n=1}^{\infty} \subseteq X$ is a Schauder basis of X, it is known ([10]) that one can find a constant $K \geq 1$ so that, for every $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{K}$ and k, $l \in \mathbb{N}$,

$$\Big\| \sum_{i=1}^{k} a_i e_i \Big\| \leq K \Big\| \sum_{i=1}^{k+l} a_i e_i \Big\|.$$

If such a constant can be found equal to 1, that Schauder basis is said to be monotone.

In Section 4 we study whether the same conclusions as in Section 3 could be obtained if we focus on surjectivity. The approach that we will take is related with the problem of finding solutions of systems of infinite linear equations in an infinite number of variables. This area has huge applications in applied mathematics, and for example those kinds of systems with a Vandermonde type matrix of coefficients can be found in the study of plasma physics ([7]) or in the General Relativity Theory of Gravitation ([11]). The search for solutions to such systems already appeared in the work of Fourier (see, for example, [19]) and the reader might find it interesting to take a look at the results in [9,17,18,22,23].

To construct the corresponding structures in this section, we will use the results from [6], where the authors carried out and simplified the conclusions by E. Schmidt ([20]) and G. Kowalewski ([13]). Those results are gathered in Notation 1.2 and Theorem 1.3 below.

Notation 1.2. *Let us consider an infinite system of equations*

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_n x = b_n, \\ \vdots \end{cases}$$

where $\alpha_n = (a_{1,n}, a_{2,n}, a_{3,n}, \ldots)$ is a sequence of scalars (real or complex) of finite ℓ_2 norm, x is the sequence of variables, b is the vector of non-homogeneous coefficients, the system $\{\alpha_n\}_{n=1}^{\infty}$ is linearly independent and, given a pair of infinite sequences $v = \{v_i\}_{i=1}^{\infty}$ and $w = \{w_i\}_{i=1}^{\infty}$, we are using the notation $vw = \sum_{i=1}^{\infty} v_i w_i$. We will define the following elements associated to such a system:

$$B_1 = b_1,$$

$$B_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{j-1} & b_1 \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_{j-1} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_j\bar{\alpha}_1 & \alpha_j\bar{\alpha}_2 & \cdots & \alpha_j\bar{\alpha}_{j-1} & b_j \end{vmatrix}, j \geq 2,$$

$$G_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_j \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_j \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j\bar{\alpha}_1 & \alpha_j\bar{\alpha}_2 & \cdots & \alpha_j\bar{\alpha}_j \end{vmatrix}, j \geq 1,$$

$$\varphi_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_j \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_j \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{j-1}\bar{\alpha}_1 & \alpha_{j-1}\bar{\alpha}_2 & \cdots & \alpha_{j-1}\bar{\alpha}_j \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_j \end{vmatrix}, j \geq 1,$$

where we are denoting, for a vector $v = (v_1, v_2, v_3, \ldots)$, $\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \ldots)$.

Theorem 1.3. *Let us consider the finite system of equations*

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_m x = b_m, \end{cases}$$

where the infinite sequences $\alpha_i = (\alpha_{i,1}, \alpha_{i,2}, \ldots)$ *satisfy* $\sum_{j=1}^{\infty} |\alpha_{i,j}|^2 < \infty$ *and they are linearly independent.*

Then, a solution is given by

$$x = \sum_{j=1}^{m} \frac{B_j}{G_{j-1}G_j}\varphi_j.$$

We remark that the element $\sum_{j=1}^{m} \frac{B_j}{G_{j-1}G_j}\varphi_j$ is well-defined, since $G_j \neq 0$ for every $j \in \mathbb{N}$. Indeed, if we had $G_j = 0$ for some $j \in \mathbb{N}$, then we would have that the rows of the corresponding matrix are linearly dependent, so in particular we could find $(\lambda_1, \ldots, \lambda_j) \in \mathbb{K}^j \setminus \{0\}$ so that

$$\sum_{i=1}^{j} \lambda_i \alpha_i \overline{\alpha}_1 = 0 \quad (1)$$

$$\sum_{i=1}^{j} \lambda_i \alpha_i \overline{\alpha}_2 = 0 \quad (2)$$

$$\vdots$$

$$\sum_{i=1}^{j} \lambda_i \alpha_i \overline{\alpha}_j = 0 \quad (j).$$

Considering the equation obtained by $\overline{\lambda}_1(1)+\ldots+\overline{\lambda}_j(j)$ we conclude that $\left|\sum_{i=1}^{j}\lambda_i\alpha_i\right| = 0$, contradicting the hypothesis that the sequences of coefficients α_i are linearly independent.

2. Results for finite dimensions

We start this section by remarking that, even though the problem of the lineability of the set of injective functions might seem innocent at first sight, in fact it entails a very deep question in analytic geometry. Let us recall the characterization of the lineability of the set of isomorphisms provided in [12] (Proposition 2.7):

Proposition 2.1. *Let n and l be two natural numbers.*

(1) *If $k \geq 1$ and the set $\{f \in C^k(\mathbb{R}^n, \mathbb{R}^n) : f$ is injective$\}$ is l-lineable, then there exist injective functions $f_1, \ldots, f_l \in C^k(\mathbb{R}^n, \mathbb{R}^n)$ so that, for every $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_1(x), \ldots, f_l(x)\}$ is a linearly independent system.*
(2) *$\{f : \mathbb{R}^n \to \mathbb{R}^n : f$ is an isomorphism (that is, linear and bijective)$\}$ is l-lineable if and only if there exist isomorphisms $f_1, \ldots, f_l : \mathbb{R}^n \to \mathbb{R}^n$ so that, for every $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_1(x), \ldots, f_l(x)\}$ is a linearly independent system.*

We shall now introduce and recall some notions from analytic geometry:

Definition 2.2. A differentiable manifold $M \subseteq \mathbb{R}^m$ of dimension n is said to be parallelizable if there exist smooth vector fields $V_1, \ldots, V_n : M \to \mathbb{R}^n$ so that, for every $p \in M$, the set $\{V_1(p), \ldots, V_n(p)\}$ is a basis of $\mathbb{R}^n$.

The parallelizability of the $(n-1)$-dimensional sphere was completely solved in [8]:

Theorem 2.3. *The sphere $\mathbb{S}^{n-1}$ is parallelizable if and only if $n = 1, 2, 4$ or 8.*

By $\mathcal{L}(X; X)$ we denote, as usual, the family of all operators on the topological vector space X, that is, the set of all continuous linear functions $X \to X$. When X is a Banach space, then we endow $\mathcal{L}(X; X)$ with the usual norm-topology, the norm being defined by $\|T\| = \sup\{\|Tx\|_X : \|x\|_X = 1\}$.

Taking a look at Proposition 2.1 we have the following corollary:

Corollary 2.4. *Let n be a natural number and denote by*

$$\mathcal{A}_n = \{A \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) : A \text{ is an isomorphism}\},$$

where $\mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ stands for the space of all linear functions $\mathbb{R}^n \to \mathbb{R}^n$. Then $\mathcal{A}_n$ is n-lineable (that is, maximal lineable, since we have that $\mathcal{A}_n$ cannot be $(n+1)$-lineable, due to [12]) if and only if $n = 1, 2, 4$ or 8.

Proof. If $\mathcal{A}_n$ is n-lineable, Proposition 2.1 guarantees the existence of isomorphisms $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}^n$ so that, for every $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_1(x), \dots, f_n(x)\}$ is a linearly independent system. Without loss of generality (multiplying by f^{-1} if necessary) we may assume that f_1 is the identity matrix.

If now $x \in \mathbb{S}^{n-1}$ we can apply Grahm–Schmidt orthogonalization to $\{x, f_2(x), \dots, f_n(x)\}$ to define

$$
\begin{aligned}
V_1(x) &= f_2(x) - (f_2(x)x)x, \\
V_2(x) &= f_3(x) - (f_3(x)x)x - \frac{f_3(x)V_1(x)}{\|V_1(x)\|_2^2} V_1(x), \\
V_3(x) &= f_4(x) - (f_4(x)x)x - \frac{f_4(x)V_1(x)}{\|V_1(x)\|_2^2} V_1(x) - \frac{f_4(x)V_2(x)}{\|V_2(x)\|_2^2} V_2(x), \\
&\vdots \\
V_{n-1}(x) &= f_n(x) - (f_n(x)x)x - \sum_{k=1}^{n-2} \frac{f_n(x)V_k(x)}{\|V_k(x)\|_2^2} V_k(x).
\end{aligned}
$$

Then, we have that $V_i : \mathbb{S}^{n-1} \to \mathbb{R}^n$ are differentiable vector fields for $2 \leq i \leq n$. In conclusion, $\mathbb{S}^{n-1}$ is parallelizable and therefore $n = 1, 2, 4$ or 8.

Reciprocally, we refer to the proof in Theorem 2.8, where the reader can access such isomorphisms $f_1, \dots, f_n : \mathbb{R}^n \to \mathbb{R}^n$ in the construction of $A^{(3)}(x_1, x_2, \dots, x_8)$. □

Remark 2.5. On the other hand, $\mathcal{A}_n$ cannot be 2-lineable for odd dimension n thanks to the *Hairy Ball Theorem* ([16]), since otherwise Proposition 2.1 would guarantee the existence of isomorphisms (in particular, continuous) $f_1, f_2 : \mathbb{R}^n \to \mathbb{R}^n$ so that, for every $x \in \mathbb{R}^n \setminus \{0\}$, $\{f_1(x), \dots, f_l(x)\}$ is a linearly independent system (and, in particular, nonzero for every vector in the sphere).

Dealing with dimension 2^n for $n \geq 4$ we can still say something more about the lineability of $\mathcal{A}_{2^n}$ (even though we already know it cannot be 2^n). For that we will give a preliminary Lemma which carries out further the ideas initiated in Proposition 2.1:

Lemma 2.6. *Assume we can find $A_1, \dots, A_l \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n)$ so that, for every $(\lambda_1, \dots, \lambda_l) \in \mathbb{R}^l \setminus \{0\}$, $\det(\lambda_1 A_1 + \ldots + \lambda_l A_l) \neq 0$. Then the set $\mathcal{A}_n$ is l-lineable.*

Proof. Let us prove that for every $\mathbf{x} \in \mathbb{R}^n \setminus \{0\}$, the set $\{A_1\mathbf{x}, \dots, A_l\mathbf{x}\}$ is linearly independent (and therefore we can apply Proposition 2.1).

Indeed, let $\lambda_1, \dots, \lambda_l \in \mathbb{R}$ and assume

$$0 = \lambda_1 A_1 \mathbf{x} + \ldots + \lambda_l A_l \mathbf{x} = (\lambda_1 A_1 + \ldots \lambda_l A_l)\, \mathbf{x}.$$

Now, by assumption, $\det(\lambda_1 A_1 + \ldots + \lambda_l A_l) \neq 0$, so that $\lambda_1 A_1 + \ldots + \lambda_l A_l$ is invertible and therefore the only solution to $\lambda_1 A_1 \mathbf{x} + \ldots + \lambda_l A_l \mathbf{x} = 0$ is $\mathbf{x} = 0$. □

We will also need the definition of an auxiliary operator to help us with the notation of the functions to be considered in the study of the lineability of $\mathcal{A}_{2^n}$ for $n \geq 4$.

Definition 2.7. For $n \in \mathbb{N}$, we define the mapping

$$\begin{array}{rccl} \Upsilon: & \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n) & \longrightarrow & \mathcal{L}(\mathbb{R}^n;\mathbb{R}^n) \\ & A=(a_{ij})_{i,j=1}^n & \longmapsto & (b_{ij})_{i,j=1}^n = \begin{cases} a_{ij} & \text{if } 1 \leq i \leq n,\ j=1 \\ -a_{ji} & \text{if } i=1,\ 1<j\leq n \\ a_{ji} & \text{otherwise.} \end{cases} \end{array}$$

With these considerations, we are ready to prove the main result in this section:

Theorem 2.8. *Let $n \geq 3$. Then, the set $\mathcal{A}_{2^n}$ is $n+5$-lineable.*

Proof. We will show, via induction on n, that we can find a matrix

$$A^{(n)}(x_1,\ldots,x_{n+5}) = \left\{a_{i,j}^{(n)}(x_1,\ldots,x_{n+5})\right\}_{i,j=1}^{2^n}$$

of dimension 2^n whose entries are expressed in term of $n+5$ variables and whose columns are pairwise orthogonal for every choice $(x_1,\ldots,x_{n+5}) \in \mathbb{R}^{n+5}\setminus\{0\}$. By abuse of notation and for the sake of clarity, we will write $a_{i,j}^{(n)}$ instead of $a_{i,j}^{(n)}(x_1,\ldots,x_{n+5})$.

For technical reasons, in the induction process we will also prove that such matrices fulfill $a_{1,j}^{(n)} = -a_{j,1}^{(n)}$ for every $1 < j \leq 2^n$. Setting

$$A^{(n)}(x_1,\ldots,x_{n+5}) = x_1 A_1^{(n)} + \ldots + x_{n+5} A_{n+5}^{(n)},$$

where each $A_j^{(n)}$ is a $2^n \times 2^n$ matrix whose entries consist only on 0's and ± 1's, we will find $n+5$ matrices of dimension $\mathbb{R}^{2^n}$ that fulfill the hypothesis of Lemma 2.6 and, hence, $\mathcal{A}_{2^n}$ would be $(n+5)$-lineable.

For $n=3$, consider the matrix

$$A^{(3)}(x_1,x_2,\ldots,x_8) = \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 & -x_5 & -x_6 & -x_7 & -x_8 \\ x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 & -x_7 \\ x_3 & x_4 & x_1 & -x_2 & -x_7 & -x_8 & x_5 & x_6 \\ x_4 & -x_3 & x_2 & x_1 & -x_8 & x_7 & -x_6 & x_5 \\ x_5 & x_6 & x_7 & x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & -x_5 & x_8 & -x_7 & x_2 & x_1 & x_4 & -x_3 \\ x_7 & -x_8 & -x_5 & x_6 & x_3 & -x_4 & x_1 & x_2 \\ x_8 & x_7 & -x_6 & -x_5 & x_4 & x_3 & -x_2 & x_1 \end{bmatrix}.$$

One can check that the columns of $A^{(3)}(x_1,x_2,\ldots,x_8)$ form an orthogonal basis.

Assume next that the result is true for n and consider

$$A^{(n)}(x_1,\ldots,x_{n+5}) = x_1 A_1^{(n)} + \ldots + x_{n+5}A_{n+5}^{(n)}.$$

Define then the matrix

$$A^{(n+1)}(x_1,\ldots,x_{n+6}) = \left[\begin{array}{c|c} A^{(n)}(x_1,\ldots,x_{n+5}) & \begin{matrix} x_{n+6} & & & 0 \\ & -x_{n+6} & & \\ & & -x_{n+6} & \\ & 0 & & \ddots \end{matrix} \\ \hline \begin{matrix} -x_{n+6} & & & 0 \\ & x_{n+6} & & \\ & & x_{n+6} & \\ 0 & & & \ddots \end{matrix} & \Upsilon\Big(A^{(n)}(x_1,\ldots,x_{n+5})\Big) \end{array}\right].$$

It is trivial that $a_{1,j}^{(n+1)} = -a_{j-1}^{(n+1)}$ for every $1 < j \leq 2^{n+1}$, by the induction hypothesis. Let us show that the columns of $A^{(n+1)}(x_1,\ldots,x_{n+6})$ are orthogonal. Indeed, let $1 \leq i < j \leq 2^{n+1}$.

(1) Assume $1 \leq i,j \leq 2^n$ or $2^n + 2 \leq i,j \leq 2^{n+1}$. Then, the ith and jth columns are orthogonal by the induction hypothesis and the construction of the matrix for the new entries. It might be useful to notice that if $2 \leq i \leq 2^n$ the i-th column of $A^{(n+1)}(x_1,\ldots,x_{n+6})$ can be expressed like

$$\begin{pmatrix} a_{1,i}^{(n)} \\ a_{2,i}^{(n)} \\ \vdots \\ a_{2^n,i}^{(n)} \\ 0 \\ \left.\vdots\right\} i-3 \\ 0 \\ x_{n+6} \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

while if $2^n + 2 \leq j \leq 2^{n+1}$, the j-th column of $A^{(n+1)}(x_1,\ldots,x_{n+6})$ looks like

$$\begin{pmatrix} 0 \\ \left.\vdots\right\} j-2^n-3 \\ 0 \\ -x_{n+6} \\ 0 \\ \left.\vdots\right\} 2^{n+1}-j-2 \\ 0 \\ -a^{(n)}_{j-2^n,1} \\ a^{(n)}_{j-2^n,2} \\ \vdots \\ a^{(n)}_{j-2^n,2^n} \end{pmatrix}.$$

(2) Assume $2^n + 2 \leq j \leq 2^{n+1}$ and let us examine the scalar product of the j-th and the $(2^n + 1)$-th columns. Notice that the $(2^n + 1)$-th column looks like

$$\begin{pmatrix} x_{n+6} \\ 0 \\ \left.\vdots\right\} 2^n-3 \\ 0 \\ a^{(n)}_{1,1} \\ a^{(n)}_{2,1} \\ \vdots \\ a^{(n)}_{2^n,1} \end{pmatrix} = \begin{pmatrix} x_{n+6} \\ 0 \\ \left.\vdots\right\} 2^n-3 \\ 0 \\ a^{(n)}_{1,1} \\ -a^{(n)}_{1,2} \\ \vdots \\ -a^{(n)}_{1,2^n} \end{pmatrix}.$$

The scalar product of the j-th and the $(2^n + 1)$-th columns is hence zero again by the induction hypothesis.

(3) Assume $2 \leq i \leq 2^n$ and $2^n + 2 \leq j \leq 2^{n+1}$. Then, the scalar product of the i-th and j-th columns yields

$$\begin{aligned} & a^{(n)}_{1,i} \cdot 0 + a^{(n)}_{2,i} \cdot 0 + \ldots + a^{(n)}_{j-2^n-1,i} \cdot 0 + a^{(n)}_{j-2^n,i} \cdot (-x_{n+6}) + a^{(n)}_{j-2^n+1,i} \cdot 0 + \ldots \\ & + a^{(n)}_{2^n,i} \cdot 0 + 0 \cdot \left(-a^{(n)}_{j-2^n,1}\right) + 0 \cdot a^{(n)}_{j-2^n,2} + \ldots + 0 \cdot a^{(n)}_{j-2^n,i-1} + x_{n+6} \cdot a^{(n)}_{j-2^n,i} \\ & + 0 \cdot a^{(n)}_{j-2^n,i+1} + \ldots + 0 \cdot a^{(n)}_{j-2^n,2^n} \\ & = 0. \end{aligned}$$

(4) Assume $2^n + 2 \leq j \leq 2^{n+1}$ and let us take a look at the scalar product of the 1st and j-th columns. It must look like

$$a^{(n)}_{j-2^n,1} \left(-x_{n+6}\right) - x_{n+6} \left(-a^{(n)}_{j-2^n,1}\right) = 0.$$

(5) If $2 \leq i \leq 2^n$, the comparison between the i-th column and the (2^n+1)-th column works the same as in the previous case.
(6) The scalar product of the 1st and (2^n+1)-th column yields as the only nonzero factors

$$a_{1,1}^{(n)} x_{n+6} - x_{n+6} a_{1,1}^{(n)} = 0,$$

and the result follows. □

Remark 2.9. Even though the authors have been able to prove that the dimension of the lineability of $\mathcal{A}_{2^n}$ increases with n, the optimal dimension remains open. The question of what can be said concerning *even* dimension in general still has the partial results from [12] as the only answer.

3. Results for infinite dimensions

Theorem 3.1. *Let X be an infinite dimensional space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) that admits a Schauder basis $\{e_n\}_{n=1}^{\infty}$ for which the* **forward shift operator**

$$\begin{array}{rccc} F: & X & \longrightarrow & X \\ & \sum_{n=1}^{\infty} a_n e_n & \longmapsto & \sum_{n=2}^{\infty} a_{n-1} e_n \end{array}$$

is well-defined. Then the set

$$\{A \in \mathcal{L}(X;X) : A \text{ is injective}\}$$

is lineable.

Proof. Consider $\lambda_1, \dots, \lambda_k \in \mathbb{K} \setminus \{0\}$ and $n_1 < \dots < n_k \in \mathbb{N}$ and define

$$f(x) = \sum_{i=1}^{k} \lambda_i F^{n_i}(x).$$

Assume $f(x) = 0$ for $x = \sum_{l=1}^{\infty} a_l e_l$. Let us prove via induction on l that $a_l = 0$ for every $l \in \mathbb{N}$. Indeed,

$$\begin{aligned} e_{n_1+1}^*(f(x)) &= \sum_{i=1}^{k} \lambda_i e_{n_1+1}^* \Big(F^{n_i}(x)\Big) = \sum_{i=1}^{k} \lambda_i e_{n_1+1}^* F^{n_i}\Big(\sum_{l=1}^{\infty} a_l e_l\Big) \\ &= \sum_{i=1}^{k} \lambda_i e_{n_1+1}^* \Big(\sum_{l=n_i+1}^{\infty} a_{l-n_i} e_l\Big) = \sum_{i=1}^{k} \lambda_i \sum_{l=n_i+1}^{\infty} a_{l-n_i} e_{n_1+1}^*(e_l) = \lambda_1 a_1, \end{aligned}$$

where e_j^* denotes the coefficient functional associated to e_j ($j \geq 1$). Since $\lambda_1 \neq 0$, we must conclude $a_1 = 0$.

Assume $a_j = 0$ for $1 \le j \le m-1$. Then we have

$$e^*_{n_1+m}(f(x)) = \sum_{i=1}^{k} \lambda_i e^*_{n_1+m}\Big(F^{n_i}(x)\Big) = \sum_{i=1}^{k} \lambda_i e^*_{n_1+m} F^{n_i}\Big(\sum_{l=m}^{\infty} a_l e_l\Big)$$
$$= \sum_{i=1}^{k} \lambda_i e^*_{n_1+m}\Big(\sum_{l=n_i+m}^{\infty} a_{l-n_i} e_l\Big) = \sum_{i=1}^{k} \lambda_i \sum_{l=n_i+m}^{\infty} a_{l-n_i} e^*_{n_1+m}(x_l) = \lambda_1 a_m.$$

Again, using the fact that $\lambda_1 \neq 0$, we can conclude that $a_m = 0$, and the conclusion follows. □

Remark 3.2. The conclusion of Theorem 3.1 can also be obtained from the more general result that if X is a separable Banach space then there exists a commutative freely $\mathfrak{c}$-generated algebra in $\mathcal{L}(X;X)$ consisting, except for zero, of injective operators (see [4, Theorem 3.7]). Here $\mathfrak{c}$ denotes the cardinality of the continuum. Nevertheless, the conclusion of Theorem 3.1 still holds for *Fréchet spaces.* For instance, it holds for the space $H(\mathbb{C})$ of entire functions, endowed with the compact-open topology. Indeed, it suffices to consider the Schauder basis $e_n(z) := z^n$ $(n \ge 0)$.

As usual, we denote by c_0 the space of scalar sequences $x = (x_k)$ such that $x_k \to 0$ as $k \to \infty$. This space becomes a Banach space under the maximum norm $\|x\|_\infty = \max_{n\ge1} |x_n|$.

Theorem 3.3. *The sequence* $\{F^n : c_0 \to c_0\}_{n=1}^{\infty}$ *of powers of the forward shift* F *with respect to the standard basis of* $\mathcal{L}(c_0;c_0)$ *is a monotone basic sequence. Furthermore, if* $F = \sum_{n=1}^{\infty} a_n F^n \in \mathcal{L}(c_0;c_0) \setminus \{0\}$, *then* F *is injective.*

Proof. Let us consider a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$, and let us study the norms of the partial sums of the series $\sum_{n=1}^{\infty} a_n F^n$. For this purpose, let $k, l \in \mathbb{N}$. Then, if $\|x\|_\infty = 1$,

$$\left\|\sum_{n=1}^{k} a_n F^n(x)\right\|_\infty \le \sum_{n=1}^{k} |a_n| \|F^n(x)\|_\infty = \sum_{n=1}^{k} |a_n| \|x\|_\infty = \sum_{n=1}^{k} |a_n|. \tag{3.1}$$

Conversely, define the function

$$\begin{aligned} \text{sign:}\quad \mathbb{R} &\longrightarrow \{-1,0,1\} \\ t &\longmapsto \begin{cases} \frac{|t|}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases} \end{aligned}$$

and consider, focusing on the number $k \in \mathbb{N}$, the vector $x = (x_n)$ as follows:

$$\begin{cases} x_n = \text{sign}(a_{k-n+1}) & \text{if } 1 \le n \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$
\begin{aligned}
a_k F^k(x) &= a_k(\overbrace{0,\ldots,0}^{k}, x_1, x_2, x_3, \ldots, x_k, 0, \ldots) \\
&= a_k(\overbrace{0,\ldots,0}^{k}, \operatorname{sign}(a_k), \operatorname{sign}(a_{k-1}), \operatorname{sign}(a_{k-2}), \ldots, \operatorname{sign}(a_1), 0, \ldots), \\
a_{k-1} F^{k-1}(x) &= a_{k-1}(\overbrace{0,\ldots,0}^{k-1}, x_1, x_2, x_3, \ldots, x_k, 0, \ldots) \\
&= a_{k-1}(\overbrace{0,\ldots,0}^{k-1}, \operatorname{sign}(a_k), \operatorname{sign}(a_{k-1}), \operatorname{sign}(a_{k-2}), \ldots, \operatorname{sign}(a_1), 0, \ldots), \\
&\vdots \\
a_1 F^1(x) &= a_1(0, x_1, x_2, x_3, \ldots, x_k, 0, \ldots) \\
&= a_1(0, \operatorname{sign}(a_k), \operatorname{sign}(a_{k-1}), \operatorname{sign}(a_{k-2}), \ldots, \operatorname{sign}(a_1), 0, \ldots).
\end{aligned}
$$

Hence, $e^*_{k+1} \sum_{n=1}^k a_n F^n(x) = \sum_{n=1}^k |a_n|$, so that $\left\| \sum_{n=1}^k a_n F^n \right\| \geq \sum_{n=1}^k |a_n|$. Together with the equation given by (3.1), we can conclude that

$$
\Big\| \sum_{n=1}^{k} a_n F^n \Big\| = \sum_{n=1}^{k} |a_n|.
$$

Therefore,

$$
\Big\| \sum_{n=1}^{k} a_n F^n \Big\| = \sum_{n=1}^{k} |a_n| \leq \sum_{n=1}^{k+l} |a_n| = \Big\| \sum_{n=1}^{k+l} a_n F^n \Big\|.
$$

Our final task is to show that every $F \in \overline{\operatorname{span}}\,\{F^n : n \geq 1\} \setminus \{0\}$ is injective. With this aim, notice that such an F has the form

$$
F = \sum_{k=N}^{\infty} c_k F^k,
$$

for certain scalars c_n ($n \geq N$) and certain N with $c_N \neq 0$, the convergence of the series being with respect to the operator norm. Fix a vector $x = (x_n) \in c_0$ and assume that $F(x) = 0$. Since norm-convergence implies pointwise convergence, we get $\sum_{k=N}^{\infty} c_k F^k(x) = 0 = (0, 0, 0, \ldots)$. Now, the jth coordinate of $c_k F^k(x)$ equals 0 if $1 \leq j \leq k$, and $c_k x_{j-k}$ if $j > k$. Consequently, we get

$$
\sum_{j=N}^{n} c_j \; x_{n+1-j} = 0 \quad \text{for all} \quad n = N, N+1, N+2, \ldots,
$$

from which we recursively obtain $x_1 = 0$, $x_2 = 0$, $x_3 = 0, \ldots$, and therefore we have that $x = 0$, which yields the desired conclusion. □

Corollary 3.4. *The set given by*

$$\{T \in \mathcal{L}(c_0; c_0) : T \text{ is injective}\}$$

is spaceable in $\mathcal{L}(c_0; c_0)$. In particular, this set is $\mathfrak{c}$-lineable.

Proof. We just remark that, for a complete metrizable topological vector space of infinite dimension, the cardinality of any Hamel basis of this space must be at least $\mathfrak{c}$, as proved in [15, p. 158]. □

4. On the property of surjectivity

The symbol ℓ_p $(1 \leq p < \infty)$ will represent, as usual, the Banach space of all p-summable scalar sequences, endowed with the norm $\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ (for each $x = (x_n)$).

Concerning families of surjective operators, we have the following result:

Theorem 4.1. *Let $1 \leq p \leq \infty$. Then, the set*

$$M_p = \{A \in \mathcal{L}(\ell_p; \ell_p) : A \text{ is surjective}\}$$

is spaceable.

Proof. Let $\{p_1 < p_2 < p_3 < \ldots\}$ be the sequence of prime numbers and consider, given $k \in \mathbb{N}$, the following operator:

$$\begin{array}{rccc} T_k : & \ell_p & \longrightarrow & \ell_p \\ & \sum_{i=1}^{\infty} a_i e_i & \mapsto & \sum_{j=1}^{\infty} a_{p_j^k} e_j, \end{array}$$

where $\{e_i\}_{i=1}^{\infty}$ is, as usual, the canonical basis in ℓ_p. Notice that, in particular, for every i, $k \in \mathbb{N}$,

$$T_k(e_i) = \begin{cases} e_j & \text{if } i = p_j^k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

We will divide the proof in three cases: $2 \leq p < \infty$, $1 \leq p < 2$ and $p = \infty$. Assume first $2 \leq p < \infty$ and let us consider the operator

$$\begin{array}{rccc} \Phi : & \ell_{p'} & \longrightarrow & \mathcal{L}(\ell_p; \ell_p) \\ & \{\lambda_k\}_{k=1}^{\infty} & \mapsto & \sum_{k=1}^{\infty} \lambda_k T_k, \end{array}$$

where $1 < p' \leq 2$ is the conjugate of p (that is, $\frac{1}{p} + \frac{1}{p'} = 1$), and notice that, if $x = \sum_{i=1}^{\infty} a_i e_i \in \ell_p$, then

$$\Phi(\{\lambda_k\}_{k=1}^{\infty})(x) = \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{\infty} a_i T_k(e_i) = \sum_{k=1}^{\infty} \lambda_k \sum_{j=1}^{\infty} a_{p_j^k} e_j.$$

Therefore, if $i \geq 1$,

$$e_i^* \left(\Phi(\{\lambda_k\}_{k=1}^{\infty})(x)\right) = \sum_{k=1}^{\infty} \lambda_k a_{p_i^k}.$$

Then, using Hölder's inequality,

$$\begin{aligned} \|\Phi(\{\lambda_k\}_{k=1}^{\infty})(x)\|_p^p &= \Big\| \sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} \lambda_k a_{p_i^k} \Big) e_i \Big\|_p^p = \sum_{i=1}^{\infty} \Big| \sum_{k=1}^{\infty} \lambda_k a_{p_i^k} \Big|^p \leq \sum_{i=1}^{\infty} \Big(\sum_{k=1}^{\infty} |\lambda_k a_{p_i^k}| \Big)^p \\ &\leq \sum_{i=1}^{\infty} \|\{\lambda_k\}_{k=1}^{\infty}\|_{p'}^p \Big\| \sum_{k=1}^{\infty} a_{p_i^k} e_k \|_p^p = \|\{\lambda_k\}_{k=1}^{\infty}\|_{p'}^p \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{p_i^k}|^p \\ &\leq \|\{\lambda_k\}_{k=1}^{\infty}\|_{p'}^p \|x\|_p^p. \end{aligned}$$

Therefore, $\|\Phi(\{\lambda_k\}_{k=1}^{\infty})\|_{\mathcal{L}(\ell_p;\ell_p)} \leq \|\{\lambda_k\}_{k=1}^{\infty}\|_{p'}$ and hence $\|\Phi\| \leq 1$.

Let us show that $\Phi(\{\lambda_k\}_{k=1}^{\infty})$ is surjective, for every $\{\lambda_k\}_{k=1}^{\infty} \in \ell_{p'} \setminus \{0\}$. Indeed, given an element $b = \sum_{i=1}^{\infty} b_i e_i \in \ell_p$, we need to show that we can find an element $x = \sum_{i=1}^{\infty} a_i e_i \in \ell_p$ so that

$$b_j = e_j^* \Big(\Phi(\{\lambda_k\}_{k=1}^{\infty})(x) \Big) = \sum_{k=1}^{\infty} \lambda_k \sum_{i=1}^{\infty} a_{p_i^k} e_j^*(e_i) = \sum_{k=1}^{\infty} \lambda_k a_{p_j^k},$$

for every $j \in \mathbb{N}$.

We are led then to consider the infinite system of linear equations

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_l x = b_l, \\ \vdots \end{cases}$$

where $\alpha_j = \sum_{k=1}^{\infty} \lambda_k e_{p_j^k}$. Notice that, in particular,

$$\alpha_i \bar{\alpha}_j = \delta_{ij} \sum_{k=1}^{\infty} |\lambda_k|^2,$$

where δ_{ij} stands for the Kronecker δ, $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

We can then make use of Theorem 1.3 and find a solution for the system formed by the first n equations,

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_n x = b_n. \end{cases}$$

Such a solution can be expressed as $\sum_{j=1}^{n} \frac{B_j}{G_{j-1}G_j}\varphi_j$, where

$$B_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_{j-1} & b_1 \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_{j-1} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_j\bar{\alpha}_1 & \alpha_j\bar{\alpha}_2 & \cdots & \alpha_j\bar{\alpha}_{j-1} & b_j \end{vmatrix} = \begin{vmatrix} \sum_{k=1}^{\infty}|\lambda_k|^2 & 0 & \cdots & 0 & b_1 \\ 0 & \sum_{k=1}^{\infty}|\lambda_k|^2 & \cdots & 0 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & b_j \end{vmatrix}$$

$$= b_j \left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{j-1},$$

$$G_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_j \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_j \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j\bar{\alpha}_1 & \alpha_j\bar{\alpha}_2 & \cdots & \alpha_j\bar{\alpha}_j \end{vmatrix} = \left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{j},$$

$$\varphi_j = \begin{vmatrix} \alpha_1\bar{\alpha}_1 & \alpha_1\bar{\alpha}_2 & \cdots & \alpha_1\bar{\alpha}_j \\ \alpha_2\bar{\alpha}_1 & \alpha_2\bar{\alpha}_2 & \cdots & \alpha_2\bar{\alpha}_j \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{j-1}\bar{\alpha}_1 & \alpha_{j-1}\bar{\alpha}_2 & \cdots & \alpha_{j-1}\bar{\alpha}_j \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \cdots & \bar{\alpha}_j \end{vmatrix} = \bar{\alpha}_j \left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{j-1}$$

$$= \sum_{l=1}^{\infty}\lambda_l \left(\sum_{k=1}^{\infty}|\lambda_k|^2\right)^{j-1} e_{p_j^l}.$$

Therefore, the solution of the system

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_n x = b_n \end{cases}$$

is given by

$$\sum_{j=1}^{n} \frac{B_j}{G_{j-1}G_j}\varphi_j = \sum_{j=1}^{n} \frac{b_j \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{j-1}}{\left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{j} \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{j-1}} \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{j-1} \sum_{l=1}^{\infty} \lambda_l e_{p_j^l}$$
$$= \frac{1}{\sum_{k=1}^{\infty} |\lambda_k|^2} \sum_{j=1}^{n} b_j \sum_{l=1}^{\infty} \lambda_l e_{p_j^l}.$$

Therefore, if the element formally defined as

$$x = \frac{1}{\sum_{k=1}^{\infty} |\lambda_k|^2} \sum_{j=1}^{\infty} b_j \sum_{l=1}^{\infty} \lambda_l e_{p_j^l}$$

converges, we would have that x is the desired solution to the infinite system

$$\begin{cases} \alpha_1 x = b_1, \\ \alpha_2 x = b_2, \\ \vdots \\ \alpha_l x = b_l, \\ \vdots \end{cases}$$

Notice that, indeed, we have that

$$\|x\|_p \leq \frac{1}{\sum_{k=1}^{\infty} |\lambda_k|^2} \|\{b_j\}_{j=1}^{\infty}\|_p \|\{\lambda_l\}_{l=1}^{\infty}\|_\infty < \infty,$$

using the fact that $\ell_{p'} \hookrightarrow \ell_2 \hookrightarrow \ell_\infty$, so that $0 < \|\{\lambda_l\}_{l=1}^{\infty}\|_\infty \leq \|\{\lambda_l\}_{l=1}^{\infty}\|_2 \leq \|\{\lambda_l\}_{l=1}^{\infty}\|_{p'} < \infty$, if $\|\{\lambda_l\}_{l=1}^{\infty}\|_{p'} \neq 0$.

Finally, let $\{\lambda^{(n)}\}_{n=1}^{\infty} \subseteq \ell_{p'}$ so that $\Phi(\lambda^{(n)}) \xrightarrow[n\to\infty]{} T \in \mathcal{L}(\ell_p; \ell_p)$. Then,

$$\sum_{k=1}^{\infty} \lambda_k^{(n)} T_k(e_{p_j^l}) = \lambda_l^{(n)} e_j \xrightarrow[n\to\infty]{} T(e_{p_j^l}).$$

In particular,

$$e_l^*(\lambda^{(n)}) \xrightarrow[n\to\infty]{} e_j^* T(e_{p_j^l})$$

for every $j \in \mathbb{N}$, so that we can formally define the element

$$\lim_{n\to\infty} \lambda^{(n)} = \sum_{l=1}^{\infty} e_1^* T(e_{p_1^l}) e_l \in \mathbb{R}^{\mathbb{N}}.$$

Let now $b = \{b_k\}_{k=1}^{\infty} \in \mathbb{S}_{\ell_p}$. Then we have

$$\|T\| \geq \left| e_1^* T\left(\sum_{k=1}^{\infty} b_k e_{p_1^k}\right)\right| = \left|\sum_{k=1}^{\infty} b_k e_1^* T(e_{p_1^k})\right| = \left|\sum_{k=1}^{\infty} b_k e_k^* (\lim_{n\to\infty} \lambda^{(n)})\right|$$
$$= \langle b, \lim_{n\to\infty} \lambda^{(n)}\rangle.$$

Therefore,

$$\|\lim_{n\to\infty} \lambda^{(n)}\|_{p'} = \max_{b\in\mathbb{S}_{\ell_p}} \langle b, \lim_{n\to\infty} \lambda^{(n)}\rangle \leq \|T\|,$$

so we can conclude that $\lim_{n\to\infty} \lambda^{(n)} = \lambda \in \ell_{p'}$. Applying the Closed Graph Theorem, we can say that $T = \Phi(\lambda)$ and hence T is surjective.

Assume next $1 \leq p < 2$ and consider the operator

$$\begin{array}{rccc} \Psi: & \ell_2 & \longrightarrow & \mathcal{L}(\ell_p;\ell_p) \\ & \{\lambda_k\}_{k=1}^{\infty} & \mapsto & \sum_{k=1}^{\infty} \lambda_k T_k. \end{array}$$

We claim that, $\overline{\Psi(\ell_2)} \subseteq \{T \in \mathcal{L}(\ell_p;\ell_p) \ : \ T \text{ is surjective}\}$. The proof for this claim follows the same ideas as the case $2 \leq p \leq \infty$, with two main steps where the procedure might differ slightly:

First of all, we can write

$$\|\Psi(\{\lambda_k\}_{k=1}^{\infty})(x)\|_p^p \leq \sum_{i=1}^{\infty} \|\{\lambda_k\}_{k=1}^{\infty}\|_{p'}^p \left\| \sum_{k=1}^{\infty} a_{p_i^k} e_k\right\|_p^p \leq \|\{\lambda_k\}_{k=1}^{\infty}\|_2^p \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} |a_{p_i}^k|^p \tag{4.2}$$
$$\leq \|\{\lambda_k\}_{k=1}^{\infty}\|_2^p \|x\|_p^p,$$

where again p' is the conjugate of p (so, in particular, $2 < p' \leq \infty$).

Also, if we formally define the element

$$x = \frac{1}{\sum_{k=1}^{\infty} |\lambda_k|^2} \sum_{j=1}^{\infty} b_j \sum_{l=1}^{\infty} \lambda_l e_{p_j^l},$$

we can write

$$\|x\|_p \leq \frac{1}{\sum_{k=1}^{\infty} |\lambda_k|^2} \|\{b_j\}_{j=1}^{\infty}\|_p \|\{\lambda_l\}_{l=1}^{\infty}\|_{\infty} < \infty. \tag{4.3}$$

The reader should keep in mind that equations (4.2) and (4.3) are supported by the fact that $\ell_2 \hookrightarrow \ell_{p'} \hookrightarrow \ell_{\infty}$.

Finally, to prove that $\lim_{n\to\infty} \lambda^{(n)} \in \ell_2$ for $\{\lambda^{(n)}\}_{n=1}^{\infty} \subseteq \ell_2$ so that $\Psi(\lambda^{(n)}) \xrightarrow[n\to\infty]{} T \in \mathcal{L}(\ell_p;\ell_p)$, we first need to consider $b \in \mathbb{S}_{\ell_2} \cap \ell_p$. In this case, we still have

$$\langle b, \lim_{n\to\infty} \lambda^{(n)}\rangle \leq \|T\|.$$

Therefore, if $b = \{b_k\}_{k=1}^\infty \in \mathbb{S}_{\ell_2}$, we would have that

$$|\langle b, \lim_{n\to\infty} \lambda^{(n)}\rangle| = \Big|\sum_{k=1}^\infty b_k e_k^*(\lim_{n\to\infty}\lambda^{(n)})\Big| = \Big|\lim_{m\to\infty}\sum_{k=1}^m b_k e_k^*(\lim_{n\to\infty}\lambda^{(n)})\Big|$$
$$= \lim_{m\to\infty}\Big|\sum_{k=1}^m b_k e_k^*(\lim_{n\to\infty}\lambda^{(n)})\Big| \le \|T\|$$

We then conclude in this case that

$$\|\lim_{n\to\infty}\lambda^{(n)}\|_2 = \max_{b\in\mathbb{S}_{\ell_2}}\langle b, \lim_{n\to\infty}\lambda^{(n)}\rangle \le \|T\|,$$

so $\lim_{n\to\infty}\lambda^{(n)} = \lambda \in \ell_2$. Applying again the Closed Graph Theorem, we can say that $T = \Psi(\lambda)$ and hence T is surjective.

The third and last case, $p = \infty$, follows the same procedure as the two previous cases. We let

$$\begin{array}{rccc} \Gamma: & \ell_1 & \to & \mathcal{L}(\ell_\infty;\ell_\infty) \\ & \{\lambda_k\}_{k=1}^\infty & \mapsto & \sum_{k=1}^\infty \lambda_k T_k, \end{array}$$

and the only somewhat different step is the point where it has to be proved that $\overline{\Gamma(\ell_1)} \subseteq M_\infty \cup \{0\}$.

To this end, consider $\{\lambda^{(n)}\}_{n=1}^\infty \subseteq \ell_1$ so that $\Gamma(\lambda^{(n)}) \xrightarrow[n\to\infty]{} T \in \mathcal{L}(\ell_\infty;\ell_\infty)$ and define, for every $m\in\mathbb{N}$, the sequence $a^{(m)} = \{a_k^{(m)}\}_{k=1}^\infty$ as follows:

$$a_k^{(m)} = \begin{cases} \frac{\overline{e_1^*T(e_{p_1^l})}}{|e_1^*T(e_{p_1^l})|} & \text{if } k = p_1^l \text{ for some } l\le m, \text{ and } e_1^*T(e_{p_1^l}) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $a^{(m)} \in \mathbb{S}_{\ell_\infty}$, so that $\|T(a^{(m)})\|_\infty \le \|T\|$. Now, notice that

$$T(\{a_i^{(m)}\}_{i=1}^\infty) = \lim_{n\to\infty}\Gamma(\lambda^{(n)})(\{a_i^{(m)}\}_{i=1}^\infty) = \lim_{n\to\infty}\sum_{i=1}^\infty a_i^{(m)}\Gamma(\lambda^{(n)})(e_i)$$
$$= \lim_{n\to\infty}\sum_{i=1}^\infty a_i^{(m)}\sum_{k=1}^\infty \lambda_k^{(n)} T_k(e_i) = \lim_{n\to\infty}\sum_{l=1}^m a_{p_1^l}\sum_{k=1}^\infty \lambda_k^{(n)}T_k(e_{p_1^l})$$
$$= \lim_{n\to\infty}\sum_{l=1}^m a_{p_1^l}\lambda_l^{(n)} e_1 = \sum_{l=1}^m a_{p_1^l} T(e_{p_1^l}).$$

Therefore,

$$\sum_{l=1}^m \lim_{n\to\infty}|\lambda_l^{(n)}| = |e_1^*T(\{a_i^{(m)}\}_{i=1}^\infty)| \le \|T\|,$$

for every $m \in \mathbb{N}$. In particular, letting $m \to \infty$, we obtain

$$\sum_{l=1}^{\infty} \lim_{n\to\infty} |\lambda_l^{(n)}| = \lim_{n\to\infty} \|\lambda^{(n)}\|_1 \leq \|T\| < \infty,$$

so that $\lim_{n\to\infty} \lambda^{(n)} \in \ell_1$. Applying the Closed Graph Theorem once more, $T = \Gamma(\lim_{n\to\infty} \lambda^{(n)})$ and therefore T is surjective. □

Remark 4.2. The same proof as for the set M_∞ works if we are interested in proving that the set $\{T \in \mathcal{L}(c_0; c_0) : T \text{ is surjective}\} \cup \{0\}$ contains an isomorphic copy of ℓ_1.

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