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EFFICIENT SOLUTIONS IN UNCERTAIN MULTIOBJECTIVE OPTIMIZATION WITH COUNTABLY MANY SCENARIOS*

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Abstract. We use a vector approach to address, from an efficient point of view, an uncertain unconstrained multiobjective optimization problem with countably many scenarios. Specifically, we introduce several efficient solution notions that work not only in the Pareto case, but also when the preferences in the image space depend on the scenario and they are defined by a convex cone in the usual way. We state basic properties of these notions and we relate the involved solution sets with other well-known solution sets of the literature. Particularly, it is shown that the so-called highly solutions are a particular case of efficient solutions. In addition, we obtain characterizations through solutions of associated scalar optimization problems and we derive existence theorems. Finally, two applications are provided to illustrate the main results.

Key words. uncertainty, multiobjective optimization, vector optimization, efficient solution, robust solution, linear scalarization, existence theorem, convex quadratic Pareto optimization

MSC codes. 90C29, 90C70, 90C17

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1. Introduction.

In the last decade, optimization problems with uncertain data have attracted more and more interest among researchers and practitioners. This is mainly due to the inaccuracy of the classical deterministic optimization models, which is a result of prediction errors in the forecasts of certain data entries (such as future demands, returns, etc.), measurement errors in the empirical data (e.g., parameters of devices or processes), and implementation errors during computation.

There exist three points of view to deal with uncertain scalar optimization problems. First, one can consider a stochastic approach, where the study of the problem is carried out by considering probability distributions on the uncertain data (see, for instance, [7, 29]). Second, in the fuzzy approach, uncertainty is managed using fuzzy sets and membership functions to represent imprecise or vague information, particularly in situations where probabilistic assumptions are difficult to justify (see [4]). Third, one can assume a pessimistic point of view and only pay attention to the worst scenario in each feasible point. This deterministic approach is called robust optimization and it guarantees that a feasible point will never be considered as a solution of the uncertain optimization problem whenever its objective value at the worst scenario can be improved for the worst value at another feasible point. We refer the reader to

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the seminal work by Ben-Tal, El Ghaoui, and Nemirovski [5], where some interesting arguments comparing and clarifying stochastic and robust approaches are presented.

In the literature there are several robust formulations for an uncertain scalar optimization problem as a result of different ways to define the notion of worst scenario. These formulations were unified by Klamroth et al. [21] and Köbis [23] via weakly efficient solutions of associated vector optimization problems. Namely, these authors introduced a new deterministic approach based on considering as a solution of the uncertain scalar optimization problem any feasible point whose value cannot be simultaneously improved in each scenario for the corresponding values at another feasible point. Accordingly, it is called the vector optimization approach (vector approach in short form). Relationships between the vector approach and several robust formulations can be found in [21, 22, 23] and [15, Chapter 6]. They are derived by scalarization schemes.

Concerning uncertain multiobjective optimization problems, as far as we know, the first deterministic reformulations were provided by Teghem et al. [31] in the 1980s and Abdelaziz, Lang, and Nadeau [1, 2] in the 1990s for a Pareto linear problem with a finite set of scenarios. These authors considered a vector approach based on replacing the uncertain multiobjective function with the deterministic one that results from putting together all scenario-criterion pairs. The solutions that they obtained by applying some kind of Pareto nondomination notion to this reformulation were called pointwise efficient solutions.

In the last decade, several works have been devoted to the study of robust counterparts of uncertain multiobjective optimization problems (see [17, 19, 32] and the references therein). Specifically, several solution concepts have been introduced which involve deterministic multiobjective optimization problems (see [24, 25, 28]) and deterministic set optimization problems (see [12, 18]). Recently, Sigler [30] and Engau and Sigler [13] focused again on the vector approach to deal with an uncertain multiobjective optimization problem ordered by components (see also [9]).

In this paper, we study the vector approach in a more general setting since the uncertainty involves not only the objective function but also the ordering cone. Its main contributions focus on the characterization of several efficient solution concepts by linear scalarization in convex problems, which complete and clarify the main results in [13] concerning problems with countably many scenarios.

The paper is structured as follows. In section 2, we introduce the problem, the main notation, and some mathematical tools that will be needed throughout the paper. Let us emphasize the topological dual space of the Cartesian product ℓ_∞^m, which allows us to clarify and extend the main results of [13, 30] concerning the countable setting. In section 3, several concepts of solution of an uncertain multiobjective optimization problem are introduced according to the vector approach. Their basic relationships are derived and a new view of the so-called highly robust efficient solutions is stated. In section 4, some weak versions of the new efficient solutions are characterized in convex problems by using linear scalarization, and existence results are derived from the sufficient conditions. Finally, the obtained results are applied to study uncertain unconstrained convex quadratic multiobjective optimization problems, where the uncertainty is periodic, and minmax robust solutions of uncertain unconstrained convex scalar optimization problems.

2. Preliminaries. Throughout, a nonempty set H in a Banach space $(Y, \|\cdot\|)$ is called a cone if $\alpha H \subseteq H$ for all $\alpha \geq 0$. We denote $\operatorname{cone} H := \bigcup_{\alpha \geq 0} \alpha H$. A cone $H \subseteq Y$ is called pointed (resp., nontrivial) if $H \cap (-H) = \{0\}$ (resp., $Y \neq H \neq \{0\}$). The topological interior and the closure of H are denoted int$H$ and cl$H$, respectively. In addition, to model the decision maker's preferences in a vector optimization problem given by a nonempty ordering set $C \subseteq Y$, we consider the binary relation (see [15]):

$$y_1, y_2 \in Y, \quad y_1 \precsim_C y_2 \overset{\text{def}}{\Longleftrightarrow} y_2 - y_1 \in C.$$

We denote $y_1 \precnsim_C y_2$ whenever $y_1 \precsim_C y_2$ and $y_1 \neq y_2$. Clearly, $\precsim_C$ and $\precnsim_C$ coincide whenever $0 \notin C$. If C is a nontrivial pointed convex cone, then $\precsim_C$ is a partial order. As is usual, this particular case will be denoted $\leq_C$ and $\lneq_C$ instead of $\precsim_C$ and $\precnsim_C$, respectively. Also, we refer to the Pareto order as the partial order $\leq_{\mathbb{R}^m_+}$, where $Y = \mathbb{R}^m$ and $\mathbb{R}^m_+$ denotes the nonnegative orthant in $\mathbb{R}^m$. Moreover, if C is solid, i.e., the topological interior int$C$ of C is nonempty, sometimes one can take advantage of the binary relation $\precsim_{\text{int}C}$. From now on we assume that int$C \neq \emptyset$ whenever $\precsim_{\text{int}C}$ is dealt with. The positive (resp., strict positive) polar cone of a cone C is denoted by C^+ (resp., C^{+s}), i.e.,

$$C^+ := \{\lambda \in Y^* : \lambda(y) \geq 0 \text{ for all } y \in C\},$$
$$(\text{resp., } C^{+s} := \{\lambda \in Y^* : \lambda(y) > 0 \text{ for all } y \in C \backslash \{0\}\}),$$

where Y^* stands for the topological dual space of Y and its dual norm is denoted by $\|\cdot\|_{Y^*}$. The product space $(Y^*)^m$ is equipped with the norm $\|(\lambda_1, \ldots, \lambda_m)\| = \|\lambda_1\|_{Y^*} + \cdots + \|\lambda_m\|_{Y^*}$.

We refer to the cardinal of a nonempty set H as $\#H$ and the complement of H as H^c. Moreover, for each set $A \subseteq Y$, we define $H \backslash A := H \cap A^c$. We assume $0 \notin \mathbb{N}$.

In this work, we consider the following uncertain (unconstrained) multiobjective optimization problem:

$$\text{(UP)} \qquad D(u)\text{-Minimize } f(x,u) \text{ subject to } x \in X,$$

where $f = (f_1, f_2, \ldots, f_m) \colon X \times \mathcal{U} \to \mathbb{R}^m$, X denotes the decision set, $\mathcal{U}$ stands for the uncertainty set, i.e., the set of scenarios, and $D \colon \mathcal{U} \rightrightarrows \mathbb{R}^m$ is a set valued map such that $D(u)$ denotes the ordering set at scenario $u \in \mathcal{U}$, which is assumed to be a nontrivial pointed convex cone. In other words, the decision maker's preferences at scenario $u \in \mathcal{U}$ are introduced as above via the partial order $\leq_{D(u)}$. Problem (UP) is called the uncertain Pareto problem whenever $D(u) = \mathbb{R}^m_+$ for all $u \in \mathcal{U}$. We refer to problem (UP) as a multiobjective optimization problem instead of a vector optimization problem since the final space of the vector objective function f is finite dimensional.

It is worth noting that the setting of problem (UP) with $D(u) = \mathbb{R}^m_+$ for all u was discussed in [13, Remark 2.1]. Moreover, the motivation to consider multiobjective optimization problems involving uncertain preferences was discussed in [32, section 8.2]. Roughly speaking, this modeling allows us studying real-world decision problems whose preferences depend on decision makers or they are related with the objectives being minimized. For instance, in the Markowitz portfolio selection problem (see [26, Problem (1)]), the aim is to find the best trade-off concerning the expected return of an investment and its corresponding risk. Clearly, the attitude to face the risk is very personal and it also depends on the setting where the choice should be made. Therefore, if the scenarios involve different decision makers or circumstances, then the preferences should be able to be changed accordingly.

When the realization of the scenario $u \in \mathcal{U}$ is known, problem (UP) is the deterministic (unconstrained) multiobjective optimization problem

$$(u\text{-DP}) \qquad D_u\text{-Minimize } f^u(x) \text{ subject to } x \in X,$$

where $D_u := D(u)$ and $f^u := f(\cdot,u)\colon X \to \mathbb{R}^m$ is $f^u(x) := f(x,u)$ for all $x \in X$. For this problem, we consider the usual concepts of efficient and weakly efficient solution (see [11, 27]).

DEFINITION 2.1. *A point $x_0 \in X$ is said to be*

- *a strict efficient solution of problem* (u-DP)*, denoted by $x_0 \in \mathrm{SE}(D_u)$, if there is not $x \in X\backslash\{x_0\}$ such that $f^u(x) \leq_{D_u} f^u(x_0)$;*
- *an efficient solution of problem* (u-DP)*, denoted by $x_0 \in \mathrm{E}(D_u)$, if there is not $x \in X\backslash\{x_0\}$ such that $f^u(x) \lneq_{D_u} f^u(x_0)$.*
- *Assume that D_u is solid. Then, x_0 is a weakly efficient solution of problem* (u-DP)*, denoted by $x_0 \in \mathrm{WE}(D_u)$, if $x_0 \in \mathrm{E}(\mathrm{int}D_u \cup \{0\})$.*

It follows that $\mathrm{SE}(D_u) \subseteq \mathrm{E}(D_u) \subseteq \mathrm{WE}(D_u)$.

When X is convex, f^u is said to be D_u-convex if for every $x_1, x_2 \in X$ and $\alpha \in (0,1)$ we have

$$f^u(\alpha x_1 + (1-\alpha)x_2) \leq_{D_u} \alpha f^u(x_1) + (1-\alpha) f^u(x_2).$$

It is not hard to check that f^u is D_u-convex if and only if the function $\langle \lambda, f^u\rangle : X \to \mathbb{R}$ is convex for all $\lambda \in D_u^+$.

From now on, we assume that $\mathcal{U}$ is a countable set (with either finitely or infinitely many scenarios). We suppose $\mathcal{U} = \{1,2,\ldots,r\}$ whenever $\mathcal{U}$ is finite. In the case $\#\mathcal{U} = \infty$, we assume that $\mathcal{U} = \mathbb{N}$ and, unless otherwise stated, we suppose that $f^x := f(x,\cdot) : \mathcal{U} \to \mathbb{R}^m$ is a bounded function (sequence) for all $x \in X$. In order to deal with both cases in a unified way, we denote

$$\mathcal{B}_{\mathcal{U}} := \mathcal{B}(\mathcal{U},\mathbb{R}^m) = \{g = (g_1,\ldots,g_m)\colon \mathcal{U} \to \mathbb{R}^m\colon g \text{ is bounded}\}.$$

Consequently, when handling the solutions of problem (UP), the set $\mathcal{B}_{\mathcal{U}}$ is a Banach space endowed with the norm

$$\|g\|_\infty := \sup_{u\in\mathcal{U}} \|g(u)\|_\infty,$$

where $\|g(u)\|_\infty := \max_{1\le i\le m} |g_i(u)|$. Clearly, $\|g\|_\infty = \max_{1\le i\le m} \|g_i\|_\infty$ and $\|g_i\|_\infty := \sup_{u\in\mathcal{U}} |g_i(u)|$. The closed ball of center g and radius $r > 0$ in $\mathcal{B}_{\mathcal{U}}$ with respect to $\|\cdot\|_\infty$ is denoted by $\mathbb{B}_\infty(g,r)$ and the closed unit ball of $\mathbb{R}^m$ with respect to the supremum norm is denoted by $\mathbb{B}_\infty$.

Notice that $\mathcal{B}(\mathbb{N},\mathbb{R})$ coincides with the classical space of bounded sequences

$$\ell_\infty := \left\{ x = (x_k)_{k\in\mathbb{N}} : \sup_{k\in\mathbb{N}} |x_k| < \infty \right\}.$$

Analogously, we have $\mathcal{B}(\mathbb{N},\mathbb{R}^m) = \ell_\infty^m$ if the norm $\|(x^1,\ldots,x^m)\| = \max_{1\le i\le m} \|x^i\|_\infty$ is used on the product space ℓ_∞^m. We recall some properties of the Banach space $(\mathcal{B}(\mathbb{N},\mathbb{R}^m), \|\cdot\|_\infty)$. To do this, we first collect the topological dual spaces $(c_0)^*$, c^*, and $(\ell_\infty)^*$ where c (resp., c_0) is the space of all convergent (resp., null) sequences. A linear functional $\mathrm{Lim}\colon \ell_\infty \to \mathbb{R}$ is called the Banach–Mazur limit (see [3, 10]) if it satisfies the following properties: for each $(x_k) \in \ell_\infty$, (i) if $x_k \ge 0$ for all k, then $\mathrm{Lim}(x_k) \ge 0$; (ii) $\mathrm{Lim}(e) = 1$, where $e = (1,1,1,\ldots)$; and (iii) $\mathrm{Lim}(x_k) = \mathrm{Lim}(x_{k+1})$. It follows that $\liminf_{k\to\infty} x_k \le \mathrm{Lim}(x_k) \le \limsup_{k\to\infty} x_k$. We denote by $\mathbb{L}$ the set of all Banach–Mazur limits.

Remark 2.2. Each Banach–Mazur limit is a positive linear extension to ℓ_∞ of the limit functional $\varphi\colon c \to \mathbb{R}$, $\varphi(x) = \lim_{k\to\infty} x_k$, for all $x = (x_k) \in c$. Moreover, this extension is not unique.

If $(x_k) \in \ell_\infty$ is periodic, i.e., there exists $p \in \mathbb{N}$ such that $x_k = x_{k+p}$ for all $k \in \mathbb{N}$, it is well-known that $\operatorname{Lim}(x_k) = \frac{1}{p}\sum_{k=1}^{p} x_k$.

LEMMA 2.3 (see [6, 8] and [3, Theorem 16.31]). *We have the following dual representations:*

(i) *For every $\lambda \in (c_0)^*$, there exists a unique element $\xi \in \ell_1$ such that $\|\xi\|_1 = \|\lambda\|_{(c_0)^*}$ and*

$$\lambda(x) = \sum_{k=1}^{\infty} \xi_k x_k \quad \text{for all } x \in c_0.$$

(ii) *For every $\lambda \in c^*$, there exists a unique pair $(\xi, \alpha) \in \ell_1 \times \mathbb{R}$ such that $\|\xi\|_1 + |\alpha| = \|\lambda\|_{c^*}$ and*

$$\lambda(x) = \sum_{k=1}^{\infty} \xi_k x_k + \alpha \lim_{k\to\infty} x_k \quad \text{for all } x \in c.$$

(iii) *For every $\lambda \in (\ell_\infty)^*$, there exists a unique element $(\xi, \alpha, \operatorname{Lim}) \in \ell_1 \times \mathbb{R} \times \mathbb{L}$ such that $\|\xi\|_1 + |\alpha| = \|\lambda\|_{(\ell_\infty)^*}$ and*

$$\lambda(x) = \sum_{k=1}^{\infty} \xi_k x_k + \alpha \operatorname{Lim}(x_k) \quad \text{for all } x \in \ell_\infty.$$

As a consequence of this result, we describe the topological dual spaces $(c_0^m)^*$, $(c^m)^*$, and $(\ell_\infty^m)^*$. To do this, given $x^\bullet = (x^1, \ldots, x^m) \in \ell_\infty^m$ where $x^i = (x_k^i) \in \ell_\infty$ for all $i \in \{1, 2, \ldots, m\}$, we denote $x_k^\bullet := (x_k^1, \ldots, x_k^m) \in \mathbb{R}^m$ for all component k. In addition, $\mathbb{L}^m$ stands for the class of mappings $\mathbb{L}\mathrm{im}\colon \ell_\infty^m \to \mathbb{R}^m$ defined by

$$\mathbb{L}\mathrm{im}(x^\bullet) := (\operatorname{Lim}^1(x^1), \ldots, \operatorname{Lim}^m(x^m)),$$

where $\mathbb{L}\mathrm{im} = (\operatorname{Lim}^1, \ldots, \operatorname{Lim}^m)$ and $\operatorname{Lim}^i \in \mathbb{L}$ for all $i \in \{1, 2, \ldots, m\}$.

THEOREM 2.4. *We have the following dual representations:*

(i) *For every $\lambda \in (c_0^m)^*$, there exists a unique element $\xi^\bullet = (\xi^1, \ldots, \xi^m) \in \ell_1^m$ such that $\sum_{i=1}^m \|\xi^i\|_1 = \|\lambda\|_{(c_0^m)^*}$ and*

$$\lambda(x^\bullet) = \sum_{i=1}^{m}\sum_{k=1}^{\infty} \xi_k^i x_k^i = \sum_{k=1}^{\infty} \langle \xi_k^\bullet, x_k^\bullet \rangle \quad \text{for all } x^\bullet = (x^1, \ldots, x^m) \in c_0^m.$$

(ii) *For every $\lambda \in (c^m)^*$, there exist unique elements $\xi^\bullet = (\xi^1, \ldots, \xi^m) \in \ell_1^m$ and $\alpha \in \mathbb{R}^m$ such that $\sum_{i=1}^m \|\xi^i\|_1 + \|\alpha\|_1 = \|\lambda\|_{(c^m)^*}$ and*

$$\begin{aligned} \lambda(x^\bullet) &= \sum_{i=1}^{m} \left(\sum_{k=1}^{\infty} \xi_k^i x_k^i + \alpha_i \lim_{k\to\infty} x_k^i \right) \\ &= \sum_{k=1}^{\infty} \langle \xi_k^\bullet, x_k^\bullet \rangle + \langle \alpha, \lim_{k\to\infty} x_k^\bullet \rangle \quad \text{for all } x^\bullet = (x^1, \ldots, x^m) \in c^m. \end{aligned} \tag{2.1}$$

(iii) *For every $\lambda \in (\ell_\infty^m)^*$, there exist unique elements $\xi^\bullet = (\xi^1, \ldots, \xi^m) \in \ell_1^m$, $\alpha \in \mathbb{R}^m$, and $\mathbb{L}\mathrm{im} = (\operatorname{Lim}^1, \ldots, \operatorname{Lim}^m) \in \mathbb{L}^m$ such that $\sum_{i=1}^m \|\xi^i\|_1 + \|\alpha\|_1 = \|\lambda\|_{(\ell_\infty^m)^*}$ and*

$$\lambda(x^\bullet) = \sum_{i=1}^{m}\left(\sum_{k=1}^{\infty}\xi_k^i x_k^i + \alpha_i \mathrm{Lim}^i(x^i)\right)$$
$$= \sum_{k=1}^{\infty}\langle \xi_k^\bullet, x_k^\bullet\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(x^\bullet)\rangle \; \textit{for all } x^\bullet = (x^1,\dots,x^m)\in \ell_\infty^m.$$

Proof. Let us only prove part (ii), as the others follow similarly.

We have that $(c^m)^*$ is isometrically isomorphic to $(c^*)^m$ (see [14, Exercise 5.21]) by the next application: for each $\lambda \in (c^m)^*$ there exists a unique element $(\lambda_1,\dots,\lambda_m)\in (c^*)^m$ such that

$$\lambda(x^\bullet) = \sum_{i=1}^{m}\lambda_i(x^i) \text{ for all } x^\bullet = (x^1,\dots,x^m)\in c^m. \tag{2.2}$$

Thus, by applying Lemma 2.3(ii) we deduce that for every λ_i there exists a unique pair $(\xi^i,\alpha_i)\in \ell_1\times\mathbb{R}$ such that $\|\xi^i\|_1 + |\alpha_i| = \|\lambda_i\|_{c^*}$ and

$$\lambda_i(x) = \sum_{k=1}^{\infty}\xi_k^i x_k + \alpha_i \lim_{k\to\infty} x_k \text{ for all } x=(x_k)\in c.$$

Hence,

$$\lambda(x^\bullet) = \sum_{i=1}^{m}\left(\sum_{k=1}^{\infty}\xi_k^i x_k^i + \alpha_i \lim_{k\to\infty} x_k^i\right) \text{ for all } x^\bullet = (x^1,\dots,x^m)\in c^m.$$

Then statement (2.1) is obtained by arranging the terms above. In addition, by (2.2) and Lemma 2.3 we have

$$\|\lambda\|_{(c^m)^*} = \sum_{i=1}^{m}\|\lambda_i\|_{c^*} = \sum_{i=1}^{m}\|\xi^i\|_1 + \sum_{i=1}^{m}|\alpha_i| = \sum_{i=1}^{m}\|\xi^i\|_1 + \|\alpha\|_1$$

and the result is stated. ∎

Remark 2.5. Although ℓ_∞^m and ℓ_∞ are isometrically isomorphic, allowing one to identify $(\ell_\infty^m)^*$ and $(\ell_\infty)^*$ we choose to retain the formulation because it clarifies the scalarization process and is essential for presenting the computational results in section 4.

3. Efficient solutions. In order to deal with the uncertain multiobjective optimization problem (UP) via the vector approach, it is usual to consider as a solution any nondominated feasible point, i.e., any point $x_0\in X$ whose image $f(x_0,u)$ cannot be improved for the value $f(x,u)$ at another alternative $x\in X$ for all scenario $u\in\mathcal{U}$ (see, for instance, [13, 22, 23]). Therefore, the next additional vector optimization problem associated to the nominal problem (UP) has to be addressed:

$$Q_D\text{-Minimize } F(x) \text{ subject to } x\in X, \tag{VP}$$

where $F: X\to \mathcal{B}_\mathcal{U}$ is the trajectory function $F(x) := f^x$ and $\emptyset \neq Q_D \subseteq \mathcal{B}_\mathcal{U}$ is the involved ordering set. In this setting, the basic approach to introduce notions of an efficient (nondominated) solution for problem (UP) is to apply the well-known solution concepts in vector optimization in terms of the (VP) problem (see [1, 2, 15, 20, 31]).

DEFINITION 3.1. *A point $x_0 \in X$ is said to be*

- *a strict efficient solution of problem* (UP) *with respect to Q_D, denoted by $x_0 \in \mathrm{SE}(Q_D)$, if there is not $x \in X\backslash\{x_0\}$ such that $f^x \precsim_{Q_D} f^{x_0}$,*
- *an efficient solution of problem* (UP) *with respect to Q_D, denoted by $x_0 \in \mathrm{E}(Q_D)$, if there is not $x \in X\backslash\{x_0\}$ such that $f^x \precnsim_{Q_D} f^{x_0}$.*
- *Assume that Q_D is solid. Then, x_0 is a weakly efficient solution of problem* (UP) *with respect to Q_D, denoted by $x_0 \in \mathrm{WE}(Q_D)$, if $x_0 \in \mathrm{E}(\mathrm{int}Q_D)$.*

Taking into account the abovementioned vector point of view, one could point out among the most interesting ordering sets Q_D the following cones:

$$\begin{aligned} P &:= \{p \in \mathcal{B}_{\mathcal{U}} \colon p(u) \in D(u) \text{ for all } u \in \mathcal{U}\}, \\ \hat{P} &:= \{p \in \mathcal{B}_{\mathcal{U}} \colon p(u) \in \mathrm{int}D(u) \cup \{0\} \text{ for all } u \in \mathcal{U}\}, \end{aligned} \tag{3.1}$$

where the cone $D(u)$ is assumed to be solid for all $u \in \mathcal{U}$ whenever $\hat{P}$ is considered. In addition, we also are interested in the cones that result from removing the point $0 \in \mathbb{R}^m$ in each component $u \in \mathcal{U}$ and adding $0 \in \mathcal{B}_{\mathcal{U}}$:

$$\begin{aligned} P' &:= \{p \in \mathcal{B}_{\mathcal{U}} \colon p(u) \in D(u)\backslash\{0\} \text{ for all } u \in \mathcal{U}\} \cup \{0\}, \\ \hat{P}' &:= \{p \in \mathcal{B}_{\mathcal{U}} \colon p(u) \in \mathrm{int}D(u) \text{ for all } u \in \mathcal{U}\} \cup \{0\}. \end{aligned}$$

Finally, the projection onto a nonempty set $\bar{\mathcal{U}} \subseteq \mathcal{U}$ of each cone above will also be considered. For instance,

$$P'_{\bar{\mathcal{U}}} := \{p \in \mathcal{B}_{\mathcal{U}} \colon p(u) \in D(u)\backslash\{0\} \text{ for all } u \in \bar{\mathcal{U}}\} \cup \{0\}$$

is the projection on $\bar{\mathcal{U}}$ of P'.

Remark 3.2. Some particular cases of the concepts in Definition 3.1 in connection with the above cones have already been introduced in the literature. For instance, [13, Definition 3.4] considers $D(u) = \mathbb{R}^m_+$ for all $u \in \mathcal{U}$ and the solution sets E_i, $i \in \{1, 2, \ldots, 6\}$. It is not hard to check that $E_1 = \mathrm{SE}(P)$, $E_2 = \mathrm{E}(P)$, $E_3 = \mathrm{E}(\bigcup_{u\in\mathcal{U}}(P \cap \hat{P}'_{\{u\}}))$, $E_4 = \mathrm{E}(P')$, $E_5 = \mathrm{E}(\bigcup_{u\in\mathcal{U}}(P' \cap \hat{P}'_{\{u\}}))$, and $E_6 = \mathrm{E}(\hat{P}')$.

The following lemma is required in what follows to give relationships between sets of efficient solutions.

LEMMA 3.3. *Consider $\emptyset \neq \bar{\mathcal{U}} \subseteq \mathcal{U}$. We have the next basic properties:*

(i) *It follows that $P'_{\bar{U}} \subseteq P_{\bar{U}}$, $\hat{P}'_{\bar{U}} \subseteq \hat{P}_{\bar{U}}$, $\hat{P}_{\bar{\mathcal{U}}} \subseteq P_{\bar{\mathcal{U}}}$, and $\hat{P}'_{\bar{\mathcal{U}}} \subseteq P'_{\bar{\mathcal{U}}}$. It is also true that $P_{\bar{U}_2} \subseteq P_{\bar{U}_1}$, $\hat{P}_{\bar{U}_2} \subseteq \hat{P}_{\bar{U}_1}$, $P'_{\bar{U}_2} \subseteq P'_{\bar{U}_1}$, and $\hat{P}'_{\bar{U}_2} \subseteq \hat{P}'_{\bar{U}_1}$ for all $\bar{U}_1 \subseteq \bar{U}_2 \subseteq \mathcal{U}$.*

(ii) *Given $g \in \mathcal{B}_{\mathcal{U}}$ and $\varepsilon > 0$, we have*

$$\mathbb{B}_\infty(g, \varepsilon) \subseteq P_{\bar{\mathcal{U}}} \iff g(u) + \varepsilon\mathbb{B}_\infty \subseteq D(u) \text{ for all } u \in \bar{\mathcal{U}}. \tag{3.2}$$

(iii) *$P_{\bar{\mathcal{U}}}, \hat{P}_{\bar{\mathcal{U}}}, P'_{\bar{\mathcal{U}}}$, and $\hat{P}'_{\bar{\mathcal{U}}}$ are convex cones. In addition, $P_{\bar{\mathcal{U}}}$ is closed whenever $D(u)$ is closed, for all $u \in \bar{\mathcal{U}}$.*

(iv) *We have that $\mathrm{int}P_{\bar{\mathcal{U}}} = \mathrm{int}P'_{\bar{\mathcal{U}}} = \mathrm{int}\hat{P}_{\bar{\mathcal{U}}} = \mathrm{int}\hat{P}'_{\bar{\mathcal{U}}}$. In addition, if $\bar{\mathcal{U}}$ is finite, then $\hat{P}'_{\bar{\mathcal{U}}} = \mathrm{int}P_{\bar{\mathcal{U}}} \cup \{0\}$.*

(v) *Suppose that $\#\mathcal{U} = \infty$. If $y \in \mathrm{int}\bigcap_{u\in\bar{\mathcal{U}}} D(u)$, then the mapping $p \in \mathcal{B}_{\mathcal{U}}$ given by $p(u) = y$, for all $u \in \mathcal{U}$, belongs to $\mathrm{int}P_{\bar{\mathcal{U}}}$. In particular, $\mathrm{int}P_{\bar{\mathcal{U}}} \neq \emptyset$.*

Proof. Part (i) is an obvious consequence of the definitions of the involved cones. (ii) Clearly,

$$\mathbb{B}_\infty(g, \varepsilon) = \{h \in \mathcal{B}_{\mathcal{U}} : \|h(u) - g(u)\|_\infty \leq \varepsilon \text{ for all } u \in \mathcal{U}\}. \tag{3.3}$$

Therefore, $h \in \mathbb{B}_\infty(g, \varepsilon)$ if and only if $h(u) \in g(u) + \varepsilon\mathbb{B}_\infty$ for all $u \in \mathcal{U}$.

We claim that $g(u)+\varepsilon\mathbb{B}_\infty \subseteq D(u)$ for all $u\in\bar{\mathcal{U}}$ whenever $\mathbb{B}_\infty(g,\varepsilon)\subseteq P_{\bar{\mathcal{U}}}$. Indeed, consider $u\in\bar{\mathcal{U}}$ and $y\in\mathbb{B}_\infty$. The function $h\in\mathcal{B}_\mathcal{U}$ given by $h(u)=g(u)+\varepsilon y$ and $h(u')=g(u')$ for all $u'\neq u$ belongs to $\mathbb{B}_\infty(g,\varepsilon)$. Thus, $h\in P_{\bar{\mathcal{U}}}$ and so $h(u)\in D(u)$. As y is an arbitrary point in $\mathbb{B}_\infty$ it follows that $g(u)+\varepsilon\mathbb{B}_\infty\subseteq D(u)$.

Conversely, assume that $g(u)+\varepsilon\mathbb{B}_\infty\subseteq D(u)$ for all $u\in\bar{\mathcal{U}}$, and consider $h\in\mathbb{B}_\infty(g,\varepsilon)$. By (3.3) it follows that $h(u)\in g(u)+\varepsilon\mathbb{B}_\infty$ and so $h(u)\in D(u)$ for all $u\in\bar{\mathcal{U}}$, i.e., $h\in P_{\bar{\mathcal{U}}}$. Therefore, claim (3.2) is proved.

(iii) Let us prove the closedness of $P_{\bar{\mathcal{U}}}$, since the other properties are obvious results of the definitions. Thus, assume that $D(u)$ is closed for all $u\in\bar{\mathcal{U}}$. Suppose reasoning by contradiction that a sequence $(p_k)_k$ in $P_{\bar{\mathcal{U}}}$ converges to $p\in\mathcal{B}_\mathcal{U}$ and $p\notin P_{\bar{\mathcal{U}}}$. Then there exists $\bar{u}\in\bar{\mathcal{U}}$ such that $p(\bar{u})\notin D(\bar{u})$. Since $D(\bar{u})$ is closed there exists $\varepsilon>0$ such that $(p(\bar{u})+\varepsilon\mathbb{B}_\infty)\cap D(\bar{u})=\emptyset$ and so $\|p_k(\bar{u})-p(\bar{u})\|_\infty>\varepsilon$ since $p_k(\bar{u})\in D(\bar{u})$ for all k. Therefore,

$$\|p_k-p\|_\infty=\sup_{u\in\mathcal{U}}\|p_k(u)-p(u)\|_\infty\geq\|p_k(\bar{u})-p(\bar{u})\|_\infty>\varepsilon \text{ for all } k,$$

which is a contradiction as $p_k\to p$. Thus, $p\in P_{\bar{\mathcal{U}}}$ and it follows that $P_{\bar{\mathcal{U}}}$ is closed.

(iv) Let $p\in\mathcal{B}_\mathcal{U}$ be a point in $\operatorname{int}P_{\bar{\mathcal{U}}}$. Then, there exists $\varepsilon>0$ such that $\mathbb{B}_\infty(p,\varepsilon)\subseteq P_{\bar{\mathcal{U}}}$, and by assertion (3.2) we see that $p\in\hat{P}'_{\bar{\mathcal{U}}}$. Thus, $\operatorname{int}P_{\bar{\mathcal{U}}}\subseteq\hat{P}'_{\bar{\mathcal{U}}}$ and by part (i) we have $\hat{P}'_{\bar{\mathcal{U}}}\subseteq\hat{P}_{\bar{\mathcal{U}}}\subseteq P_{\bar{\mathcal{U}}}$ and $\hat{P}'_{\bar{\mathcal{U}}}\subseteq P'_{\bar{\mathcal{U}}}\subseteq P_{\bar{\mathcal{U}}}$. It follows that $\operatorname{int}P_{\bar{\mathcal{U}}}=\operatorname{int}\hat{P}'_{\bar{\mathcal{U}}}=\operatorname{int}\hat{P}_{\bar{\mathcal{U}}}=\operatorname{int}P'_{\bar{\mathcal{U}}}$.

Assume that $\bar{\mathcal{U}}$ is finite. In order to state the inclusion $\hat{P}'_{\bar{\mathcal{U}}}\subseteq\operatorname{int}P_{\bar{\mathcal{U}}}\cup\{0\}$, consider p such that $p(u)\in\operatorname{int}D(u)$ for all $u\in\bar{\mathcal{U}}$. As $\bar{\mathcal{U}}$ is finite, there exists $\varepsilon>0$ such that $p(u)+\varepsilon\mathbb{B}_\infty\subseteq D(u)$ for all $u\in\bar{\mathcal{U}}$. By assertion (3.2) we see that $p\in\operatorname{int}P_{\bar{\mathcal{U}}}$ and so $\hat{P}'_{\bar{\mathcal{U}}}\subseteq\operatorname{int}P_{\bar{\mathcal{U}}}\cup\{0\}$. In addition, the converse inclusion follows as $\operatorname{int}P_{\bar{\mathcal{U}}}\subseteq\hat{P}'_{\bar{\mathcal{U}}}$ and the proof of part (iv) is completed.

(v) Consider the mapping $p(u)=y$ for all $u\in\mathcal{U}$, where $y\in\operatorname{int}\bigcap_{u\in\bar{\mathcal{U}}}D(u)$. There exists $\varepsilon>0$ such that $p(u)+\varepsilon\mathbb{B}_\infty\subseteq D(u)$ for all $u\in\bar{\mathcal{U}}$. Clearly, by (3.2) we see that $\mathbb{B}_\infty(p,\varepsilon)\subseteq P_{\bar{\mathcal{U}}}$ and the result is proved. ∎

Remark 3.4. Concerning Lemma 3.3(iv), we deduce if $\bar{\mathcal{U}}$ is not finite, the equality $\operatorname{int}P_{\bar{\mathcal{U}}}\cup\{0\}=\hat{P}'_{\bar{\mathcal{U}}}$ is not true in general. For instance, consider $m=2$, $\bar{\mathcal{U}}=\mathcal{U}=\mathbb{N}$, $D(n)=\mathbb{R}^2_+$ and the mapping $p\in\hat{P}'$, $p(n)=(1/n,1)$, for all n. It follows that $p\notin\operatorname{int}P\cup\{0\}$. Indeed, for each $\varepsilon>0$ we have that

$$p(n)+\varepsilon\mathbb{B}_\infty\subseteq D(n)\iff\varepsilon\leq 1/n.$$

Thus, if $\mathbb{B}_\infty(p,\varepsilon)\subseteq P$ for some $\varepsilon>0$, by statement (3.2) we deduce that $\varepsilon\leq 0$, which is a contradiction. As a result of this example, we see that, in general, $\hat{P}'_{\bar{\mathcal{U}}}\backslash\{0\}$ is not an open set. Otherwise, $p\in\hat{P}'\backslash\{0\}=\operatorname{int}(\hat{P}'\backslash\{0\})=\operatorname{int}P\subseteq\operatorname{int}P\cup\{0\}$, a contradiction.

In what follows, we introduce new notions of an efficient solution of problem (UP) that provide smaller efficient solution sets than the ones in Definition 3.1. Roughly speaking, the basic idea is that a point $x\in X$ could be rejected as an efficient solution of problem (UP) if there exists a selection $s\colon\mathcal{U}\to X$ such that for each scenario $u\in\mathcal{U}$ the value $f(s(u),u)$ improves the value $f(x,u)$. Thus, we replace $f^x\colon\mathcal{U}\to\mathbb{R}^m$ in Definition 3.1 by a trajectory $f(s(\cdot),\cdot)\colon\mathcal{U}\to\mathbb{R}^m$ determined by choosing for each scenario $u\in\mathcal{U}$ a feasible point $s(u)\in X$.

Throughout, we consider selections s from $\mathcal{U}$ to the feasible set X satisfying the following property:

$$\mathcal{S}^b\colon=\{s\colon\mathcal{U}\to X\colon(f(s(u),u))\in\mathcal{B}_\mathcal{U}\}.$$

Thus, if $s \in \mathcal{S}^b$ we denote by $f^s : \mathcal{U} \to \mathbb{R}^m$ the mapping $f^s(u) = f(s(u), u)$ for all $u \in \mathcal{U}$. Another important set concerns constant selections:

$$\mathcal{S}^c :\, = \{s \in \mathcal{S}^b : s \text{ is constant}\}.$$

Each element in $\mathcal{S}^c$ is denoted as its corresponding value, i.e., $x \in \mathcal{S}^c$ stands for the selection $s(u) = x$ for all $u \in \mathcal{U}$.

Definition 3.5. *Consider $\mathcal{S}^c \subseteq \mathcal{S} \subseteq \mathcal{S}^b$. A point $x_0 \in X$ is said to be*

- *a highly strict efficient solution of problem* (UP) *with respect to* $(Q_D, \mathcal{S})$, *denoted by* $x_0 \in \mathrm{HSE}(Q_D, \mathcal{S})$, *if there is not* $s \in \mathcal{S}\backslash\{x_0\}$ *such that* $f^s \precsim_{Q_D} f^{x_0}$,
- *a highly efficient solution of problem* (UP) *with respect to* $(Q_D, \mathcal{S})$, *denoted by* $x_0 \in \mathrm{HE}(Q_D, \mathcal{S})$, *if there is not* $s \in \mathcal{S}\backslash\{x_0\}$ *such that* $f^s \precsim_{Q_D} f^{x_0}$ (with $f^{x_0} \not\precsim_{Q_D} f^s$).
- *Assume that* Q_D *is solid. Then,* x_0 *is a highly weakly efficient solution of problem* (UP) *with respect to* $(Q_D, \mathcal{S})$, *denoted by* $x_0 \in \mathrm{HWE}(Q_D, \mathcal{S})$, *if* $x_0 \in \mathrm{HE}(\mathrm{int} Q_D, \mathcal{S})$.

Notice that Definition 3.5 covers the usual concepts of efficient solution to the following vector optimization problem associated to the nominal problem (UP):

$$\text{(GVP)} \qquad Q_D\text{-Minimize } \widetilde{F}(s) \text{ subject to } s \in \mathcal{S},$$

where $\widetilde{F} : \mathcal{S}^b \to \mathcal{B}_{\mathcal{U}}$ is the function $\widetilde{F}(s) := f(s(\cdot), \cdot)$. Clearly, problem (GVP) reduces to problem (VP) by taking $\mathcal{S} = \mathcal{S}^c$.

Next, some relationships between the sets of efficient solutions of problem (UP) introduced in Definitions 3.1, 3.5, and 2.1 are stated.

Theorem 3.6. *Consider $\emptyset \neq \bar{\mathcal{U}} \subseteq \mathcal{U}$ and a set $\mathcal{S}^c \subseteq \mathcal{S} \subseteq \mathcal{S}^b$. The following properties hold true:*

(i) $\mathrm{SE}(Q_D) = \mathrm{HSE}(Q_D, \mathcal{S}^c)$, $\mathrm{E}(Q_D) = \mathrm{HE}(Q_D, \mathcal{S}^c)$, $\mathrm{WE}(Q_D) = \mathrm{HWE}(Q_D, \mathcal{S}^c)$, *and* $\mathrm{HSE}(Q_D, \mathcal{S}) \subseteq \mathrm{HE}(Q_D, \mathcal{S}) \subseteq \mathrm{HWE}(Q_D, \mathcal{S})$. *In addition, if* $0 \notin Q_D$, *then*

$$\mathrm{HSE}(Q_D, \mathcal{S}) = \mathrm{HE}(Q_D, \mathcal{S}) = \{x \in X : f^s \not\precsim_{Q_D} f^x \text{ for all } s \in \mathcal{S}\}.$$

(ii) *Let* $Q_D^1 \subseteq Q_D^2 \subseteq \mathcal{B}_{\mathcal{U}}$ *be two ordering sets and* $\mathcal{S}^c \subseteq \mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}^b$. *Then* $\mathrm{HSE}(Q_D^2, \mathcal{S}_2) \subseteq \mathrm{HSE}(Q_D^1, \mathcal{S}_1)$, $\mathrm{HE}(Q_D^2, \mathcal{S}_2) \subseteq \mathrm{HE}(Q_D^1, \mathcal{S}_1)$, $\mathrm{HWE}(Q_D^2, \mathcal{S}_2) \subseteq \mathrm{HWE}(Q_D^1, \mathcal{S}_1)$.

(iii) $\mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) \subseteq \mathrm{HE}(\hat{P}_{\bar{\mathcal{U}}}, \mathcal{S}) \cup \mathrm{HE}(P'_{\bar{\mathcal{U}}}, \mathcal{S}) \subseteq \mathrm{HE}(\hat{P}'_{\bar{\mathcal{U}}}, \mathcal{S})$.

(iv) $\mathrm{HWE}(\hat{P}'_{\bar{\mathcal{U}}}, \mathcal{S}) = \mathrm{HWE}(\hat{P}_{\bar{\mathcal{U}}}, \mathcal{S}) = \mathrm{HWE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) = \mathrm{HWE}(P'_{\bar{\mathcal{U}}}, \mathcal{S})$. *In addition, if* $\bar{\mathcal{U}}$ *is finite, then* $\mathrm{HWE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) = \mathrm{HE}(\hat{P}'_{\bar{\mathcal{U}}}, \mathcal{S})$.

(v) *If* $\mathcal{S}$ *contains all selections* $s : \mathcal{U} \to X$ *that are constant except possibly at a scenario* $u \in \bar{\mathcal{U}}$ *it follows that*

$$\mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) = \bigcap_{u \in \bar{\mathcal{U}}} \mathrm{E}(D_u),$$

$$\mathrm{HE}(\hat{P}_{\bar{\mathcal{U}}}, \mathcal{S}) = \bigcap_{u \in \bar{\mathcal{U}}} \mathrm{WE}(D_u),$$

$$\mathrm{HE}(P \cap \hat{P}_{\bar{\mathcal{U}}}, \mathcal{S}) = \bigcap_{u \in \bar{\mathcal{U}}} \mathrm{WE}(D_u) \cap \bigcap_{u \in \mathcal{U} \backslash \bar{\mathcal{U}}} \mathrm{E}(D_u).$$

(vi) $\mathrm{SE}(P_{\{u\}}) = \mathrm{SE}(D_u)$, $\mathrm{E}(P_{\{u\}}) = \mathrm{E}(D_u)$, $\mathrm{WE}(P_{\{u\}}) = \mathrm{E}(\hat{P}'_{\{u\}}) = \mathrm{WE}(D_u)$ *for all* $u \in \mathcal{U}$.

Proof. Parts (i)–(iv) and (vi) can be easily stated by Lemma 3.3 and the corresponding definitions.

(v) Let us start proving the first equality. Suppose that $x_0 \in \mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S})$ and consider $\bar{u} \in \bar{\mathcal{U}}$ and $x \in X$ such that $f^{\bar{u}}(x) \leq_{D_{\bar{u}}} f^{\bar{u}}(x_0)$. Define $s \in \mathcal{S}$ as follows: $s(\bar{u}) = x$ and $s(u) = x_0$ for all $u \in \mathcal{U} \backslash \{\bar{u}\}$. Clearly, $f^s \precsim_{P_{\bar{\mathcal{U}}}} f^{x_0}$ and then $f^s(u) = f^{x_0}(u)$ for all $u \in \bar{\mathcal{U}}$, as x_0 is a highly efficient solution of problem (UP) with respect to $(P_{\bar{\mathcal{U}}}, \mathcal{S})$. Therefore,

$$f^{\bar{u}}(x) = f(x, \bar{u}) = f(s(\bar{u}), \bar{u}) = f(x_0, \bar{u}) = f^{\bar{u}}(x_0)$$

and we have $x_0 \in \mathrm{E}(D_{\bar{u}})$. As $\bar{u} \in \bar{\mathcal{U}}$ was arbitrarily chosen, we obtain $x_0 \in \bigcap_{u \in \bar{\mathcal{U}}} \mathrm{E}(D_u)$.

Conversely, consider $x_0 \in \bigcap_{u \in \bar{\mathcal{U}}} \mathrm{E}(D_u)$ and $s \in \mathcal{S}$ such that $f^s \precsim_{P_{\bar{\mathcal{U}}}} f^{x_0}$. Therefore, $f^u(s(u)) \leq_{D_u} f^u(x_0)$ and so $f^u(s(u)) = f^u(x_0)$ since $x_0 \in \mathrm{E}(D_u)$ for all $u \in \bar{\mathcal{U}}$. Thus, $f^s(u) = f^{x_0}(u)$ for all $u \in \bar{\mathcal{U}}$, and it follows that $x_0 \in \mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S})$.

The remaining equalities in (v) are particular cases of the first one. For instance, the second one can be deduced from the first one by considering $\mathrm{int} D_u \cup \{0\}$ instead of $D(u)$ in the definition of the cone P. ∎

By parts (i), (iii), and (iv) of Theorem 3.6 we deduce that

$$\mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) \subseteq \mathrm{HE}(\hat{P}_{\bar{\mathcal{U}}}, \mathcal{S}) \subseteq \mathrm{HE}(\hat{P}'_{\bar{\mathcal{U}}}, \mathcal{S}) \subseteq \mathrm{HWE}(P_{\bar{\mathcal{U}}}, \mathcal{S}).$$

Hence, to approximate the set of highly efficient solutions of problem (UP) with respect to $(P_{\bar{\mathcal{U}}}, \mathcal{S})$, it is preferable to consider highly efficient solutions with respect to the cones $\hat{P}_{\bar{\mathcal{U}}}$ and $\hat{P}'_{\bar{\mathcal{U}}}$ instead of highly weakly efficient solutions of problem (UP) with respect to $P_{\bar{\mathcal{U}}}$.

In [17, 19], Ide and Schöbel introduced the intuitive concepts of highly and flimsily robust efficient solutions of an uncertain Pareto problem and finitely many scenarios. Namely, a point $x_0 \in X$ is said to be a highly (resp., flimsily) robust efficient solution of problem (UP) where $D(u) = \mathbb{R}^m_+$, for all $u \in \mathcal{U}$, if $x_0 \in \bigcap_{u \in \mathcal{U}} \mathrm{E}(D_u)$ (resp., $x_0 \in \bigcup_{u \in \mathcal{U}} \mathrm{E}(D_u)$). By the first equality of Theorem 3.6(v) it is clear that such efficient solutions can be rewriten as a particular case of highly efficient solutions of problem (UP) with respect to $(P, \mathcal{S})$ by considering $D(u) = \mathbb{R}^m_+$ for all $u \in \mathcal{U}$ and a certain family $\mathcal{S}$.

Theorem 3.6 clarifies and extends the main results of [13, section 3]. Specifically, we have the following inclusions and equalities:

$$\begin{gathered} \hat{P}' \subseteq \bigcup_{u \in \mathcal{U}} (P' \cap \hat{P}'_{\{u\}}) \subseteq \bigcup_{u \in \mathcal{U}} (P \cap \hat{P}'_{\{u\}}) \subseteq P, \\ \bigcup_{u \in \mathcal{U}} (P' \cap \hat{P}'_{\{u\}}) \subseteq P' \subseteq P, \\ P = \bigcap_{u \in \mathcal{U}} P_{\{u\}}, \quad P' = \bigcap_{u \in \mathcal{U}} P'_{\{u\}}, \quad \hat{P}' = \bigcap_{u \in \mathcal{U}} \hat{P}'_{\{u\}}. \end{gathered} \tag{3.4}$$

Therefore, by applying parts (i), (ii), and (vi) of Theorem 3.6 and Remark 3.2 with $D_u = \mathbb{R}^m_+$, for all $u \in \mathcal{U}$, and $\mathcal{S} = \mathcal{S}^c$ we obtain Propositions 3.6 and 3.10 given in [13],

$$\begin{gathered} E_1 \subseteq E_2 \subseteq E_3 \subseteq E_5 \subseteq E_6, \\ E_2 \subseteq E_4 \subseteq E_5, \\ \bigcup_{u \in \mathcal{U}} \mathrm{SE}(D_u) \subseteq E_1, \quad \bigcup_{u \in \mathcal{U}} \mathrm{E}(D_u) \subseteq E_4, \quad \bigcup_{u \in \mathcal{U}} \mathrm{WE}(D_u) \subseteq E_6. \end{gathered}$$

Notice that by (3.4) and parts (i), (ii), and (vi) of Theorem 3.6, one can derive some relationships between the set of efficient solutions and flimsily robust efficient solutions of problem (UP).

Theorem 3.6(v) shows that the notions of highly efficient solution in Definition 3.5 could not depend on the family $\mathcal{S}$. For instance, $\mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S}) = \mathrm{HE}(P_{\bar{\mathcal{U}}}, \mathcal{S}^b)$ provided that $\mathcal{S}$ contains all functions that are constant except possibly at a scenario.

4. Characterizations through linear scalarization.

In this section, weakly efficient solutions of problem (UP) with respect to the cone P defined in (3.1) are characterized as solutions of certain associated scalar optimization problems. For this aim, a linear scalarization approach is considered, which works whenever the involved problem is convex.

4.1. Necessary conditions.

As usual, we derive necessary conditions for weakly efficient solutions via linear scalarization when problem (UP) satisfies some convexity assumptions. There are in the literature different concepts to deal with convex vector optimization problems (see [16] and the references therein). Next, we formulate the more important ones in the setting of the Banach space $\mathcal{B}_{\mathcal{U}}$.

Concerning problem (VP) we denote

$$\mathcal{B}_{\mathcal{U}}(f) := \{f^x : \mathcal{U} \to \mathbb{R}^m : x \in X\} \subseteq \mathcal{B}_{\mathcal{U}}.$$

Definition 4.1. *Consider problem* (VP). $f : X \times \mathcal{U} \to \mathbb{R}^m$ *is said to be*

- Q_D*-convex if* $f^{\alpha x_1 + (1-\alpha)x_2} \precsim_{Q_D} \alpha f^{x_1} + (1-\alpha) f^{x_2}$ *for all* $x_1, x_2 \in X$*, for all* $\alpha \in (0,1)$*, where* X *is a convex set,*
- Q_D*-convexlike if* $\mathcal{B}_{\mathcal{U}}(f) + Q_D$ *is a convex set,*
- Q_D*-subconvexlike if* Q_D *is solid and* $\mathcal{B}_{\mathcal{U}}(f) + \mathrm{int} Q_D$ *is a convex set,*
- *generalized* Q_D*-subconvexlike if* Q_D *is solid and* $\mathrm{cone}\mathcal{B}_{\mathcal{U}}(f) + \mathrm{int} Q_D$ *is a convex set,*
- *nearly* Q_D*-subconvexlike if* $\mathrm{clcone}(\mathcal{B}_{\mathcal{U}}(f) + Q_D)$ *is a convex set.*

Remark 4.2. When the ordering set Q_D is a convex cone we have the following relationships:

$$\begin{aligned} f \text{ is } Q_D\text{-convex} &\Rightarrow f \text{ is } Q_D\text{-convexlike} \\ &\Rightarrow f \text{ is nearly } Q_D\text{-subconvexlike.} \end{aligned}$$

If, in addition, Q_D is solid, then

$$\begin{aligned} f \text{ is } Q_D\text{-convexlike } &\Rightarrow f \text{ is } Q_D\text{-subconvexlike} \\ &\Rightarrow f \text{ is generalized } Q_D\text{-subconvexlike} \\ &\Longleftrightarrow f \text{ is nearly } Q_D\text{-subconvexlike.} \end{aligned}$$

Indeed, it follows that the concepts in Definition 4.1 correspond to function F in problem (VP) to be Q_D-convex, Q_D-convexlike, Q_D-subconvexlike, generalized Q_D-subconvexlike, and nearly Q_D-subconvexlike, and then the relationships above are particular cases of well-known implications between generalized convexity notions of a vector-valued function (see [16, section 2] and the references therein). In particular, notice that the generalized Q_D-subconvexlikeness is the weakest one whenever Q_D is solid.

The next proposition provides a sufficient condition to check if f is P-convex.

Proposition 4.3. *Suppose that* X *is convex. If* f^u *is* D_u*-convex for all* $u \in \mathcal{U}$*, then* f *is* P*-convex.*

Proof. Consider $x_1, x_2 \in X$ and $\alpha \in (0,1)$. As f^u is D_u-convex, it follows that

$$\alpha f(x_1,u) + (1-\alpha) f(x_2,u) \in f(\alpha x_1 + (1-\alpha) x_2, u) + D_u \text{ for all } u \in \mathcal{U}.$$

Define $p : \mathcal{U} \to \mathbb{R}^m$, $p(u) := \alpha f(x_1,u) + (1-\alpha) f(x_2,u) - f(\alpha x_1 + (1-\alpha) x_2, u)$. Clearly, $p(u) \in D(u)$ for all $u \in \mathcal{U}$, and

$$\|p(u)\|_\infty \leq \alpha \|f^{x_1}\|_\infty + (1-\alpha)\|f^{x_2}\|_\infty + \|f^{\alpha x_1 + (1-\alpha)x_2}\|_\infty < +\infty \text{ for all } u \in \mathcal{U},$$

since $f^x \in \mathcal{B}_\mathcal{U}$ for all $x \in X$. Therefore,

$$\alpha f^{x_1} + (1-\alpha) f^{x_2} = f^{\alpha x_1 + (1-\alpha) x_2} + p \in f^{\alpha x_1 + (1-\alpha) x_2} + P$$

and f is P-convex. ∎

In what follows, necessary conditions for weakly efficient solutions of problem (UP) are stated, which are based in the case $\#\mathcal{U} = r < +\infty$ on solutions of the family of scalar optimization problems

$$(\mathrm{SP}\lambda_1) \qquad \operatorname{Min}\{\Phi_{\lambda_1}(x) : x \in X\},$$

where $\lambda_1 = (\xi^1, \xi^2, \ldots, \xi^r) \in (\mathbb{R}^m)^r$ and $\Phi_{\lambda_1} : X \to \mathbb{R}$,

$$\Phi_{\lambda_1}(x) := \sum_{k=1}^{r} \langle \xi^k, f(x,k)\rangle \text{ for all } x \in X,$$

and in the case $\#\mathcal{U} = +\infty$ on solutions of the families

$$(\mathrm{SP}\lambda_2) \qquad \operatorname{Min}\{\Phi_{\lambda_2}(x) : x \in X\}$$

and

$$(\mathrm{SP}\lambda_3) \qquad \operatorname{Min}\{\Phi_{\lambda_3}(x) : x \in X\},$$

where $\lambda_2 = (\xi^\bullet, \alpha) \in \ell_1^m \times \mathbb{R}^m$, $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$, $(\xi^\bullet, \alpha) \neq (0,0)$, and

$$\Phi_{\lambda_2}(x) := \sum_{k=1}^{\infty} \langle \xi_k^\bullet, f(x,k)\rangle + \left\langle \alpha, \lim_{k\to\infty} f(x,k) \right\rangle,$$

$$\Phi_{\lambda_3}(x) := \sum_{k=1}^{\infty} \langle \xi_k^\bullet, f(x,k)\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(f^x)\rangle \text{ for all } x \in X$$

where $\mathbb{L}\mathrm{im}(f^x)$ denotes $\mathbb{L}\mathrm{im}((f(x,k))_k)$. Notice that all problems correspond to linear scalarizations of the vector optimization problem (VP) and $Q_D = P$, which are defined by points λ_i in the topological dual space $\mathcal{B}_\mathcal{U}^*$. Namely, if $\#\mathcal{U} = r < +\infty$,

$$\lambda_1(y^\bullet) = \sum_{k=1}^{r} \langle \xi^k, y^k \rangle \text{ for all } y^\bullet = (y^1, y^2, \ldots, y^r) \in (\mathbb{R}^m)^r,$$

and if $\#\mathcal{U} = +\infty$,

$$\lambda_2(y^\bullet) = \sum_{k=1}^{\infty} \langle \xi_k^\bullet, y_k^\bullet \rangle + \left\langle \alpha, \lim_{k\to\infty} y_k^\bullet \right\rangle,$$

$$\lambda_3(y^\bullet) = \sum_{k=1}^{\infty} \langle \xi_k^\bullet, y_k^\bullet \rangle + \langle \alpha, \mathbb{L}\mathrm{im}(y^\bullet)\rangle \text{ for all } y^\bullet = (y^1, y^2, \ldots, y^m) \in \ell_\infty^m.$$

Clearly, $\lambda_2(y^\bullet) = \lambda_3(y^\bullet)$ for all $y^\bullet$ such that the sequence $(y^\bullet_k)_k$ converges and $\lambda_2 = \lambda_3$ as long as $\alpha = 0$. The set of solutions of problem $(\mathrm{SP}_{\lambda_i})$ is denoted by $\mathrm{argmin}\Phi_{\lambda_i}$:

$$\mathrm{argmin}\Phi_{\lambda_i} := \{x_0 \in X : \Phi_{\lambda_i}(x_0) \le \Phi_{\lambda_i}(x) \text{ for all } x \in X\}.$$

Next, optimality conditions for weakly efficient solutions of problem (UP) with respect to P in the case $\#\mathcal{U} = +\infty$ are obtained. For this aim, the formulation of the positive and the strict positive polar cone of P will be required. We denote

$$\underline{D} := \{y \in \mathbb{R}^m : \exists p^\bullet \in P \text{ s.t. } \lim_{k\to\infty} p^\bullet_k = y\},$$

and for each functional $\mathbb{L}\mathrm{im} \in \mathbb{L}^m$,

$$\mathbb{L}\mathrm{im}(P) := \{\mathbb{L}\mathrm{im}(p^\bullet) : p^\bullet \in P\}.$$

It follows that $\mathbb{L}\mathrm{im}(P)$ is a convex cone and $\underline{D} \subseteq \mathbb{L}\mathrm{im}(P)$.

THEOREM 4.4. *Assume that* $\mathcal{U} = \mathbb{N}$. *We have*

$$\begin{aligned} P^+ &= \{(\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m : \xi^\bullet_n \in D_n^+, \alpha \in \mathbb{L}\mathrm{im}(P)^+\}, \\ P^{+s} &= \{(\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m : \xi^\bullet_n \in D_n^{+s}, \alpha \in \mathbb{L}\mathrm{im}(P)^+\}. \end{aligned} \tag{4.1}$$

Proof. Concerning statement (4.1), let us only prove inclusion $\subseteq$ since the other one is obvious. Consider $\lambda \in P^+$. By Lemma 2.4(iii) there exists $(\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ such that

$$\lambda(x^\bullet) = \sum_{k=1}^{\infty} \langle \xi^\bullet_k, x^\bullet_k\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(x^\bullet)\rangle \text{ for all } x^\bullet \in \ell_\infty^m,$$

and $\lambda(p^\bullet) \ge 0$ for all $p^\bullet \in P$. For each n consider an arbitrary element $d_n \in D_n$ and the point $q^{n\bullet} \in P$ defined by $q^{n\bullet}_n = d_n$ and $q^{n\bullet}_k = 0$ for all $k \ne n$. Clearly, $\langle \xi^\bullet_n, d_n\rangle = \lambda(q^{n\bullet}) \ge 0$ and so $\xi^\bullet_n \in D_n^+$ for all n.

In addition, let $d \in \mathbb{L}\mathrm{im}(P)$. There exists a point $p^\bullet \in P$ satisfying $d = \mathbb{L}\mathrm{im}(p^\bullet)$. For each n define $q^{n\bullet} \in P$ by $q^{n\bullet}_k = p^\bullet_k$ if $k \le n$ and $q^{n\bullet}_k = 0$ otherwise. Clearly, $p^\bullet - q^{n\bullet} \in P$, $\mathbb{L}\mathrm{im}(q^{n\bullet}) = 0$, and we have that

$$0 \le \lambda(p^\bullet - q^{n\bullet}) = \sum_{k=n+1}^{\infty} \langle \xi^\bullet_k, p^\bullet_k\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(p^\bullet - q^{n\bullet})\rangle = \sum_{k=n+1}^{\infty} \langle \xi^\bullet_k, p^\bullet_k\rangle + \langle \alpha, d\rangle$$

and by considering the limit $n \to \infty$ we obtain $\langle \alpha, d\rangle \ge 0$. Thus, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ and the proof of part (i) finishes.

Part (ii) can be stated by the same arguments as in part (i). □

Remark 4.5. (i) It is quite easy to choose elements in P^+ in the case $\mathcal{U} = \mathbb{N}$ and $\alpha = 0$. Consider, for instance, any point $q^\bullet_k = (q^1_k, q^2_k, \ldots, q^m_k) \in D_k^+ \backslash \{0\}$ and define $\xi^\bullet$ as follows: $\xi^\bullet_k := \frac{1}{2^k \|q^\bullet_k\|_\infty} q^\bullet_k$. Clearly, $\xi^\bullet_k \in D_k^+ \backslash \{0\}$ since D_k^+ is a cone. In addition, we have that

$$\|\xi^i\|_1 = \sum_{k=1}^{\infty} |\xi^i_k| = \sum_{k=1}^{\infty} \frac{1}{2^k \|q^\bullet_k\|_\infty} |q^i_k| \le \sum_{k=1}^{\infty} \frac{1}{2^k} < +\infty \text{ for all } i \in \{1, 2, \ldots, m\},$$

and it follows that $\xi^\bullet \in \ell_1^m$.

(ii) Consider the convex cone $Q_D = \bigcap_n D_n$. We have $\alpha \in Q_D^+$ whenever $\alpha \in \mathbb{L}\mathrm{im}(P)^+$. Indeed, for each $q \in Q_D$ one can define the linear functional $p^\bullet \in P$ such that $p_k^\bullet = q$ for all k, and so

$$\langle \alpha, q\rangle = \langle \alpha, \mathbb{L}\mathrm{im}(p^\bullet)\rangle \geq 0.$$

In addition, if $D_k = \mathbb{R}^m_+$ for all k, then $\mathbb{L}\mathrm{im}(P)^+ = \mathbb{R}^m_+$. Indeed, inclusion $\subseteq$ has just been discussed, and the converse one derives because of the components of $\mathbb{L}\mathrm{im}$ are positive linear functionals.

Next, for each $x_0 \in X$ we denote $f - f^{x_0} : X \times \mathcal{U} \to \mathbb{R}^m$ by $(f - f^{x_0})(x,u) := f(x,u) - f(x_0,u)$.

THEOREM 4.6. *Consider $\mathcal{U} = \mathbb{N}$, P solid, and $x_0 \in \mathrm{WE}(P)$. Suppose that $f - f^{x_0}$ is nearly P-subconvexlike. Then there exists $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$, $(\xi^\bullet, \alpha) \neq (0,0)$, such that $\xi_k^\bullet \in D_k^+$ for all k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$, and $x_0 \in \mathrm{argmin}\Phi_{\lambda_3}$. Moreover, if the sequence $(f(x,k))_k$ converges for each $x \in X$, then $x_0 \in \mathrm{argmin}\Phi_{\lambda_2}$, where $\lambda_2 = (\xi^\bullet, \alpha)$.*

Proof. Let $x_0 \in \mathrm{WE}(P)$ and suppose that $\mathcal{U} = \mathbb{N}$. Then, there is not any $x \in X$ such that $f^x \precnsim_{\mathrm{int}P} f^{x_0}$ and so

$$(\{f^x : x \in X\} - f^{x_0}) \cap (-\mathrm{int}P) = \emptyset.$$

As P is a convex cone, it follows that $P + \mathrm{int}P = \mathrm{int}P$. Thus, from the equality above it follows that

$$(\mathcal{B}_{\mathcal{U}}(f - f^{x_0}) + P) \cap (-\mathrm{int}P) = \emptyset.$$

In addition, since $\mathrm{int}P$ is open, $0 \notin \mathrm{int}P$, and $\mathrm{int}P \cup \{0\}$ is a cone, we deduce that

$$\mathrm{clcone}(\mathcal{B}_{\mathcal{U}}(f - f^{x_0}) + P) \cap (-\mathrm{int}P) = \emptyset.$$

The set $\mathrm{clcone}(\mathcal{B}_{\mathcal{U}}(f - f^{x_0}) + P)$ is convex, as $f - f^{x_0}$ is nearly P-subconvexlike. Since P is convex, we have that $-\mathrm{int}P$ is convex too. Thus, the Eidelheit's separation theorem [20, Theorem 3.16] can be applied to deduce that there exist $\lambda \in (\ell_\infty^m)^* \backslash \{0\}$ and $a \in \mathbb{R}$ such that

$$-\lambda(p_1) \leq a \leq \lambda(f^x - f^{x_0} + p_2) \text{ for all } x \in X \text{ and } p_1, p_2 \in P. \tag{4.2}$$

By taking $x = x_0$ and $p_1 = p_2 = 0$ in (4.2) we see that $a = 0$. Therefore, $\lambda \in P^+ \backslash \{0\}$ and $\lambda(f^{x_0}) \leq \lambda(f^x)$ for all $x \in X$. Then, by Theorem 4.4 we deduce that there exists $(\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ such that $\lambda = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im})$, $\xi_k^\bullet \in D_k^+$ for all k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ and

$$\sum_{k=1}^{\infty} \langle \xi_k^\bullet, f(x_0,k)\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(f^{x_0})\rangle \leq \sum_{k=1}^{\infty} \langle \xi_k^\bullet, f(x,k)\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(f^x)\rangle \text{ for all } x \in X, \tag{4.3}$$

i.e., $x_0 \in \mathrm{argmin}\Phi_{\lambda_3}$. By Theorem 2.4(iii), as $\lambda \neq 0$, we see that

$$\sum_{i=1}^{m} \|\xi^i\|_1 + \|\alpha\|_1 \neq 0,$$

and then $(\xi^\bullet, \alpha) \neq (0,0)$. Finally, statement $x_0 \in \mathrm{argmin}\Phi_{\lambda_2}$ whenever the sequence $(f(\cdot,k))_k$ pointwise converges, where $\lambda_2 = (\xi^\bullet, \alpha)$, is an obvious consequence of (4.3) and the definition of the elements in $\mathbb{L}^m$, which completes the proof. ∎

We finish the case $\#\mathcal{U}=+\infty$ by showing the particular setting of Theorem 4.6 corresponding to a Pareto uncertain multiobjective optimization problem with countable infinite scenarios and $\mathbb{R}^m_+$-convex objective functions. In what follows we denote $f(\cdot,k)=(f_1(\cdot,k),f_2(\cdot,k),\dots,f_m(\cdot,k)):X\to\mathbb{R}^m$ for all $k\in\mathcal{U}$.

THEOREM 4.7. *Assume that X is convex and $\mathcal{U}=\mathbb{N}$ and $f_i(\cdot,k):X\to\mathbb{R}$ is convex for all $i\in\{1,2,\dots,m\}$ and $k\in\mathcal{U}$. If $x_0\in\mathrm{WE}(P)$, then there exists $\lambda_3=(\xi^\bullet,\alpha,\mathbb{L}\mathrm{im})\in\ell^m_1\times\mathbb{R}^m_+\times\mathbb{L}^m$, $(\xi^\bullet,\alpha)\neq(0,0)$, such that $\xi^\bullet_k\in\mathbb{R}^m_+$ for all k and $x_0\in\mathrm{argmin}\Phi_{\lambda_3}$. Moreover, if the sequence $(f(x,k))_k$ converges for each $x\in X$, then $x_0\in\mathrm{argmin}\Phi_{\lambda_2}$, where $\lambda_2=(\xi^\bullet,\alpha)$.*

Proof. We claim that assumptions of Theorem 4.6 are fulfilled. Indeed, the convexity of the functions $f_i(\cdot,k):X\to\mathbb{R}$ for all $i\in\{1,2,\dots,m\}$ is equivalent to saying that $f(\cdot,k):X\to\mathbb{R}^m$ is $\mathbb{R}^m_+$-convex, and by Proposition 4.3 we see that f is P-convex, which is equivalent to the P-convexity of $f-f^{x_0}$. Thus, by Remark 4.2, we deduce that $f-f^{x_0}$ is nearly P-subconvexlike. In addition, by Lemma 3.3(v) we deduce that P is solid. Hence, Theorem 4.6 can be applied and the result is obtained since the positive dual cone of $\mathbb{R}^m_+$ is $\mathbb{R}^m_+$ and $\mathbb{L}\mathrm{im}(P)^+=\mathbb{R}^m_+$ by Remark 4.5(ii). ☐

The next result deals with necessary optimality conditions for weakly efficient solutions of problem (UP) with respect to P in the case $\#\mathcal{U}<+\infty$. Its proof is omitted as it is the same as the proof of Theorem 4.6. Notice that, in this case, $P^+=D_1^+\times D_2^+\times\cdots\times D_r^+$ and P is solid whenever D_k is solid for all k because of Lemma 3.3(iv).

THEOREM 4.8. *Consider $\mathcal{U}=\{1,2,\dots,r\}$ and $x_0\in\mathrm{WE}(P)$. Suppose that D_k is solid for all $k\in\mathcal{U}$ and $f-f^{x_0}$ is nearly P-subconvexlike. Then there exists $\lambda_1=(\xi^1,\xi^2,\dots,\xi^r)\in(\mathbb{R}^m)^r$, $\xi^k\in D_k^+$, for all $k\in\mathcal{U}$ and $\xi^k\neq0$ for some $k\in\mathcal{U}$, such that $x_0\in\mathrm{argmin}\Phi_{\lambda_1}$.*

By considering $D_u=\mathbb{R}^m_+$ for all $u\in\mathcal{U}$, Theorems 4.6 and 4.8 improve the necessary conditions in [30, Theorem VI.9 and Corollary VI.11] involving an uncertain countable set. Namely, weaker assumptions than P-convexlikeness or $\mathbb{R}^m_+$-convex objective functions are considered (see Remark 4.2 and Proposition 4.3) and we compute a formulation of the linear scalarization function.

On the other hand, our results generalize such necessary conditions which are established in terms of E_6, i.e., efficient solutions of problem (UP) with respect to $\hat{P}'$. It is due to $\mathrm{int}P\subseteq\hat{P}'$ by the proof of Lemma 3.3(iv) and $E_6=\mathrm{E}(\hat{P}')\subseteq\mathrm{E}(\mathrm{int}P)=\mathrm{WE}(P)$ by Theorem 3.6(ii).

Moreover, both theorems can be applied to other concepts of a robust solution of problem (UP) as highly efficient solutions and minmax solutions. For instance, in the first case, by applying parts (ii) and (v) of Theorem 3.6 we have that

$$\mathrm{HE}(P,\mathcal{S})\subseteq\mathrm{HE}(P,\mathcal{S}^c)=\mathrm{E}(P)\subseteq\mathrm{WE}(P)$$

and the necessary conditions for weakly efficient solutions of problem (UP) with respect to P are also for highly efficient solutions. Particularly, Theorems 4.6 and 4.8 improve and develop the necessary conditions in [30, Corollary VI.21] that involve an uncertain countable set in the same way as we discuss above in the first paragraph after Theorem 4.8. Finally, necessary optimality conditions for minmax robust solutions will be presented as applications in section 5.

4.2. Sufficient conditions, characterizations, and existence results. Next, sufficient conditions for efficient solutions of problem (UP) are stated. They are also based on solutions of the families of scalar optimization problems $(\mathrm{SP}_{\lambda_i})$.

The following lemma is well-known in the setting of an abstract vector optimization problem (see, for instance, [20, Theorem 5.18]). We proved it for the reader's convenience. Recall that $F: X \to \mathcal{B}_{\mathcal{U}}$ denotes the trajectory function, i.e., $F(x) = f^x$, for all $x \in X$.

LEMMA 4.9. *Consider problem* (UP) *and a nonempty cone* $C \subseteq \mathcal{B}_{\mathcal{U}}$. *We have*

$$\bigcup_{\phi \in C^{+s}} \operatorname{argmin}(\phi \circ F) \subseteq \mathrm{E}(C). \tag{4.4}$$

If, in addition, $\operatorname{argmin}(\phi \circ F) = \{x\}$ *and* $\phi \in C^+$, *then* $x \in \mathrm{SE}(C)$.

Proof. Assume $\phi \in C^{+s}$ and $\bar{x} \in \operatorname{argmin}(\phi \circ F)$. If $\bar{x} \notin \mathrm{E}(C)$, then there exists $x \in X \backslash \{\bar{x}\}$ such that $f^x \precnsim_C f^{\bar{x}}$. Therefore $f^{\bar{x}} - f^x \in C \backslash \{0\}$ and $\phi(f^{\bar{x}} - f^x) > 0$, since $\phi \in C^{+s}$. Clearly,

$$(\phi \circ F)(x) = \phi(f^x) < \phi(f^{\bar{x}}) = (\phi \circ F)(\bar{x}),$$

which is a contradiction. Thus, $x \notin \mathrm{E}(C)$ and statement (4.4) is proved.

To state the second part of the lemma, consider $\operatorname{argmin}(\phi \circ F) = \{\bar{x}\}$ with $\phi \in C^+$, and suppose $\bar{x} \notin \mathrm{SE}(C)$. As in the previous case, there exists $x \in X \backslash \{\bar{x}\}$ such that $f^{\bar{x}} - f^x \in C$ and so $\phi(f^{\bar{x}} - f^x) \geq 0$, since $\phi \in C^+$. It follows that

$$(\phi \circ F)(x) = \phi(f^x) \leq \phi(f^{\bar{x}}) = (\phi \circ F)(\bar{x}) \leq (\phi \circ F)(x') \text{ for all } x' \in X,$$

since $\bar{x} \in \operatorname{argmin}(\phi \circ F)$. Thus, $x \in \operatorname{argmin}(\phi \circ F) = \{\bar{x}\}$, which is a contradiction as $x \neq \bar{x}$, and the proof finishes. ∎

We begin obtaining sufficient conditions for efficient solutions of problem (UP) in the case $\mathcal{U} = \mathbb{N}$.

THEOREM 4.10. *Let* $\bar{\mathcal{U}} \subseteq \mathcal{U} = \mathbb{N}$. *Consider* $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$.

(i) *If* $\xi_k^\bullet \in D_k^+$ *for all* $k \in \mathcal{U}$, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$, *and* $\operatorname{argmin}\Phi_{\lambda_3} = \{x\}$, *then* $x \in \mathrm{SE}(P)$.

(ii) *If* $\xi_k^\bullet \in D_k^+ \backslash \{0\}$ *for all* $k \in \bar{\mathcal{U}}$, $\xi_k^\bullet \in D_k^{+s}$, *for all* $k \in \mathcal{U} \backslash \bar{\mathcal{U}}$, *and* $\alpha \in \mathbb{L}\mathrm{im}(P \cap \hat{P}_{\bar{\mathcal{U}}})^+$, *then* $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(P \cap \hat{P}_{\bar{\mathcal{U}}})$.

(iii) *If* $\xi_k^\bullet \in D_k^+ \backslash \{0\}$ *for all* $k \in \mathcal{U}$ *and* $\alpha \in \mathbb{L}\mathrm{im}(C)^+$, *then* $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(C)$, *where* $C = \bigcup_{k \in \mathcal{U}} (P \cap \hat{P}'_{\{k\}})$.

(iv) *If* $\xi_k^\bullet \in D_k^+$ *for all* $k \in \mathcal{U}$, $\alpha \in \mathbb{L}\mathrm{im}(P' \cap \hat{P}'_{\bar{\mathcal{U}}})^+$, *and there exists* $k_0 \in \bar{\mathcal{U}}$ *such that* $\xi_{k_0}^\bullet \neq 0$ *or* $k_0 \in \mathcal{U} \backslash \bar{\mathcal{U}}$ *and* $\xi_{k_0}^\bullet \in D_{k_0}^{+s}$, *then* $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(P' \cap \hat{P}'_{\bar{\mathcal{U}}})$.

Proof. (i) This assertion is a result of the second part of Lemma 4.9 since $\lambda_3 \in P^+$ by (4.1).

(ii) Let $\bar{\lambda}_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in (\ell_\infty^m)^*$ satisfy the assumptions of the theorem. We claim that $\bar{\lambda}_3 \in (P \cap \hat{P}_{\bar{\mathcal{U}}})^{+s}$. Indeed, consider $p \in P \cap \hat{P}_{\bar{\mathcal{U}}}$, $p \neq 0$. Then, $p(k) \in \operatorname{int} D_k \cup \{0\}$ for all $k \in \bar{\mathcal{U}}$, $p(k) \in D_k$, for all $k \in \mathcal{U} \backslash \bar{\mathcal{U}}$, and there exists $k_0 \in \mathcal{U}$ such that $p(k_0) \neq 0$. Clearly, $\langle \alpha, \mathbb{L}\mathrm{im}(p) \rangle \geq 0$ as $\alpha \in \mathbb{L}\mathrm{im}(P \cap \hat{P}_{\bar{\mathcal{U}}})^+$ and $p \in P \cap \hat{P}_{\bar{\mathcal{U}}}$. Moreover, $\langle \xi_k^\bullet, p(k) \rangle \geq 0$ since $\xi_k^\bullet \in D_k^+$ and $p(k) \in D_k$ for all $k \in \mathcal{U}$. In addition, if $k_0 \in \bar{\mathcal{U}}$, then $\langle \xi_{k_0}^\bullet, p(k_0) \rangle > 0$ as $p(k_0) \in \operatorname{int} D_{k_0}$ and $\xi_{k_0}^\bullet \in D_{k_0}^+ \backslash \{0\}$. Analogously, if $k_0 \in \mathcal{U} \backslash \bar{\mathcal{U}}$, then $\langle \xi_{k_0}^\bullet, p(k_0) \rangle > 0$ as $p(k_0) \in D_{k_0} \backslash \{0\}$ and $\xi_{k_0}^\bullet \in D_{k_0}^{+s}$. Therefore,

$$\bar{\lambda}_3(p) = \sum_{k=1}^{\infty} \langle \xi_k^\bullet, p(k) \rangle + \langle \alpha, \mathbb{L}\mathrm{im}(p) \rangle \geq \langle \xi_{k_0}^\bullet, p(k_0) \rangle > 0$$

and $\bar{\lambda}_3 \in (P \cap \hat{P}_{\bar{\mathcal{U}}})^{+s}$. By applying Lemma 4.9 we deduce

$$\operatorname{argmin}\Phi_{\bar{\lambda}_3} \subseteq \bigcup_{\lambda_3 \in (P \cap \hat{P}_{\bar{\mathcal{U}}})^{+s}} \operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(P \cap \hat{P}_{\bar{\mathcal{U}}})$$

and the proof of the second part finishes.

Parts (iii) and (iv) can be stated by the same arguments as part (ii) and the proof is completed. ∎

The previous proof also works to the case $\#\mathcal{U} < +\infty$.

Corollary 4.11. *Consider $\bar{\mathcal{U}} \subseteq \mathcal{U} = \{1, 2, \dots, r\}$ and $\lambda_1 = (\xi^1, \xi^2, \dots, \xi^r) \in (\mathbb{R}^m)^r$.*

(i) *If $\xi^k \in D_k^+$ for all $k \in \mathcal{U}$ and $\operatorname{argmin}\Phi_{\lambda_1} = \{x\}$, then $x \in \mathrm{SE}(P)$.*
(ii) *If $\xi^k \in D_k^+ \backslash \{0\}$ for all $k \in \bar{\mathcal{U}}$ and $\xi^k \in D_k^{+s}$, for all $k \in \mathcal{U} \backslash \bar{\mathcal{U}}$, then $\operatorname{argmin}\Phi_{\lambda_1} \subseteq \mathrm{E}(P \cap \hat{P}_{\bar{\mathcal{U}}})$.*
(iii) *If $\xi^k \in D_k^+ \backslash \{0\}$ for all $k \in \mathcal{U}$, then $\operatorname{argmin}\Phi_{\lambda_1} \subseteq \mathrm{E}(C)$, where $C = \bigcup_{k \in \mathcal{U}}(P \cap \hat{P}'_{\{k\}})$.*
(iv) *If $\xi^k \in D_k^+$ for all $k \in \mathcal{U}$ and there exists $k_0 \in \bar{\mathcal{U}}$ such that $\xi^{k_0} \neq 0$ or $k_0 \in \mathcal{U} \backslash \bar{\mathcal{U}}$ and $\xi^{k_0} \in D_{k_0}^{+s}$, then $\operatorname{argmin}\Phi_{\lambda_1} \subseteq \mathrm{E}(P' \cap \hat{P}'_{\bar{\mathcal{U}}})$.*

We emphasize that [13, Theorem 4.5] is a particular case of Theorem 4.10. Indeed, suppose that $\#\mathcal{U} = +\infty$ (the case $\#\mathcal{U} < +\infty$ follows from Corollary 4.11 by considering the same arguments). Notice that [13] concerns the uncertain Pareto problem (UP) and the scalarization

$$W_\lambda(x) := \sum_{i=1}^{m} \sum_{k=1}^{\infty} \lambda_i(k) f_i(x, k),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m) \in \ell_1^m$ (by following notation given in [13]). Therefore, $W_\lambda = \Phi_{\lambda_3}$ with $\lambda_3 = (\xi^\bullet, 0, 0) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ and $\xi^\bullet = \lambda$. Observe that $\lambda(k) = (\lambda_1(k), \lambda_2(k), \dots, \lambda_m(k)) = \xi_k^\bullet$. Consider the sets

$$\begin{aligned}
\Lambda_1 &:= \{\lambda \in \ell_1^m : \lambda(k) \in \mathbb{R}_+^m \text{ for all } k \in \mathbb{N}\},\\
\Lambda_2 &:= \{\lambda \in \ell_1^m : \lambda(k) \in \operatorname{int}\mathbb{R}_+^m \text{ for all } k \in \mathbb{N}\},\\
\Lambda_3 &:= \{\lambda \in \ell_1^m : \lambda(k) \in \mathbb{R}_+^m \backslash \{0\} \text{ for all } k \in \mathbb{N}\},\\
\Lambda_4 &:= \{\lambda \in \ell_1^m : \lambda(k) \in \mathbb{R}_+^m \text{ for all } k \in \mathbb{N} \text{ and } \lambda(k_0) \in \operatorname{int}\mathbb{R}_+^m, \text{ for some } k_0 \in \mathbb{N}\},\\
\Lambda_6 &:= \{\lambda \in \ell_1^m : \lambda(k) \in \mathbb{R}_+^m \text{ for all } k \in \mathbb{N} \text{ and } \lambda(k_0) \in \mathbb{R}_+^m \backslash \{0\}, \text{ for some } k_0 \in \mathbb{N}\}.
\end{aligned}$$

For each $\lambda \in \Lambda_2$ (resp., $\lambda \in \Lambda_3$), the assumptions of Theorem 4.10(ii) for $\bar{\mathcal{U}} = \emptyset$ (resp., Theorem 4.10(iii)) are fulfilled and then $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(P) = E_2$ (resp., $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(\bigcup_{k \in \mathcal{U}}(P \cap \hat{P}'_{\{k\}})) = E_3$; see Remark 3.2).

Analogously, for each $\lambda \in \Lambda_4$ (resp., $\lambda \in \Lambda_6$), the assumptions of Theorem 4.10(iv) for $\bar{\mathcal{U}} = \emptyset$ (resp., $\bar{\mathcal{U}} = \mathcal{U}$) are fulfilled and then $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(P') = E_4$ (resp., $\operatorname{argmin}\Phi_{\lambda_3} \subseteq \mathrm{E}(\hat{P}') = E_6$; see Remark 3.2).

Finally, if $\lambda \in \Lambda_1$, then λ_3 satisfies the hypotheses of Theorem 4.10(i) and so $x \in \mathrm{SE}(P) = E_1$ provided that $\operatorname{argmin}\Phi_{\lambda_3} = \{x\}$. Next, a characterization of the set of weakly efficient solutions of problem (UP) with respect to P is stated. First we consider the case $\#\mathcal{U} = +\infty$.

Theorem 4.12. *Consider $\mathcal{U} = \mathbb{N}$ and $x_0 \in X$. Suppose that P is solid and $f - f^{x_0}$ is nearly P-subconvexlike. Then, $x_0 \in \mathrm{WE}(P)$ if and only if there exists*

$\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ *such that* $(\xi^\bullet, \alpha) \neq (0,0)$, $\xi_k^\bullet \in D_k^+$ *for all* k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$, *and* $x_0 \in \mathrm{argmin}\Phi_{\lambda_3}$. *Moreover, if the sequence* $(f(\cdot,k))_k$ *pointwise converges, then* λ_3 *and* Φ_{λ_3} *can be replaced with* $\lambda_2 = (\xi^\bullet, \alpha)$ *and* Φ_{λ_2}*, respectively.*

Proof. The necessary condition follows because of Theorem 4.6, and the sufficient one by Lemma 4.9 and statement (4.1), since $(\mathrm{int}P \cup \{0\})^{+s} = P^+ \backslash \{0\}$. ◻

Next, the corresponding characterization for the case $\#\mathcal{U} < +\infty$ of Theorem 4.12 is obtained as a result of Theorem 4.8, Corollary 4.11(iv), and taking into account by Lemma 3.3(iv) that $\mathrm{int}P \cup \{0\} = \hat{P}'$.

Corollary 4.13. *Consider* $\mathcal{U} = \{1, 2, \ldots, r\}$, $x_0 \in X$ *and suppose that* D_k *is solid for all* $k \in \mathcal{U}$ *and* $f - f^{x_0}$ *is nearly* P*-subconvexlike. Then* $x_0 \in \mathrm{WE}(P)$ *if and only if there exists* $\lambda_1 = (\xi^1, \xi^2, \ldots, \xi^r) \in (\mathbb{R}^m)^r$ *with* $\xi^k \in D_k^+$ *for all* $k \in \mathcal{U}$ *and* $\xi^k \neq 0$ *for some* $k \in \mathcal{U}$*, such that* $x_0 \in \mathrm{argmin}\Phi_{\lambda_1}$.

As another application of Theorem 4.12, the following result characterizes weakly efficient solutions of problem (UP) when the uncertainty is periodic and the involved objective functions are cone-convex. Notice that the case $p = 1$ leads to a deterministic multiobjective optimization problem and the result then yields the usual weighted sum method.

Corollary 4.14. *Assume that* X *is convex,* D_k *is solid, and* $f(\cdot,k) : X \to \mathbb{R}^m$ *is* D_k*-convex for all* $k \in \mathbb{N}$. *Suppose that there exists* $p \geq 1$ *such that* $f(\cdot,k) = f(\cdot,k+p)$ *and* $D_k = D_{k+p}$ *for all* $k \in \mathbb{N}$. *Then,* $x_0 \in \mathrm{WE}(P)$ *if and only if there exists* $\lambda_1 = (\mu_1, \mu_2, \ldots, \mu_p) \in (\mathbb{R}^m)^p$ *with* $\mu_k \in D_k^+$ *for all* $k \in \{1, 2, \ldots, p\}$ *and* $\mu_k \neq 0$ *for some* $k \in \{1, 2, \ldots, p\}$*, such that* $x_0 \in \mathrm{argmin}\Phi_{\lambda_1}$.

Proof. Clearly, the cone P in (3.1) is solid and the function $(f - f^{x_0})(\cdot,k) : \mathbb{R}^n \to \mathbb{R}^m$ is D_k-convex. Thus, by Proposition 4.3 and Remark 4.2 we see that $f - f^{x_0}$ is nearly P-subconvexlike. Then, by Theorem 4.12 we have that $x_0 \in \mathbb{R}^n$ is a weakly efficient solution of problem (UP) with respect to the cone P if and only if there exists $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ such that $(\xi^\bullet, \alpha) \neq (0,0)$, $\xi_k^\bullet \in D_k^+$ for all k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ and $x_0 \in \mathrm{argmin}\Phi_{\lambda_3}$. Since the objective function is periodic we know that

$$\mathbb{L}\mathrm{im}(f^x) = \mathbb{L}\mathrm{im}((f(x,k))_k) = \frac{1}{p}\sum_{k=1}^{p} f(x,k) \text{ for all } x \in \mathbb{R}^n.$$

In addition, we claim that $D_k \subseteq \mathbb{L}\mathrm{im}(P)$ for all $k \in \{1, 2, \ldots, p\}$. Indeed, take a point $d_k \in D_k$ and define $q^\bullet = (q_n^\bullet)$, $q_{k+sp}^\bullet = pd_k$ for all $s \in \mathbb{N} \cup \{0\}$ and $q_n^\bullet = 0$ otherwise. Clearly, $q^\bullet \in P$,

$$\mathbb{L}\mathrm{im}(q^\bullet) = \frac{1}{p}\sum_{r=1}^{p} q_r^\bullet = d_k$$

and the assertion is stated since d_k is an arbitrary element of D_k. Therefore, we have that $\alpha \in \mathbb{L}\mathrm{im}(P)^+ \subseteq \bigcap_{k=1}^p D_k^+$ and it follows that

$$\Phi_{\lambda_3}(x) = \sum_{k=1}^{\infty} \langle \xi_k^\bullet, f(x,k)\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(f^x)\rangle = \sum_{k=1}^{p} \langle \mu_k, f(x,k)\rangle \text{ for all } x \in X,$$

where $\mu_k := \sum_{s=0}^{\infty} \xi_{k+sp}^\bullet + \frac{1}{p}\alpha \in D_k^+$, for all $k \in \{1, 2, \ldots, p\}$.

It is not hard to check that D_k^+ is pointed, since D_k is solid, for all $k \in \{1,2,\ldots,p\}$. Thus, $(\mu_1,\mu_2,\ldots,\mu_p) \neq 0$ since $(\xi^\bullet,\alpha) \neq (0,0)$, $\sum_{s=0}^{\infty} \xi^\bullet_{k+sp} \in D_k^+$, and $\alpha \in D_k^+$ for all $k \in \{1,2,\ldots,p\}$.

Reciprocally, suppose that $x_0 \in \operatorname{argmin}\Phi_{\lambda_1}$ and $\lambda_1 = (\mu_1,\mu_2,\ldots,\mu_p) \in (\mathbb{R}^m)^p$ with $\mu_k \in D_k^+$ for all $k \in \{1,2,\ldots,p\}$ and $\mu_k \neq 0$ for some $k \in \{1,2,\ldots,p\}$. Define $\xi^\bullet = (\xi_k^\bullet)$, $\xi_k^\bullet = \mu_k$, for all $k \in \{1,2,\ldots,p\}$, $\xi_k^\bullet = 0$ otherwise, and consider an arbitrary element $\mathbb{L}\mathrm{im} \in \mathbb{L}^m$. Clearly, $\lambda_3 := (\xi^\bullet, 0, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ satisfies the assumptions of the sufficient condition in Theorem 4.12, and we conclude that $x_0 \in \mathrm{WE}(P)$ since $\Phi_{\lambda_3} = \Phi_{\lambda_1}$, which finishes the proof. ∎

Next, by putting together Theorem 4.10(iv) with $\bar{\mathcal{U}} = \mathcal{U}$ and Theorem 4.6 we obtain a characterization of the set of efficient solutions of problem (UP) with respect to $\hat{P}'$ in the case $\#\mathcal{U} = +\infty$. The next lemma is needed.

Lemma 4.15. *Assume that* $\mathcal{U} = \mathbb{N}$ *and* P *is solid. Then,* $\mathbb{L}\mathrm{im}(P)^+ = \mathbb{L}\mathrm{im}(\hat{P}')^+$.

Proof. Since $\hat{P}' \subseteq P$, we have that

$$\mathbb{L}\mathrm{im}(P)^+ \subseteq \mathbb{L}\mathrm{im}(\hat{P}')^+.$$

Conversely, consider $\alpha \in \mathbb{L}\mathrm{im}(\hat{P}')^+$ and $p \in P$. As P is solid and convex, by the accessibility lemma there exists a sequence $(p_n) \subset \operatorname{int}P$ such that $p_n \to p$. By Lemma 3.3(iv) we have that $\operatorname{int}P = \operatorname{int}\hat{P}'$. Hence, the continuity of the operator $\mathbb{L}\mathrm{im}$ implies that

$$\langle \alpha, \mathbb{L}\mathrm{im}(p)\rangle = \langle \alpha, \mathbb{L}\mathrm{im}(\lim_n p_n)\rangle = \langle \alpha, \lim_n \mathbb{L}\mathrm{im}(p_n)\rangle = \lim_n \langle \alpha, \mathbb{L}\mathrm{im}(p_n)\rangle \geq 0,$$

where the inequality follows because of $p_n \in \hat{P}'$. Therefore, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ and the proof finishes. ∎

Corollary 4.16. *Consider* $\mathcal{U} = \mathbb{N}$ *and* $x_0 \in X$. *Suppose that* P *is solid and* $f - f^{x_0}$ *is nearly* P*-subconvexlike. If* $x_0 \in \mathrm{E}(\hat{P}')$*, then there exists* $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ *such that* $(\xi^\bullet, \alpha) \neq (0,0)$, $\xi_k^\bullet \in D_k^+$ *for all* k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$*, and* $x_0 \in \operatorname{argmin}\Phi_{\lambda_3}$. *Conversely, if there exists* $\lambda_3 = (\xi^\bullet, \alpha, \mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ *such that* $\xi_k^\bullet \in D_k^+$*, for all* k, $\alpha \in \mathbb{L}\mathrm{im}(P)^+$, $x_0 \in \operatorname{argmin}\Phi_{\lambda_3}$*, and* $\xi_{k_0}^\bullet \neq 0$ *for some* k_0*, then* $x_0 \in \mathrm{E}(\hat{P}')$. *Moreover, if the sequence* $(f(\cdot,k))_k$ *pointwise converges, then* λ_3 *and* Φ_{λ_3} *can be replaced with* $\lambda_2 = (\xi^\bullet, \alpha)$ *and* Φ_{λ_2}*, respectively.*

Proof. By the proof of Lemma 3.3(iv) we see that $\operatorname{int}P \subseteq \hat{P}'$, and then by Theorem 3.6(i)–(ii) it follows that $\mathrm{E}(\hat{P}') \subseteq \mathrm{E}(\operatorname{int}P) = \mathrm{WE}(P)$. Therefore, the necessary condition follows because of Theorem 4.6. Concerning the sufficient condition, by applying Theorem 4.10(iv) with $\bar{\mathcal{U}} = \mathcal{U}$ and Lemma 4.15 we deduce that $x_0 \in \mathrm{E}(\hat{P}')$. The last part of the theorem is true since $\mathbb{L}\mathrm{im}(\xi^\bullet) = \lim_{k\to\infty} \xi_k^\bullet$ as long as the sequence $\{\xi_k^\bullet\}$ converges, which finishes the proof. ∎

Let us illustrate the role of the parameter $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ in the previous characterizations. In particular, we show that the sufficient conditions given in [13] are not necessary.

Example 4.17. Consider problem (UP) with the data $X = \mathbb{R}$, $m = 1$, $\mathcal{U} = \mathbb{N}$, $D(k) = [0,+\infty)$, for all $k \in \mathcal{U}$, and $f(x,k) = 0$ if $x \neq 0$ and $f(0,k) = 1/k$, for all $k \in \mathcal{U}$. It is easy to check that

$$\operatorname{int}P = \left\{ (a_k)_k \in \ell_\infty : \inf_k \{a_k\} > 0 \right\}$$

and $x_0 := 0 \in \mathrm{WE}(P)$. Furthermore,

$$\operatorname{cone}\mathcal{B}_{\mathcal{U}}(f - f^{x_0}) + \operatorname{int}P = \operatorname{cone}((-1/k)_k) + \operatorname{int}P,$$

which is a convex set. Thus, $f - f^{x_0}$ is nearly P-subconvexlike. In addition, the sequence $(f(\cdot,k))_k$ pointwise converges to the function $g(x) = 0$ for all $x \in \mathbb{R}$. Concerning the characterization of Theorem 4.12, we have that

$$\Phi_{\lambda_2}(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty}\xi_k f(x,k) & \text{if} \quad \xi \neq 0, \\ 0 & \text{if} \quad \xi = 0, \end{cases}$$

$$\operatorname{argmin}\Phi_{\lambda_2} = \begin{cases} \mathbb{R}\backslash\{0\} & \text{if} \quad \xi \neq 0, \\ \mathbb{R} & \text{if} \quad \xi = 0. \end{cases}$$

Therefore, the only way to obtain the weakly efficient solution $x = 0$ is to consider $\alpha > 0$ and $\xi = 0$. In other words, by scalarization Φ_{λ_1} one cannot deduce all weakly efficient solutions of this problem.

The parameter $\alpha \in \mathbb{L}\mathrm{im}(P)^+$ cannot be dropped in the necessary condition for the solutions in $\mathrm{E}(\hat{P}')$. Indeed, consider the same setting as above and the objective function $f(x,k) = 1$, for all $x \notin \mathbb{N}$ and $f(n,k) = 1$ if $n \leq k$ and $f(n,k) = 1/2$ if $n > k$, for all $n \in \mathbb{N}$, for all $k \in \mathcal{U}$. Clearly, $x_0 \in \mathrm{E}(\hat{P}')$, $\operatorname{cone}\mathcal{B}_{\mathcal{U}}(f - f^{x_0}) + \operatorname{int}P$ is convex and the sequence $(f(\cdot,k))_k$ pointwise converges to the function $g(x) = 1$ for all $x \in \mathbb{R}$. Given $((\xi_k)_k, \alpha) \in (\ell_1 \times \mathbb{R}_+)\backslash\{(0,0)\}$, $\xi_k \geq 0$, we have

$$\Phi_{\lambda_2}(x) = \begin{cases} \displaystyle\sum_{k=1}^{\infty}\xi_k + \alpha & \text{if} \quad x \notin \mathbb{N} \text{ or } x = 1, \\ \displaystyle\frac{1}{2}\sum_{k=1}^{n-1}\xi_k + \sum_{k=n}^{\infty}\xi_k + \alpha & \text{if} \quad x = n \in \mathbb{N}\backslash\{1\}. \end{cases}$$

In the case $\alpha = 0$ there exists $k_0 \in \mathbb{N}$ such that $\xi_{k_0} > 0$ and so

$$\Phi_{\lambda_2}(k_0 + 1) = \frac{1}{2}\sum_{k=1}^{k_0}\xi_k + \sum_{k=k_0+1}^{\infty}\xi_k < \sum_{k=1}^{\infty}\xi_k = \Phi_{\lambda_2}(x_0).$$

Therefore, the only way to obtain $x_0 \in \operatorname{argmin}\Phi_{\lambda_2}$ is to consider $\alpha \neq 0$. For instance, $(0,1) \in (\ell_1 \times \mathbb{R}_+)\backslash\{(0,0)\}$ provides $\Phi_{\lambda_2}(x) = g(x) = 1$ for all $x \in \mathbb{R}$, and $\operatorname{argmin}\Phi_{\lambda_2} = \mathbb{R}$.

Finally, we state Weierstrass theorems for some classes of efficient solutions of problem (UP). The following well-known statements concerning the nonemptiness of C^+ and C^{+s} will be required.

LEMMA 4.18. *Let $C \subset \mathbb{R}^m$ be a nonempty convex cone. We have that*
(i) *if C is solid and $C \neq \mathbb{R}^m$, then $C^+\backslash\{0\} \neq \emptyset$,*
(ii) *if C is closed and pointed, then $C^{+s} \neq \emptyset$.*

THEOREM 4.19. *Consider problem* (UP), *$\bar{\mathcal{U}} \subseteq \mathcal{U} = \mathbb{N}$, and suppose that X is compact and $f(\cdot,k) : X \to \mathbb{R}^m$ is continuous, for all $k \in \mathcal{U}$. It follows that*
(i) *if $f \in \mathcal{B}(X \times \mathcal{U}, \mathbb{R}^m)$, D_k is closed, for all $k \in \mathcal{U}\backslash\bar{\mathcal{U}}$, and solid, for all $k \in \bar{\mathcal{U}}$, then $\mathrm{E}(P \cap \hat{P}_{\bar{\mathcal{U}}}) \neq \emptyset$,*
(ii) *if $f \in \mathcal{B}(X \times \mathcal{U}, \mathbb{R}^m)$ and D_k is solid, for all $k \in \mathcal{U}$, then $\mathrm{E}(\bigcup_{k \in \mathcal{U}}(P \cap \hat{P}'_{\{k\}})) \neq \emptyset$,*
(iii) *if there exists $k_0 \in \bar{\mathcal{U}}$ such that D_{k_0} is solid, or there exists $k_0 \in \mathcal{U}\backslash\bar{\mathcal{U}}$ such that D_{k_0} is closed, then $\mathrm{E}(P' \cap \hat{P}'_{\bar{\mathcal{U}}}) \neq \emptyset$.*

Proof. (i) By Lemma 4.18 we see that there exists a sequence $(\zeta_k)_{k\in\mathcal{U}} \subset \mathbb{R}^m$ fulfilling $\zeta_k \in D_k^{+s}$ for all $k\in\mathcal{U}\backslash\bar{\mathcal{U}}$ and $\zeta_k \in D_k^{+}\backslash\{0\}$, for all $k\in\bar{\mathcal{U}}$. For each $k\in\mathcal{U}$, define $\xi_k^\bullet := (1/(2^k\|\zeta_k\|))\zeta_k$. Clearly, $\xi_k^\bullet \in D_k^{+s}$, for all $k\in\mathcal{U}\backslash\bar{\mathcal{U}}$, and $\xi_k^\bullet \in D_k^{+}\backslash\{0\}$, for all $k\in\bar{\mathcal{U}}$, since D_k^{+s} and D_k^{+} are cones. In addition, $\xi^\bullet = (\xi_k^\bullet)_{k\in\mathcal{U}} \in \ell_1^m$ as

$$\sum_{k=1}^{\infty}|\xi_k^i| \le \sum_{k=1}^{\infty}\|\xi_k^\bullet\| = \sum_{k=1}^{\infty}\frac{1}{2^k} < +\infty \text{ for all } i\in\{1,2,\dots,m\}.$$

Therefore, the result is deduced by Theorem 4.10(ii) after applying the Weierstrass theorem to the optimization problem

$$\text{Min}\{\Phi_{\lambda_3}(x) : x\in X\},$$

where $\lambda_3 = (\xi^\bullet, 0, 0)$. For it we claim that Φ_{λ_3} is continuous. Indeed, it is sufficient to check that the partial sum $S_n := \sum_{k=1}^{n}\langle\xi_k^\bullet, f(\cdot,k)\rangle : X\to\mathbb{R}$ uniformly converges to $S := \sum_{k=1}^{+\infty}\langle\xi_k^\bullet, f(\cdot,k)\rangle : X\to\mathbb{R}$. Since $f\in\mathcal{B}(X\times\mathcal{U},\mathbb{R}^m)$ there exists $M>0$ such that $\|f(x,k)\|\le M$ for all $x\in X$ and $k\in\mathcal{U}$. Thus,

$$\left|\sum_{k=1}^{n}\langle\xi_k^\bullet, f(x,k)\rangle - \sum_{k=1}^{+\infty}\langle\xi_k^\bullet, f(x,k)\rangle\right| \le \sum_{k=n+1}^{+\infty}\|\xi_k^\bullet\|\|f(x,k)\| \le M\sum_{k=n+1}^{+\infty}\frac{1}{2^k} \text{ for all } x\in X,$$

and clearly $S_n\to S$ uniformly, which finishes the proof.

Part (ii) can be stated by the same arguments as part (i).

(iii) By Lemma 4.18 and the assumptions we can define a sequence $\xi^\bullet = (\xi_k^\bullet)_{k\in\mathcal{U}} \in \ell_1^m$ such that $\xi_{k_0}^\bullet \in D_{k_0}^{+}\backslash\{0\}$ for some $k_0\in\bar{\mathcal{U}}$, or $\xi_{k_0}^\bullet \in D_{k_0}^{+s}$ for some $k_0\in\mathcal{U}\backslash\bar{\mathcal{U}}$, and $\xi_k^\bullet = 0$, for all $k\neq k_0$. Clearly, $\lambda_3 = (\xi,0,0)$ defines the continuous function $\Phi_{\lambda_3}(x) = \langle\xi_{k_0}^\bullet, f(x,k_0)\rangle$ for all $x\in X$, and by Theorem 4.10(iv) and the Weierstrass theorem we obtain that $\emptyset\neq\text{argmin}\Phi_{\lambda_3}\subseteq \text{E}(P'\cap\hat{P}'_{\bar{\mathcal{U}}})$, which finishes the proof. □

The following Weierstrass result for the case $\#\mathcal{U} < +\infty$ can be proved like Theorem 4.19 by considering Φ_{λ_1} and Corollary 4.11 instead of Φ_{λ_3} and Theorem 4.10.

Corollary 4.20. *Consider problem* (UP), $\#\mathcal{U}<+\infty$, $\bar{\mathcal{U}}\subseteq\mathcal{U}$ *and suppose that* X *is compact and* $f(\cdot,k) : X\to\mathbb{R}^m$ *is continuous for all* $k\in\mathcal{U}$. *It follows that*

(i) *if* D_k *is closed, for all* $k\in\mathcal{U}\backslash\bar{\mathcal{U}}$, *and solid, for all* $k\in\bar{\mathcal{U}}$, *then* $\text{E}(P\cap\hat{P}_{\bar{\mathcal{U}}})\neq\emptyset$,

(ii) *if* D_k *is solid, for all* $k\in\mathcal{U}$, *then* $\text{E}(\bigcup_{k\in\mathcal{U}}(P\cap\hat{P}'_{\{k\}}))\neq\emptyset$,

(iii) *if there exists* $k_0\in\bar{\mathcal{U}}$ *such that* D_{k_0} *is solid, or there exists* $k_0\in\mathcal{U}\backslash\bar{\mathcal{U}}$ *such that* D_{k_0} *is closed, then* $\text{E}(P'\cap\hat{P}'_{\bar{\mathcal{U}}})\neq\emptyset$.

Consequently, Theorems 4.19(i) and 4.20(i) with $\bar{\mathcal{U}}=\emptyset$ reduce to [13, Theorem 4.8] by defining $D_u=\mathbb{R}_+^m$ for all $u\in\mathcal{U}$.

5. Applications.

We illustrate some obtained results in two specific problems.

5.1. Unconstrained convex quadratic Pareto problems and periodic uncertainty.

Let (UQP) be the problem (UP) with the next data: $X=\mathbb{R}^n$, $\mathcal{U}=\mathbb{N}$, $D(k)=\mathbb{R}_+^m$ for all $k\in\mathbb{N}$ and

$$f_i(x,k) := \frac{1}{2}x'A_{i,k}x + a'_{i,k}x + \alpha_{i,k} \text{ for all } x\in\mathbb{R}^n \text{ and } i\in\{1,2,\dots,m\},$$

where $A_{i,k}$ is a symmetric positive semidefinite $n\times n$ matrix, $a_{i,k}\in\mathbb{R}^n$, and $\alpha_{i,k}\in\mathbb{R}$. Assume that there exists $p\in\mathbb{N}$ such that $A_{i,k}=A_{i,k+p}$, $a_{i,k}=a_{i,k+p}$, and $\alpha_{i,k}=\alpha_{i,k+p}$

for all $k \in \mathbb{N}$ and $i \in \{1,2,\ldots,m\}$. We suppose $p \geq 2$ (otherwise, there is not any uncertainty since the objective function is the same at each scenario).

To illustrate this model, consider a factory whose activity is to manufacture a product for a company. Specifically, the company always asks the factory for s units, although it is not known a priori what exact day the s units will be required. In addition, the production cost is periodic as it involves n seasonal variables like the costs of the raw materials and the inventory. Hence, the factory plans its production according to a periodic convex quadratic cost function $f : X \times \mathbb{N} \to \mathbb{R}^m$, where $f(x,k) = f(x,k+p)$ since the costs coincide each p days.

The set of weakly efficient solutions of the above problem is stated in the following result. Clearly, the problem does not have any memory as the solutions coincide with the ones that one would have been obtained by taking into account any arbitrary period $\{k+1,k+2,\ldots,k+p\}$.

THEOREM 5.1. *A point* $x_0 \in \mathbb{R}^n$ *is a weakly efficient solution of problem* (UQP) *if and only if there exist* p *vectors* $\{\gamma_1,\gamma_2,\ldots,\gamma_p\} \subset \mathbb{R}^m_+$ *not all zero such that*

$$\sum_{k=1}^{p}\sum_{i=1}^{m} \gamma_k^i (A_{i,k}x_0 + a_{i,k}) = 0. \tag{5.1}$$

Proof. Clearly the function $(f - f^{x_0})(\cdot,k) : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex as each component $(f_i - f_i^{x_0})(\cdot,k) : \mathbb{R}^n \to \mathbb{R}$ is a convex function. Thus, by Proposition 4.3 and Remark 4.2 we see that $f - f^{x_0}$ is nearly P-subconvexlike. Then, by Theorem 4.12 and Remark 4.5(ii) we have that $x_0 \in \mathbb{R}^n$ is a weakly efficient solution of problem (UQP) if and only if there exists $\lambda_3 = (\xi^\bullet,\alpha,\mathbb{L}\mathrm{im}) \in \ell_1^m \times \mathbb{R}^m \times \mathbb{L}^m$ such that $(\xi^\bullet,\alpha) \neq (0,0)$, $\xi_k^\bullet \in \mathbb{R}^m_+$ for all k, $\alpha \in \mathbb{R}^m_+$ and $x_0 \in \operatorname{argmin}\Phi_{\lambda_3}$. Since the objective function is periodic we know that

$$\mathbb{L}\mathrm{im}(f^x) = \mathbb{L}\mathrm{im}((f(x,k))_k) = \frac{1}{p}\sum_{k=1}^{p} f(x,k) \quad \text{for all } x \in \mathbb{R}^n.$$

Therefore, we deduce that

$$\begin{aligned}\Phi_{\lambda_3}(x) &= \sum_{k=1}^{\infty}\langle \xi_k^\bullet, f(x,k)\rangle + \langle \alpha, \mathbb{L}\mathrm{im}(f^x)\rangle \\ &= \frac{1}{2}x'\left(\sum_{k=1}^{p}\sum_{i=1}^{m}\gamma_k^i A_{i,k}\right)x + \left(\sum_{k=1}^{p}\sum_{i=1}^{m}\gamma_k^i a_{i,k}\right)x + \sum_{k=1}^{p}\sum_{i=1}^{m}\gamma_k^i \alpha_{i,k} \quad \text{for all } x \in X,\end{aligned}$$

where

$$\gamma_k^i = \frac{\alpha_i}{p} + \sum_{j=0}^{\infty}\xi_{pj+k}^i \quad \text{for all } i \in \{1,2,\ldots,m\} \text{ and } k \in \{1,2,\ldots,p\}. \tag{5.2}$$

It follows that $\gamma_k := (\gamma_k^1,\gamma_k^2,\ldots,\gamma_k^m) \in \mathbb{R}^m_+$ for all $k \in \{1,2,\ldots,p\}$, and they are not all zero because of $(\xi^\bullet,\alpha) \neq (0,0)$. Clearly, Φ_{λ_3} is a real convex quadratic function. Hence, $x_0 \in \operatorname{argmin}\Phi_{\lambda_3}$ if and only if

$$\sum_{k=1}^{p}\sum_{i=1}^{m}\gamma_k^i A_{i,k}x_0 = -\sum_{k=1}^{p}\sum_{i=1}^{m}\gamma_k^i a_{i,k}$$

and assertion (5.1) is obtained. Thus, the necessary condition is stated.

Concerning the sufficient condition, suppose that equality (5.1) is true and define $(\xi^\bullet,\alpha)\in\ell_1^m\times\mathbb{R}^m\setminus\{(0,0)\}$ as follows: $\alpha:=0$, $\xi_k^i:=\gamma_k^i$, for all $k\in\{1,2,\dots,p\}$, $i\in\{1,2,\dots,m\}$, and $\xi_k^i:=0$ otherwise. Hence, equality (5.2) is satisfied and assertion (5.1) implies that $x_0\in\operatorname{argmin}\Phi_{\lambda_3}$, where $\lambda_3=(\xi,\alpha,\mathbb{L}\mathrm{im})$ and $\mathbb{L}\mathrm{im}$ is an arbitrary element of $\mathbb{L}^m$, and the proof finishes. ∎

Remark 5.2. For a very valuable interpretation of the weights $\{\gamma_k^i\}$ in Theorem 5.1, see [13, Remark 4.2].

5.2. Minmax robust solutions of uncertain unconstrained convex scalar optimization problems. Consider problem (UP) with the following data: X is convex, $\mathcal{U}=\mathbb{N}$, $m=1$, and $D(k)=\mathbb{R}_+$ for all $k\in\mathbb{N}$. One of the most popular approaches to solving this problem is to obtain the solutions of the next optimization problem, which are named minmax robust solutions (see [5] and the references therein):

$$\text{Minimize } \sup\{f(x,k):k\in\mathbb{N}\} \text{ subject to } x\in X.$$

In what follows we state that minmax robust solutions are weakly efficient solutions. Hence, they can be obtained as elements of $\operatorname{argmin}\Phi_{\lambda_3}$ whenever problem (UP) is convex.

LEMMA 5.3. *If $x_0\in X$ is a minmax solution of problem* (UP)*, then $x_0\in\mathrm{WE}(P)$.*

Proof. Suppose by contradiction that $x_0\notin\mathrm{WE}(P)$. Then, there exists $x\in X$ such that $f^x\leq_{\mathrm{int}P\cup\{0\}}f^{x_0}$, $f^x\neq f^{x_0}$. Thus, $f^{x_0}-f^x\in\mathrm{int}P$ and by Lemma 3.3(ii) there exists $\varepsilon>0$ such that $f(x_0,k)-f(x,k)\geq\varepsilon$ for all $k\in\mathbb{N}$. It follows that

$$\begin{aligned}\sup\{f(x_0,k):k\in\mathbb{N}\}&\geq\sup\{f(x,k)+\varepsilon:k\in\mathbb{N}\}\\&=\sup\{f(x,k):k\in\mathbb{N}\}+\varepsilon\\&>\sup\{f(x,k):k\in\mathbb{N}\},\end{aligned}$$

which is a contradiction since x_0 is a minmax solution of problem (UP). Therefore, $x_0\in\mathrm{WE}(P)$ and the proof is completed. ∎

Remark 5.4. It is not true that every weakly efficient solution of problem (UP) is a minmax solution. For instance, in the first problem of Example 4.17, $x_0=0$ is a weakly efficient solution, but it is not a minmax solution since

$$\sup\{f(x,k):k\in\mathbb{N}\}=\begin{cases}0 & \text{if}\quad x\neq0,\\ 1 & \text{if}\quad x=0.\end{cases}$$

As an obvious consequence of Lemma 5.3 and Theorem 4.7 we deduce the following result.

THEOREM 5.5. *Suppose that $f(\cdot,k):X\to\mathbb{R}$ is convex for all $k\in\mathbb{N}$. If $x_0\in X$ is a minmax robust solution of problem* (UP)*, then there exists $\lambda_3=((\xi_k)_k,\alpha,\mathrm{Lim})\in\ell_1\times\mathbb{R}_+\times\mathbb{L}$, $((\xi_k)_k,\alpha)\neq(0,0)$, such that $\xi_k\geq0$ for all k and $x_0\in\operatorname{argmin}\Phi_{\lambda_3}$, i.e.,*

$$\sum_{k=1}^{\infty}\xi_kf(x_0,k)+\alpha\mathrm{Lim}(f^{x_0})\leq\sum_{k=1}^{\infty}\xi_kf(x,k)+\alpha\mathrm{Lim}(f^x)\ \text{ for all } x\in X.$$

Moreover, if the sequence $(f(\cdot,k))_k$ pointwise converges, then $\mathrm{Lim}(f^x)$ *can be replaced above with* $\lim_{k\to\infty}f(x,k)$ *for all $x\in X$.*

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