A general model for dealing with ranking voting systems

Bonifacio Llamazares*

Departamento de Economía Aplicada, Instituto de Matemáticas (IMUVA), Universidad de Valladolid, Avda. Valle de Esgueva 6, 47011 Valladolid, Spain.

Abstract

A key problem in decision-making is selecting a winning candidate or establishing a global ranking for a set of candidates when individuals' preferences are expressed through linear orders. Scoring rules are a specific case of positional voting systems (PVSs) that are widely used in sports competitions. Likewise, some scoring rules, such as the Borda rule and plurality, have also been extensively analyzed in the field of social choice. However, the choice of the scoring vector may significantly influence the results, leading to the development of models that avoid subjective vector selection. In this paper, we introduce a general model that encompasses some previous proposals present in the literature. Our model does not have an important deficiency that some other models do, such as the fact that the relative order between two candidates may change even if there is no variation in the positions obtained by those candidates. We give an explicit formula for calculating candidate scores, enabling direct determination of winners or rankings without solving the model for each candidate, and we also analyze the fulfillment of some well-known properties. Likewise, through theoretical analysis and examples, we identify and rule out specific PVSs that may yield questionable outcomes.
Keywords: Decision support systems, ranking voting systems, positional voting systems, uncertain weights, surrogate weights, dominated winner paradox, absolute winner paradox.

1. Introduction

A central issue in the fields of decision-making and social choice is to select a winning candidate or establish a global ranking for a set of candidates (or alternatives) based on the individual rankings provided by a group of voters. This problem has been extensively studied within the field of social choice, particularly

*Corresponding author. Tel.: +34-983-186544; fax: +34-983-423299.
Email address: `b.llamazares@uva.es` (Bonifacio Llamazares)

following the publication of Arrow's theorem (Arrow, 1963). The determination of the winning candidates can be carried out through the so-called voting systems, which can be formalized as follows. Given a set of $m \geq 3$ candidates, $\mathcal{A} = \{A_1, \ldots, A_m\}$, each voter expresses their preferences through linear orders (complete, transitive, and antisymmetric binary relations) on the set $\mathcal{A}$. In this framework, a voting system (also called social choice rule) is a function $C : \bigcup\limits_{n\geq 2} \mathcal{P}^n \longrightarrow 2^{\mathcal{A}} \setminus \{\varnothing\}$, where $\mathcal{P}$ is the set of linear orders on $\mathcal{A}$ and each vector of linear orders on $\mathcal{A}$ representing the preferences of voters is called a profile.

Positional voting systems (PVSs) are an important class of voting systems where the winning candidates (or the ranking of the candidates) are chosen from the number of times each candidate appears in each position (see Llamazares & Peña (2015a), Bossert & Suzumura (2020), and references therein, for an analysis of the most commonly used PVSs). The most well-known PVSs are the scoring rules (see, for instance, Kondratev et al. (2024) for a deep analysis of geometric scoring rules in the context of sports competitions). In these functions, fixed scores are assigned to the different ranks obtained by the candidates and these ones are ordered according to the total number of points they receive. To be more specific, given the scoring vector $\mathbf{w} = (w_1, \ldots, w_k)$ (where $k \leq m$ is the number of scoring positions), the score obtained by the candidate A_o is $Z_o = \sum_{j=1}^{k} v_{oj} w_j$, where v_{oj} is the number of jth place ranks that candidate A_o occupies and w_j is the score given to the jth place.

It is evident that the chosen weight vector can determine the winning candidate (or the overall ranking of the candidates). For example, in the 2008 Formula One World Championship, the winner was Lewis Hamilton with 98 points, followed by Felipe Massa with 97. If the scoring system had not been changed in 2003 and the 2002 system had continued to be applied, the winner would have been Felipe Massa with 83 points, followed by Lewis Hamilton with 80 points.

To avoid a subjective selection of the scoring vector, Cook & Kress (1990) introduced a model based on Data Envelopment Analysis (DEA) to evaluate each candidate using the most favorable scoring vector for him/her. However, as is common in DEA methodology, many candidates often turn out to be efficient, meaning they achieve the maximum score. Consequently, various models have been proposed in the literature to distinguish among efficient candidates (see, for instance, Green et al., 1996; Hashimoto, 1997; Obata & Ishii, 2003; Foroughi et al., 2005; Foroughi & Tamiz, 2005; Wang & Chin, 2007; Wang et al., 2008).

Besides the above models, which are primarily based on DEA methodology, there exist other similar models where unknown or uncertain weights are also used (see, among others, Hashimoto & Wu, 2004; Contreras et al., 2005; Wang et al., 2007; Contreras, 2011; Foroughi & Aouni, 2012; Llamazares & Peña, 2013; Llamazares, 2016; Viappiani, 2020, 2024).

However, in some of the models proposed in the literature, the imposed conditions result in the winner (or the relative order between two candidates) depending on the results obtained by other candidates, which does not seem very reasonable (Llamazares & Peña, 2009). For example, imagine that in the Formula One World Championship, driver A finishes ahead of driver B, but the following year, with both achieving the same results as the previous year, driver B finishes ahead of driver A. It seems unlikely that a method with this deficiency would be implemented in practice.

Consequently, from our point of view, the following property should be required of any ranking voting system: when there is no variation in the results obtained by two candidates, the relative order between those candidates must not change. It is important to emphasize that this property should not be confused with the Independence of Irrelevant Alternatives (IIA).[1] For example, scoring rules satisfy this property but not IIA.

Some models that satisfy the property we just discussed in the previous paragraphs are those proposed by Contreras et al. (2005), Llamazares & Peña (2013) and Llamazares (2016).[2] In the present paper, we introduce a model that generalizes the models proposed by these authors. Additionally, an explicit expression is provided for the score obtained by each candidate, which makes it possible to determine the winner (or a ranking of the candidates) without having to solve the model for each candidate.

A significant contribution of this paper lies in the development of a unified modeling framework that encompasses several existing PVSs. Unlike prior studies that focus on specific models, our approach synthesizes these disparate methods into a single generalized model. This unification not only provides a clearer understanding of the relationships among previously proposed models but also facilitates the derivation of explicit closed-form expressions for the candidate scores.

Some particular cases of special interest are also studied in detail, and in some cases, the explicit solution obtained is related to the score obtained when using scoring rules given by certain weights (surrogate weights). We also analyze the fulfillment of some common properties in the field of social choice. Specifically, we study whether the introduced PVSs satisfy the properties of monotonicity and homogeneity, and whether they are vulnerable to the dominated winner paradox and to the absolute winner paradox. In this regard, the generalized framework proposed in this paper enables novel theoretical insights into the behavior

[1]The Independence of Irrelevant Alternatives (IIA) states that the relative order between two alternatives should not change due to the presence or absence of other alternatives.

[2]In these models, the score obtained by each candidate only depends on the positions achieved by that candidate.

of PVSs.

The rest of the paper is organized as follows. In Section 2 we recall some well-known scoring rules, surrogate weights and models proposed in the literature, including the pioneering model of Cook & Kress (1990). In Section 3 we propose a general model that encompasses some of those seen in the previous section and provide a closed expression of the solution. Section 4 is dedicated to presenting the explicit expressions of the solutions for some particular cases of special interest involving scores obtained using surrogate weights. In Section 5, we analyze some properties of the introduced PVSs, such as monotonicity, homogeneity, and the vulnerability to the dominated winner paradox and the absolute winner paradox. In Section 6, we present some examples, both academic and from the sports domain, that allow us to rule out some of the introduced positional voting systems. Finally, some concluding remarks are provided in Section 7.

2. Preliminaries

This section provides a brief overview of some concepts and models used in the paper. In the first subsection, we review some well-known scoring rules and some surrogate weights that have been used in the field of multicriteria decision-making and that play a significant role in the closed-form expressions obtained in our work. In the second subsection, building on the pioneering model of Cook & Kress (1990), we present some models that have been studied in the literature and can be seen as particular cases of our general model.

2.1. Well-known scoring rules and surrogate weights

The two best-known and most studied scoring rules are possibly plurality and the Borda rule. Two families of scoring rules that incorporate the underlying idea of these procedures and that play a significant role in this paper are the following:

1. j-approval voting: this scoring rule assigns 1 point to the top-j places and zero points thereafter. The normalized weight vector is $(1/j, \dots, 1/j, 0, \dots, 0)$ and, when $j = 1$, we get the plurality rule.
2. j-truncated Borda rule (Fishburn, 1974): this scoring rule assigns j points to a first place, $j-1$ points to a second place, ..., 1 point to a jth place, and zero points thereafter. The normalized weight vector is $\left(\frac{2j}{j(j+1)}, \frac{2(j-1)}{j(j+1)}, \dots, \frac{2}{j(j+1)}, 0, \dots, 0\right)$ and, when $j = m-1$, we obtain the Borda rule.

Surrogate weights are frequently employed in multiattribute decision making problems when only ordinal information regarding the importance of attributes is accessible (typically, an ordinal ranking of the attributes where, we can suppose, without loss of generality, that $w_1 \geq w_2 \geq \cdots \geq w_k$). In this brief subsection we recall several sets of surrogate weights that have been proposed in the literature and that are of significant importance in this work (see, for instance, Roszkowska, 2013; Danielson & Ekenberg, 2017):

1. Equal weights (EW):
$$w_j = \frac{1}{k}, \quad j = 1, \ldots, k.$$
2. Rank sum (RS) weights (Stillwell et al., 1981):
$$w_j = \frac{k+1-j}{\sum_{r=1}^{k}(k+1-r)} = \frac{2(k+1-j)}{k(k+1)}, \quad j = 1, \ldots, k.$$
Notice that the procedure obtained by using rank sum weights corresponds to the normalized truncated Borda rule.
3. Rank reciprocal (RR) weights (Stillwell et al., 1981):
$$w_j = \frac{1/j}{\sum_{r=1}^{k} 1/r}, \quad j = 1, \ldots, k.$$
4. Rank exponent (RE) weights (Stillwell et al., 1981):
$$w_j = \frac{(k+1-j)^p}{\sum_{r=1}^{k}(k+1-r)^p}, \quad j = 1, \ldots, k.$$
5. Rank order centroid (ROC) weights (Barron, 1992):
$$w_j = \frac{1}{k}\sum_{r=j}^{k}\frac{1}{r}, \quad j = 1, \ldots, k.$$

2.2. Some models proposed in the literature with uncertain weights

In their pioneering work, Cook & Kress (1990) introduced Data Envelopment Analysis (DEA) in the field of ranking voting systems so that each candidate could be evaluated with the scoring vector that was most favorable to them. In his way, the score of candidate A_o is obtained through the following model:

$$\begin{aligned} Z_o^{\mathrm{CK}}(\varepsilon) = \max \ & \sum_{j=1}^{k} v_{oj} w_j, \\ \text{s.t.} \ & \sum_{j=1}^{k} v_{ij} w_j \leq 1, \quad i = 1, \ldots, m, \\ & w_j - w_{j+1} \geq d(j, \varepsilon), \quad j = 1, \ldots, k-1, \\ & w_k \geq d(k, \varepsilon), \end{aligned} \tag{1}$$

where $\varepsilon \geq 0$ and the functions $d(j,\varepsilon)$, called the discrimination intensity functions, are nonnegative and nondecreasing in ε. Furthermore, $d(j,0) = 0$ for all $j \in \{1,\dots,k\}$.

One important drawback of this model is that several candidates may be efficient, i.e., that they attain the maximum score ($Z_o^{\text{CK}}(\varepsilon) = 1$). For this reason, several alternative models have appeared in the literature (see, among others, Green et al., 1996; Hashimoto, 1997; Obata & Ishii, 2003; Foroughi et al., 2005; Foroughi & Tamiz, 2005; Wang & Chin, 2007; Wang et al., 2008). However, some of them have an important shortcomming: the number of first, second, …, kth ranks obtained by inefficient candidates may change the order of efficient candidates (Llamazares & Peña, 2009). Nevertheless, there are other models that do not present the aforementioned problem and, in addition, allow knowing the score obtained by the candidates through closed expressions, which avoids having to solve the models for each candidate. Next we present some of these models.

1. The model analyzed by Contreras et al. (2005) is the following:

$$\begin{aligned} Z_o^{\text{CHM}}(\varepsilon) = \max\ & \sum_{j=1}^{k} v_{oj} w_j, \\ \text{s.t.}\ & \sum_{j=1}^{k} w_j = 1, \\ & w_j - w_{j+1} \geq \alpha_j (w_{j+1} - w_{j+2}), \quad j = 1,\dots,k-2, \\ & w_{k-1} - w_k \geq \alpha_{k-1} w_k, \\ & w_k \geq \varepsilon \geq 0, \end{aligned} \tag{2}$$

   where $\alpha_j \geq 0$ for all $j = 1,\dots,k-1$. Note that the constraint $\sum_{j=1}^{k} w_j = 1$ can be replaced by $\sum_{j=1}^{k} w_j \leq 1$, since in optimality, the latter constraint is always saturated.

2. Llamazares & Peña (2013) suggested and studied the following model:

$$\begin{aligned} Z_o^{\text{LP}}(\varepsilon) = \max\ & \sum_{j=1}^{k} v_{oj} w_j, \\ \text{s.t.}\ & \sum_{j=1}^{k} (n - v_{oj}) w_j \leq m - 1, \\ & w_j - w_{j+1} \geq d(j,\varepsilon), \quad j = 1,\dots,k-1, \\ & w_k \geq d(k,\varepsilon), \end{aligned} \tag{3}$$

   where n is the number of voters. This model is very similar to the one proposed by Cook & Kress

(1990), with the exception that the constraints $\sum_{j=1}^{k} v_{ij} w_j \leq 1$ ($i = 1, \ldots, m$) have been replaced by $\sum_{j=1}^{k} (n - v_{oj}) w_j \leq m - 1$ (for a justification of this change, see Llamazares & Peña (2013)).

3. The model proposed and analyzed by Llamazares (2016) is

$$
\begin{aligned}
Z_o^{\mathrm{L}}(\varepsilon) = \max\ & \sum_{j=1}^{k} v_{oj} w_j, \\
\text{s.t.}\ & \sum_{j=1}^{k} (n - v_{oj}) w_j \leq m - 1, \\
& w_j - w_{j+1} \geq w_{j+1} - w_{j+2}, \quad j = 1, \ldots, k-2, \\
& w_{k-1} - w_k \geq w_k, \\
& w_k \geq \varepsilon,
\end{aligned}
\tag{4}
$$

where constraints from both previous models are combined (in the case of the first model, considering $\alpha_j = 1$ for all $j = 1, \ldots, k-1$).

It is important to highlight that, in the above models, the constraints that define the feasible set of weights can be of two types. On one hand, there are those that determine the difference between the weights, and on the other hand, those that ensure that the feasible set is bounded. For instance, in the above models, the constraints $\sum_{j=1}^{k} w_j \leq 1$ and $\sum_{j=1}^{k} (n - v_{oj}) w_j \leq m - 1$ belong to the latter category. Taking these observations into account, in the following section we will propose a general model that considers these two types of constraints.

3. The general model

Models (1), (2), (3), and (4) presented in the previous section can be encompassed within the following model:

$$
\begin{aligned}
Z_o(\varepsilon) = \max\ & \mathbf{v}_o^{\mathrm{T}} \mathbf{w}, \\
\text{s.t.}\ & \mathbf{A}\mathbf{w} \leq \mathbf{b}, \\
& \mathbf{U}\mathbf{w} \geq \mathbf{d},
\end{aligned}
\tag{5}
$$

where $\mathbf{v}_o^{\mathrm{T}}$ is the row matrix that gathers the number of rank positions obtained by A_o, $\mathbf{d}$ is the column matrix that collects the discrimination intensity functions $d(j, \varepsilon)$, $\mathbf{U}\mathbf{w} \geq \mathbf{d}$ are the constraints that determine the relationships among the weights,[3] while $\mathbf{A}\mathbf{w} \leq \mathbf{b}$ are the ones that ensure that the feasible set is bounded.

[3]The notation $\geq$ between vectors has the usual meaning; that is, $\mathbf{x} \geq \mathbf{y}$ denotes $x_i \geq y_i$ for all $i \in \{1, \ldots, k\}$. Analogously, $\mathbf{x} > \mathbf{y}$ means that $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

In Cook and Kress' original model, $\mathbf{A}$ is a matrix of size $m \times k$, whereas in the remaining models shown in the previous section, matrix $\mathbf{A}$ only has one row; that is, in Contreras, Hinojosa and Mármol's model, the matrix $\mathbf{A}$ is given by $\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}$, whereas in Llamazares and Peña's model, and Llamazares' model, the matrix $\mathbf{A}$ is $\begin{pmatrix} n - v_{o1} & \dots & n - v_{ok} \end{pmatrix}$.

As we will see in Theorem 1, the main advantage of having matrix $\mathbf{A}$ with only one row is that, if certain assumptions about matrices $\mathbf{A}$ and $\mathbf{U}$ are fulfilled, it is possible to provide a closed-form expression for the scores obtained by the candidates and, consequently, the candidates can be ranked without having to solve the model for each one of them. Given that when $\mathbf{A}$ has only one row $\mathbf{b}$ is a number, it will be denoted by b. Moreover, given a row matrix $\mathbf{M}$, $(\mathbf{M})_j$ will denote the jth element of $\mathbf{M}$.

Theorem 1. *Consider Model (5), where* $\mathbf{U}$ *is an invertible matrix of order k such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative and* $\mathbf{A}$ *is an* $1 \times k$ *nonnegative matrix such that* $\mathbf{AU}^{-1}$ *is a positive matrix (that is,* $(\mathbf{AU}^{-1})_j > 0$ *for all* $j \in \{1, \dots, k\}$*). Then*

$$Z_o(\varepsilon) = (b - \mathbf{AU}^{-1}\mathbf{d}) \max_{j=1,\dots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{AU}^{-1})_j} + \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{d}. \tag{6}$$

Proof. Consider the change of variables $\mathbf{s} = \mathbf{Uw} - \mathbf{d}$, from where we have $\mathbf{w} = \mathbf{U}^{-1}\mathbf{s} + \mathbf{U}^{-1}\mathbf{d}$. Therefore, Model (5) can be expressed as

$$\begin{aligned} Z_o(\varepsilon) = \max\ & \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{s} + \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{d}, \\ \text{s.t. } & \mathbf{AU}^{-1}\mathbf{s} \le b - \mathbf{AU}^{-1}\mathbf{d}, \\ & \mathbf{s} \ge \mathbf{0}. \end{aligned} \tag{7}$$

Since $\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{d}$ is constant, Model (7) is equivalent to the following one:

$$\begin{aligned} \widetilde{Z}_o(\varepsilon) = \max\ & \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{s}, \\ \text{s.t. } & \mathbf{AU}^{-1}\mathbf{s} \le b - \mathbf{AU}^{-1}\mathbf{d}, \\ & \mathbf{s} \ge \mathbf{0}, \end{aligned} \tag{8}$$

with $Z_o(\varepsilon) = \widetilde{Z}_o(\varepsilon) + \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{d}$. Moreover, given that if a linear program has an optimal solution, then its dual also has an optimal solution and the optimal values for both problems are equal, it is sufficient to solve the dual of Model (8), i.e.,

$$\begin{aligned} \min\ & (b - \mathbf{AU}^{-1}\mathbf{d})X, \\ \text{s.t. } & (\mathbf{AU}^{-1})_j X \ge (\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j, \quad j = 1, \dots, k, \\ & X \ge 0. \end{aligned}$$

Since $(\mathbf{AU}^{-1})_j > 0$ and $(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j \geq 0$ for all $j \in \{1, \ldots, k\}$, the optimal solution is

$$X^* = \max_{j=1,\ldots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{AU}^{-1})_j}$$

and, consequently,

$$Z_o(\varepsilon) = (b - \mathbf{AU}^{-1}\mathbf{d}) \max_{j=1,\ldots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{AU}^{-1})_j} + \mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{d}. \qquad \square$$

Note that the value of $Z_o(\varepsilon)$ is obtained by calculating the maximum of certain quantities. This is because model (5) is linear, so the optimum is found at the vertex of the feasible set where the objective function takes on a maximum value. It is also important to note that if $(\mathbf{AU}^{-1})_j = 0$ for some $j \in \{1, \ldots, k\}$, then model (7) (and consequently, also model (5)) is unbounded, and, in this case, candidate A_o will be a winner.

Models (1), (2), (3), and (4) are obtained by taking specific values for $\mathbf{A}$, $\mathbf{U}$, and $\mathbf{d}$. Regarding $\mathbf{d}$, some typical values used in the literature are as follows:

(c1) $d(j, \varepsilon) = 0$ for all $j \in \{1, \ldots, k\}$.

(c2) $d(j, \varepsilon) = 0$ for all $j \in \{1, \ldots, k-1\}$ and $d(k, \varepsilon) = \varepsilon$.

(c3) $d(j, \varepsilon) = \varepsilon$ for all $j \in \{1, \ldots, k\}$.

(c4) $d(j, \varepsilon) = \varepsilon/j$ for all $j \in \{1, \ldots, k\}$.

Notice that in the four cases $\mathbf{d}$ can be written as $\mathbf{d} = \varepsilon\mathbf{c}$, where $\mathbf{c} = \mathbf{0}$, $\mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$, $\mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$, and $\mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 1 & 1/2 & \ldots & 1/(k-1) & 1/k \end{pmatrix}$, respectively. Given that, by using the notation $M_o = \max_{j=1,\ldots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{AU}^{-1})_j}$, the score of candidate A_o given by expression (6) can be rewritten as

$$Z_o(\varepsilon) = bM_o + (\mathbf{v}_o^{\mathrm{T}} - M_o\mathbf{A})\mathbf{U}^{-1}\mathbf{d},$$

when $\mathbf{d} = \varepsilon\mathbf{c}$ we get

$$Z_o(\varepsilon) = bM_o + \varepsilon(\mathbf{v}_o^{\mathrm{T}} - M_o\mathbf{A})\mathbf{U}^{-1}\mathbf{c}. \tag{9}$$

Obviously, when $\varepsilon = 0$, cases *(c2)*, *(c3)*, and *(c4)* coincide with *(c1)*, and we have $Z_o(0) = bM_o$.

The issue that arises in cases *(c2)*, *(c3)*, and *(c4)* when $\varepsilon > 0$ is determining the most suitable value for this parameter. A solution that has been used in the literature is to choose the maximum possible value of

ε (Cook & Kress, 1990). However, in some cases, this corresponds to using a scoring rule (Green et al., 1996) and, at other times, it does not seem to be the best solution (Llamazares & Peña, 2013; Llamazares, 2016). Another possibility that has been proposed in the literature (Llamazares & Peña, 2013) is to consider the average value of the scores obtained when ε varies in its interval of values. Note that the feasible set of model (7) is empty when $b < \mathbf{A}\mathbf{U}^{-1}\mathbf{d}$, and has a single element ($\mathbf{s} = \mathbf{0}$) if $b = \mathbf{A}\mathbf{U}^{-1}\mathbf{d}$. Hence, the maximum possible value of ε is obtained when $b = \mathbf{A}\mathbf{U}^{-1}\mathbf{d}$ and, for the previously seen cases, where $\mathbf{d} = \varepsilon\mathbf{c}$ with $\mathbf{c} > \mathbf{0}$,[4] the maximum value is $\varepsilon'_o = b/\mathbf{A}\mathbf{U}^{-1}\mathbf{c}$.

Given that $Z_o(\varepsilon)$ is an affine function of ε (see expression (9)), its average value in the interval $[0, \varepsilon'_o]$,

$$\overline{Z}_o = \frac{1}{\varepsilon'_o}\int_0^{\varepsilon'_o} Z_o(\varepsilon)\, d\varepsilon,$$

is obtained by evaluating the function $Z_o(\varepsilon)$ at the midpoint of that interval; that is,

$$\begin{aligned}\overline{Z}_o &= bM_o + \frac{b}{2\mathbf{A}\mathbf{U}^{-1}\mathbf{c}}(\mathbf{v}_o^{\mathrm{T}} - M_o\mathbf{A})\mathbf{U}^{-1}\mathbf{c}\\ &= b\left(M_o + \frac{\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c}}{2\mathbf{A}\mathbf{U}^{-1}\mathbf{c}} - \frac{M_o}{2}\right)\\ &= \frac{b}{2}\left(M_o + \frac{\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c}}{\mathbf{A}\mathbf{U}^{-1}\mathbf{c}}\right). \end{aligned} \tag{10}$$

Note that, once again, since $Z_o(\varepsilon)$ is an affine function, the value of $\overline{Z}_o$ can be obtained as the average of the values $Z_o(0) = bM_o$ and $Z_o(\varepsilon'_o) = (b\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}\mathbf{U}^{-1}\mathbf{c})$. Notice also that both values have an easy interpretation: The value bM_o is the maximum value that candidate A_o can obtain when he/she is evaluated with the weights of the feasible set $\{\mathbf{w} \mid \mathbf{A}\mathbf{w} \le \mathbf{b}, \mathbf{U}\mathbf{w} \ge \mathbf{0}\}$, whereas the value $(b\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}\mathbf{U}^{-1}\mathbf{c})$ is the score obtained by A_o when he/she is evaluated with the vector $(b\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}\mathbf{U}^{-1}\mathbf{c})$. In general, $(b\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}\mathbf{U}^{-1}\mathbf{c})$ is not a weight vector,[5] but it is when $\mathbf{A} = \begin{pmatrix}1 & 1 & \dots & 1 & 1\end{pmatrix}$, which means that all candidates are being evaluated with the same scoring rule. As we will see in Section 4, when we consider certain matrices for $\mathbf{U}$, this scoring vector corresponds to one of the surrogate weights seen in Section 2.1. Therefore, when $\mathbf{A} = \begin{pmatrix}1 & 1 & \dots & 1 & 1\end{pmatrix}$, the value $\overline{Z}_o$ is the average of two scores. The first one is the maximum score achievable in a specific set of weights (what corresponds to the original idea of Cook & Kress (1990)), while the second score is obtained through a specific weight vector, which are surrogate weights in some cases.

[4]From now on, throughout the paper, we will assume that $\mathbf{c} > \mathbf{0}$, since the case $\mathbf{c} = \mathbf{0}$ can be obtained by taking $\varepsilon = 0$.

[5]Besides, in the case $A = \begin{pmatrix}n - v_{o1} & \dots & n - v_{ok}\end{pmatrix}$, this vector is different for each of the candidates.

The following section will explicitly present the scores obtained by the candidates for some widely used cases in the literature.

4. Some particular cases

In this section we are going to show explicit expressions of the scores obtained by the candidates when specific values of $\mathbf{U}$, $\mathbf{A}$, and $\mathbf{c}$ are considered. In the first subsection, we demonstrate an interesting relationship that exists between the models obtained when using two particular cases of matrix $\mathbf{A}$, while in the second subsection, we provide explicit expressions for candidate scores for particular cases of $\mathbf{U}$ that have been widely used in the literature.

4.1. Particular cases for the matrix $\mathbf{A}$

As mentioned earlier, two matrices of size $1 \times k$ have been used in the literature for the constraint $\mathbf{A}\mathbf{w} \leq \mathbf{b}$: $\mathbf{A}_1 = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \end{pmatrix}$ (with $b = 1$, see, for instance, Contreras et al. (2005)), and $\mathbf{A}_2 = \begin{pmatrix} n - v_{o1} & \dots & n - v_{ok} \end{pmatrix}$ (with $b = m - 1$, see, for instance, Llamazares & Peña (2013)). As we will demonstrate in the following theorem, both matrices yield the same ranking when $\mathbf{d} = \mathbf{0}$. It is important to highlight that this result generalizes the one obtained in Llamazares (2016) (Proposition 1) for a specific case of the matrix $\mathbf{U}$.

Theorem 2. *Let* $\mathbf{U}$ *be an invertible matrix of order* k *such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative and* $\mathbf{A}_1\mathbf{U}^{-1}$ *and* $\mathbf{A}_2\mathbf{U}^{-1}$ *are both positive matrices. Then the models*

$$
\begin{aligned}
Z_o^1 = \max\ & \mathbf{v}_o^{\mathrm{T}}\mathbf{w}, & \qquad Z_o^2 = \max\ & \mathbf{v}_o^{\mathrm{T}}\mathbf{w}, \\
\text{s.t. } & \mathbf{A}_1\mathbf{w} \leq 1, & \text{s.t. } & \mathbf{A}_2\mathbf{w} \leq m - 1, \\
& \mathbf{U}\mathbf{w} \geq \mathbf{0}, & & \mathbf{U}\mathbf{w} \geq \mathbf{0},
\end{aligned}
$$

yield the same ranking of the candidates.

Proof. By Theorem 1 we know that $Z_o^1 = \max_{j=1,\dots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}$ and $Z_o^2 = (m-1) \max_{j=1,\dots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_2\mathbf{U}^{-1})_j}$. Notice that $\mathbf{A}_2 = n\mathbf{A}_1 - \mathbf{v}_o^{\mathrm{T}}$ and, consequently,

$$
\frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_2\mathbf{U}^{-1})_j} = \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{n(\mathbf{A}_1\mathbf{U}^{-1})_j - (\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j} = \left(n\frac{(\mathbf{A}_1\mathbf{U}^{-1})_j}{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j} - 1 \right)^{-1} = \left(n\left(\frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j} \right)^{-1} - 1 \right)^{-1}.
$$

Given that, for any $x, y > 0$, we have

$$
(nx^{-1} - 1)^{-1} > (ny^{-1} - 1)^{-1} \ \Leftrightarrow\ nx^{-1} - 1 < ny^{-1} - 1 \ \Leftrightarrow\ x > y,
$$

then, for any $p, q \in \{1, \dots, m\}$ and for any $i, j \in \{1, \dots, k\}$ we get

$$\frac{(\mathbf{v}_p^{\mathrm{T}}\mathbf{U}^{-1})_i}{(\mathbf{A}_2\mathbf{U}^{-1})_i} > \frac{(\mathbf{v}_q^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_2\mathbf{U}^{-1})_j} \;\Leftrightarrow\; \frac{(\mathbf{v}_p^{\mathrm{T}}\mathbf{U}^{-1})_i}{(\mathbf{A}_1\mathbf{U}^{-1})_i} > \frac{(\mathbf{v}_q^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}.$$

When $p = q$, the previous relationship indicates that for each candidate A_p the maximum is achieved at the same index for the two matrices $\mathbf{A}_1$ and $\mathbf{A}_2$. Therefore, if we denote by i_p the index where the candidate A_p reaches the maximum, given two candidates A_p and A_q we have:

$$Z_p^2 > Z_q^2 \;\Leftrightarrow\; \frac{(\mathbf{v}_p^{\mathrm{T}}\mathbf{U}^{-1})_{i_p}}{(\mathbf{A}_2\mathbf{U}^{-1})_{i_p}} > \frac{(\mathbf{v}_q^{\mathrm{T}}\mathbf{U}^{-1})_{i_q}}{(\mathbf{A}_2\mathbf{U}^{-1})_{i_q}} \;\Leftrightarrow\; \frac{(\mathbf{v}_p^{\mathrm{T}}\mathbf{U}^{-1})_{i_p}}{(\mathbf{A}_1\mathbf{U}^{-1})_{i_p}} > \frac{(\mathbf{v}_q^{\mathrm{T}}\mathbf{U}^{-1})_{i_q}}{(\mathbf{A}_1\mathbf{U}^{-1})_{i_q}} \;\Leftrightarrow\; Z_p^1 > Z_q^1. \qquad \square$$

However, when $\mathbf{d} \neq \mathbf{0}$, the use of matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ may generate different rankings among the candidates (an example of this will be seen in Section 6, once the expressions allowing us to obtain the scores of the candidates for some specific cases are presented in the following subsection). It is also worth noting that the use of matrix $\mathbf{A}_2$ poses a serious shortcomming as the order of the candidates may depend on the value of n (the number of voters), which does not seem very reasonable (an example of this problem will also be seen in Section 6).

Taking into account the result shown in Theorem 2 and the above comments, in the following subsection we will only present the explicit expressions of the scores for matrix $\mathbf{A}_1$.

4.2. Particular cases for the matrix $\mathbf{U}$

We will consider two generic matrices that capture the structure of most models discussed in the literature. Owing to their broad applicability, these matrices have been used in previous studies. For example, Llamazares (2024) employed them to determine the centroid of simplices defined by constraints expressed in terms of these matrices.

The scores that will be shown are those corresponding to $Z_o(0) = bM_o$ (that is, the case *(c1)* of Section 3 where $\mathbf{d} = \mathbf{0}$), and $\overline{Z}_o$ (expression (10)) in the cases *(c2)*, *(c3)*, and *(c4)* seen previously also in Section 3. Additionally, only the scores obtained by the candidates for matrix $\mathbf{A}_1$ are provided and, as previously mentioned, the score $\overline{Z}_o$ is the average of $Z_o(0)$ and the score obtained using a scoring rule associated with a weight vector obtained when the value of epsilon is at its maximum[6]. If this weight vector corresponds to a surrogate weight, the surrogate will be indicated in parentheses.

[6]It is worth noting that in this case the feasible set collapses into a single vector. The explicit expression of these weight vectors can be seen in Llamazares (2024).

4.2.1. First general matrix

The first general matrix that we consider is given by

$$\mathbf{U} = \begin{pmatrix} \beta_1 & -\beta_2 & \dots & 0 & 0 \\ 0 & \beta_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_{k-1} & -\beta_k \\ 0 & 0 & \dots & 0 & \beta_k \end{pmatrix},$$

where $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_k$. These conditions are necessary to guarantee a decreasing sequence of weights ($w_j \geq w_{j+1},\ j = 1, \dots, k-1$), given that the constraints $\mathbf{Uw} \geq \mathbf{0}$ are as follows

$$\beta_1 w_1 \geq \beta_2 w_2 \geq \beta_3 w_3 \geq \cdots \geq \beta_k w_k \geq 0.$$

It is worth noting that these constraints have appear in the literature in the form $w_j \geq \alpha_j w_{j+1},\ (j = 1, \dots, k-1)$ (see, for instance, Ahn, 2017). However, the expressions $\beta_j w_j \geq \beta_{j+1} w_{j+1},\ (j = 1, \dots, k-1)$ have the advantage of allowing us to obtain $\mathbf{U}^{-1}$ in a more compact manner. It is straightforward to verify that this matrix is[7]

$$\mathbf{U}^{-1} = \begin{pmatrix} 1/\beta_1 & 1/\beta_1 & \dots & 1/\beta_1 & 1/\beta_1 \\ 0 & 1/\beta_2 & \dots & 1/\beta_2 & 1/\beta_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1/\beta_{k-1} & 1/\beta_{k-1} \\ 0 & 0 & \dots & 0 & 1/\beta_k \end{pmatrix}.$$

Here are the scores obtained by the candidates for specific values of β_j:[8]

1. When $\beta_j = 1$ $(j = 1, \dots, k)$, $\mathbf{Uw} \geq \mathbf{d}$ are the constraints introduced by Cook & Kress (1990) in their pioneering model. In this case we get

   (1c1) $Z_o(0) = \max\limits_{j=1,\dots,k} \dfrac{1}{j} \sum\limits_{r=1}^{j} v_{or}.$

[7]Explicit formulas for finding the inverse of triangular matrices are available in Baliarsingh & Dutta (2015) and Baliarsingh et al. (2018).

[8]To avoid an excessive proliferation of formulas, we will only show the expressions for certain values of β_j where surrogate weights primarily appear. However, there are other values of β_j that have also been used in the literature (for instance, $\beta_j = j$ $(j = 1, \dots, k)$, see Noguchi et al. (2002)) and for which the expressions for the scores can also be easily obtained.

(1c2) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{1}{k}\sum_{r=1}^{k} v_{or}\right)$ (equal weights).

(1c3) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{2}{k(k+1)}\sum_{r=1}^{k}(k+1-r)v_{or}\right)$ (RS weights).

(1c4) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{1}{k}\sum_{r=1}^{k}\left(\sum_{s=r}^{k}\frac{1}{s}\right)v_{or}\right)$ (ROC weights).

The score $Z_o(0)$ corresponds to selecting the highest value from the scores provided by the normalized j-approval voting, while the score of (*1c2*), (*1c3*), and (*1c4*) correspond to averaging $Z_o(0)$ with the scores obtained when equal weights, RS weights, and ROC weights are used.

2. When $\beta_j = 1/(k+1-j)$ $(j = 1, \ldots, k)$, $\mathbf{Uw} \geq \mathbf{0}$ are the constraints recently introduced by Llamazares (2024) and they are equivalent to

$$w_1 \geq \frac{k}{k-1}w_2 \geq \frac{k}{k-2}w_3 \geq \cdots \geq kw_k \geq 0.$$

In this case the scores are as follows:

(2c1) $Z_o(0) = \max_{j=1,\ldots,k} \frac{2}{j(2k+1-j)}\sum_{r=1}^{j}(k+1-r)v_{or}.$

(2c2) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{2}{k(k+1)}\sum_{r=1}^{k}(k+1-r)v_{or}\right)$ (RS weights).

(2c3) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{6}{k(k+1)(2k+1)}\sum_{r=1}^{k}(k+1-r)^2 v_{or}\right)$ (RE weights with $p = 2$).

(2c4) $\overline{Z}_o = \frac{1}{2}\left(Z_o(0) + \frac{4}{k(3k+1)}\sum_{r=1}^{k}\left((k+1-r)\sum_{s=r}^{k}\frac{1}{s}\right)v_{or}\right).$

4.2.2. Second general matrix

The second general matrix we consider is

$$\mathbf{U} = \begin{pmatrix} \delta_1 & -\delta_1-\delta_2 & \delta_2 & \ldots & 0 & 0 \\ 0 & \delta_2 & -\delta_2-\delta_3 & \ldots & 0 & 0 \\ 0 & 0 & \delta_3 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & \delta_{k-1} & -\delta_{k-1}-\delta_k \\ 0 & 0 & 0 & \ldots & 0 & \delta_k \end{pmatrix}$$

where $0 < \delta_1 \le \delta_2 \le \cdots \le \delta_k$. The constraints $\mathbf{Uw} \ge \mathbf{0}$ are

$$\delta_1(w_1 - w_2) \ge \delta_2(w_2 - w_3) \ge \cdots \ge \delta_{k-1}(w_{k-1} - w_k) \ge \delta_k w_k \ge 0,$$

and the inverse matrix of $\mathbf{U}$ is

$$\mathbf{U}^{-1} = \begin{pmatrix} \frac{1}{\delta_1} & \frac{1}{\delta_1} + \frac{1}{\delta_2} & \frac{1}{\delta_1} + \frac{1}{\delta_2} + \frac{1}{\delta_3} & \dots & \sum_{r=1}^{k-1} \frac{1}{\delta_r} & \sum_{r=1}^{k} \frac{1}{\delta_r} \\ 0 & \frac{1}{\delta_2} & \frac{1}{\delta_2} + \frac{1}{\delta_3} & \dots & \sum_{r=2}^{k-1} \frac{1}{\delta_r} & \sum_{r=2}^{k} \frac{1}{\delta_r} \\ 0 & 0 & \frac{1}{\delta_3} & \dots & \sum_{r=3}^{k-1} \frac{1}{\delta_r} & \sum_{r=3}^{k} \frac{1}{\delta_r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \dots & \frac{1}{\delta_{k-1}} & \frac{1}{\delta_{k-1}} + \frac{1}{\delta_k} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\delta_k} \end{pmatrix}.$$

The simplest case is obtained when $\delta_j = 1$ $(j = 1, \dots, k)$, and the constraints obtained for these values have been widely used in the literature (see, for instance, Stein et al., 1994; Contreras et al., 2005; Llamazares, 2016; Ahn, 2017). For these values of δ_j, the scores obtained by the candidates are as follows:

(3c1) $Z_o(0) = \max\limits_{j=1,\dots,k} \dfrac{2}{j(j+1)} \displaystyle\sum_{r=1}^{j} (j+1-r) v_{or}.$

(3c2) $\overline{Z}_o = \dfrac{1}{2}\left(Z_o(0) + \dfrac{2}{k(k+1)} \displaystyle\sum_{r=1}^{k} (k+1-r) v_{or} \right)$ (RS weights).

(3c3) $\overline{Z}_o = \dfrac{1}{2}\left(Z_o(0) + \dfrac{3}{k(k+1)(k+2)} \displaystyle\sum_{r=1}^{k} (k+1-r)(k+2-r) v_{or} \right).$

(3c4) $\overline{Z}_o = \dfrac{1}{2}\left(Z_o(0) + \dfrac{4}{k(k+3)} \displaystyle\sum_{r=1}^{k} \left(\sum_{s=r}^{k} \dfrac{s+1-r}{s} \right) v_{or} \right).$

The score $Z_o(0)$ corresponds to selecting the highest value from the scores provided by the normalized truncated Borda rules while the score of (*3c2*) corresponds to averaging $Z_o(0)$ with the score obtained when RS weights are used.

Finally, Table 1 presents the main characteristics of the models introduced in Subsection 2.2 with respect to the matrices $\mathbf{A}$ and $\mathbf{U}$, and the existence of a known explicit solution to the model.

Table 1: Main features of the models presented in Subsection 2.2.

Model	Matrix **A** used	Matrix **U** used	Known explicit solution
Cook and Kress' model	Each row lists candidate positions	First general matrix with $\beta_j = 1$	No
Contreras, Hinojosa and Mármol's model	$\mathbf{A}_1$	Second general matrix with $\delta_j = \prod_{l=0}^{j-1} \alpha_l$	Yes (in some cases)
Llamazares and Peña's model	$\mathbf{A}_2$	First general matrix with $\beta_j = 1$	Yes
Llamazares' model	$\mathbf{A}_2$	Second general matrix with $\delta_j = 1$	Yes

5. Properties

In this section, we analyze the fulfillment of some well-known properties in the field of social choice by the PVSs introduced in Sections 3 and 4. Typical properties that are usually required include anonymity and neutrality. In this regard, note that the given expressions for the scores $Z_o(0)$ and $\overline{Z}_o$ depend only on the vector $\mathbf{v}_o$, which gathers the number of rank positions obtained by A_o. Therefore, these expressions are symmetric functions both over individuals (anonymity) and over candidates (neutrality) and, as in the case of scoring rules, these properties are trivially satisfied. Next we will focus on the following properties: monotonicity, homogeneity, dominated winner paradox and absolute winner paradox. Besides, in the case of PVSs, the study of these voting systems is often conducted more easily by using the number of positions obtained by each candidate until the ith place (see, for instance, Fishburn, 1974; Moulin, 1988; Stein et al., 1994; Llamazares & Peña, 2009, 2013, 2015a,b; Viappiani, 2020, 2024). These values, $V_{oi} = \sum_{l=1}^{i} v_{ol}$, are usually called cumulative standings, and the vector $\mathbf{V}_o = (V_{o1}, V_{o2}, \ldots, V_{ok})$ represents the candidate A_o's cumulative standings.[9]

Next we show some lemmas that will be useful for demonstrating the properties. The first one is due to Carrizosa et al. (1995), and states that the extreme points of the set $\{\mathbf{w} \in \mathbb{R}^k \mid \mathbf{w} \geq \mathbf{0}, \mathbf{U}\mathbf{w} \geq \mathbf{0}, \mathbf{A}_1\mathbf{w} = 1\}$

[9] Although the cumulative standings of each candidate depend on the profile $\mathbf{p}$, in order to avoid cumbersome notation we shall omit $\mathbf{p}$ in the notation of these values when there is no possible confusion. When it will be necessary, we will use the notation V_{oi} and $\mathbf{V}_o$ for the profile $\mathbf{p}$, V'_{oi} and $\mathbf{V}'_o$ for the profile $\mathbf{p}'$ and so on.

(which is the feasible set of model (5) with $\mathbf{d} = \mathbf{0}$, $\mathbf{A} = \mathbf{A}_1$, and $\mathbf{b} = 1$) are the columns, normalized to sum to 1, of the inverse of the matrix that determines the relationships among the weights.

Lemma 1. *Let* $\mathbf{U}$ *be an invertible matrix of order k such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative. Then, the extreme points of the set* $\{\mathbf{w} \in \mathbb{R}^k \mid \mathbf{w} \geq \mathbf{0}, \mathbf{U}\mathbf{w} \geq \mathbf{0}, \mathbf{A}_1\mathbf{w} = 1\}$ *are the columns of* $\mathbf{U}^{-1}$ *normalized to add 1.*

The second lemma shows that if the restrictions $w_1 \geq w_2 \geq \cdots \geq w_k$ are guaranteed in our model, then the column vectors of $\mathbf{U}^{-1}$ also fulfill these restrictions. Note that the constraints $w_1 \geq w_2 \geq \cdots \geq w_k$ are naturally imposed in the context of scoring rules, and in fact, all the examples shown in subsection 4.2 ensure these inequalities.

Lemma 2. *Let* $\mathbf{U}$ *be an invertible matrix of order k such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative. If the restrictions* $\mathbf{U}\mathbf{w} \geq \mathbf{0}$ *guarantees the constraints* $w_1 \geq w_2 \geq \cdots \geq w_k$*, then each column of* $\mathbf{U}^{-1}$ *(whether they are normalized to sum to 1 or not) also satisfies those constraints.*

Proof. By Lemma 1, the columns of $\mathbf{U}^{-1}$, normalized to add 1, are the extreme points of a set of vectors that satisfy the constraints $w_1 \geq w_2 \geq \cdots \geq w_k$. The thesis of the lemma is obtained by considering that the extreme points belong to the set (therefore they satisfy the constraints), and obviously they are also satisfied even if the columns are not normalized. □

In the third lemma, we show that the scores given by our models to the candidates, $Z_o(0)$ and $\overline{Z}_o$, can be written in terms of the cumulative standings instead of the standings collected in the vector $\mathbf{v}_o$.

Lemma 3. *Consider the model*

$$\begin{aligned} Z_o(\varepsilon) = \max\ & \mathbf{v}_o^{\mathrm{T}}\mathbf{w}, \\ \text{s.t. } & \mathbf{A}_1\mathbf{w} \leq 1, \\ & \mathbf{U}\mathbf{w} \geq \varepsilon\mathbf{c}, \end{aligned} \tag{11}$$

where the restrictions $\mathbf{U}\mathbf{w} \geq \mathbf{0}$ *guarantees the constraints* $w_1 \geq w_2 \geq \cdots \geq w_k$*. If* $\mathbf{U}$ *is an invertible matrix of order k such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative, then the values* $Z_o(0) = M_o = \max\limits_{j=1,\dots,k} \dfrac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}$ *and* $\overline{Z}_o = \dfrac{1}{2}\left(M_o + \dfrac{\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c}}{\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c}}\right)$ *can be written as*

$$Z_o(0) = M_o = \max_{j=1,\dots,k}\left\{\sum_{i=1}^{k-1} V_{oi}(u_{ij} - u_{i+1j}) + V_{ok}u_{kj}\right\}, \tag{12}$$

$$\overline{Z}_o = \frac{1}{2}\left(\max_{j=1,\dots,k}\left\{\sum_{i=1}^{k-1} V_{oi}(u_{ij} - u_{i+1j}) + V_{ok}u_{kj}\right\} + \sum_{i=1}^{k-1} V_{oi}(u_i^{\mathbf{c}} - u_{i+1}^{\mathbf{c}}) + V_{ok}u_k^{\mathbf{c}}\right) \tag{13}$$

where $V_{oi} = \sum_{l=1}^{i} v_{ol}$ $(i = 1, \dots, k)$*, and for each* $i = 1, \dots, k-1$ *and* $j = 1, \dots, k$*,* $\sum_{i=1}^{k} u_{ij} = 1$*,* $u_{ij} - u_{i+1j} \geq 0$*,* $u_{kj} \geq 0$*,* $\sum_{i=1}^{k} u_i^{\mathbf{c}} = 1$*,* $u_i^{\mathbf{c}} - u_{i+1}^{\mathbf{c}} \geq 0$*, and* $u_k^{\mathbf{c}} \geq 0$*.*

Proof. Consider first the value $\frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}$. Notice that for each $j \in \{1, \dots, k\}$, $(\mathbf{A}_1\mathbf{U}^{-1})_j$ is the sum of the elements of the j-th column of $\mathbf{U}^{-1}$. Therefore, for any $j \in \{1, \dots, k\}$,

$$\frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j} = \sum_{i=1}^{k} v_{oi}u_{ij},$$

where $u_{ij} \geq 0$, $\sum_{i=1}^{k} u_{ij} = 1$, and, by Lemma 2, $u_{1j} \geq u_{2j} \geq \cdots \geq u_{kj}$. Besides, it is easy to check that, by using cumulative standings, the expression $\sum_{i=1}^{k} v_{oi}u_{ij}$ can be written as

$$\sum_{i=1}^{k-1} V_{oi}(u_{ij} - u_{i+1j}) + V_{ok}u_{kj},$$

where $V_{oi} = \sum_{l=1}^{i} v_{ol}$ $(i = 1, \dots, k)$, and $u_{ij} - u_{i+1j} \geq 0$ $(i = 1, \dots, k-1)$. Consequently,

$$Z_o(0) = M_o = \max_{j=1,\dots,k} \left\{ \sum_{i=1}^{k-1} V_{oi}(u_{ij} - u_{i+1j}) + V_{ok}u_{kj} \right\}.$$

With respect to $\overline{Z}_o$, notice that the value $(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c})$ is the score obtained by A_o when he/she is evaluated with the weight vector $(\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c})$. Furthermore, the vector $\mathbf{U}^{-1}\mathbf{c}$ (and, consequently, $(\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c})$) satisfies the constraints $w_1 \geq w_2 \geq \cdots \geq w_k$ since each column of $\mathbf{U}^{-1}$ meets them, and $\mathbf{c} > \mathbf{0}$. Therefore, $(\mathbf{U}^{-1}\mathbf{c})/(\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c})$ can be written as $\sum_{i=1}^{k} v_{oi}u_i^{\mathbf{c}}$, where where $u_i^{\mathbf{c}} \geq 0$, $\sum_{i=1}^{k} u_i^{\mathbf{c}} = 1$, and $u_1^{\mathbf{c}} \geq u_2^{\mathbf{c}} \geq \cdots \geq u_k^{\mathbf{c}}$. By using cumulative standings we get

$$\frac{\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c}}{\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c}} = \sum_{i=1}^{k-1} V_{oi}(u_i^{\mathbf{c}} - u_{i+1}^{\mathbf{c}}) + V_{ok}u_k^{\mathbf{c}},$$

and, consequently,

$$\overline{Z}_o = \frac{1}{2}\left(\max_{j=1,\dots,k} \left\{ \sum_{i=1}^{k-1} V_{oi}(u_{ij} - u_{i+1j}) + V_{ok}u_{kj} \right\} + \sum_{i=1}^{k-1} V_{oi}(u_i^{\mathbf{c}} - u_{i+1}^{\mathbf{c}}) + V_{ok}u_k^{\mathbf{c}} \right). \qquad \square$$

5.1. Monotonicity

The monotonicity can be considered a basic property in the field of social choice and formalizes the idea that increasing support for a candidate never hurts and may help it to win (Brams & Fishburn, 2002).

Definition 1. A PVS is monotonic if, for every profile $\mathbf{p}$, when some voters raise a candidate in their rankings without changing the orders of the remaining candidates, this candidate cannot have a worse outcome than in the initial situation.

Proposition 1. *Consider Model (11). If* $\mathbf{U}$ *is an invertible matrix of order* k *such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative and the restrictions* $\mathbf{U}\mathbf{w} \geq \mathbf{0}$ *guarantees the constraints* $w_1 \geq w_2 \geq \cdots \geq w_k$*, then the PVSs obtained from* $Z_o(0)$ *and* $\overline{Z}_o$ *satisfy the monotonicity property.*

Proof. Let $\mathbf{p}$ be a profile, and let A_p and A_q be two candidates such that $Z_p(0) \geq Z_q(0)$. Now, suppose that some voters raise A_p in their rankings without changing the order of the remaining candidates. Let $\mathbf{p}'$ be this new profile. It is easy to check that $\mathbf{V}'_p \geq \mathbf{V}_p$ and $\mathbf{V}'_q \leq \mathbf{V}_q$. Therefore, according to expression (12) and given that $u_{ij} - u_{i+1j} \geq 0 \;\; (i = 1, \ldots, k-1; \;\; j = 1, \ldots, k)$ and $u_{kj} \geq 0 \;\; (j = 1, \ldots, k)$, we get $Z'_p(0) \geq Z_p(0) \geq Z_q(0) \geq Z'_q(0)$.

By means of reasoning analogous to the previous one, it is obtained, using expression (13), that $\overline{Z}'_p \geq \overline{Z}'_q$ as long as $\overline{Z}_p \geq \overline{Z}_q$. □

5.2. Homogeneity

Homogeneity is a property that was introduced by Smith (1973), and the same author provided a convincing justification: "if each voter suddenly splits into r voters, each of whom has the same preferences as the original, it would be hard to imagine how the collective preference would change".

Definition 2. A PVS satisfies the homogeneity property if the replication of the same profile $\mathbf{p}$ lead to the same social choice.

The fulfillment of this property in different voting systems has been studied by several authors (see, for example, Fishburn (1977), or more recently, Viappiani (2024)). The PVSs introduced in this paper satisfy this property.

Proposition 2. *Consider Model (11). If* $\mathbf{U}$ *is an invertible matrix of order* k *such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative, then the PVSs obtained from* $Z_o(0)$ *and* $\overline{Z}_o$ *satisfy the homogeneity property.*

Proof. Suppose that given a profile $\mathbf{p}$, it is replicated r times (with r an integer greater than 1), so that

$\mathbf{p}' = r\,\mathbf{p}$. It is evident that, for each candidate A_o, we have $\mathbf{v}'_o = r\,\mathbf{v}_o$ and, consequently

$$Z'_o(0) = \max_{j=1,\dots,k} \frac{(r\,\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j} = rZ_o(0),$$

$$\overline{Z}'_o = \frac{1}{2}\left(\max_{j=1,\dots,k} \frac{(r\,\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j} + \frac{r\,\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1}\mathbf{c}}{\mathbf{A}_1\mathbf{U}^{-1}\mathbf{c}} \right) = r\,\overline{Z}_o.$$

Therefore, profiles $\mathbf{p}$ and $\mathbf{p}'$ lead to the same overall ranking of candidates in both PVSs. □

5.3. Dominated winner paradox

The dominated winner paradox was introduced by Fishburn (1974) (see also Felsenthal, 2012) and occurs when a candidate who is dominated by another becomes the winner, while the candidate who dominates him/her does not. Similar to the monotonicity property, immunity to this paradox is a commonly required feature in voting systems used in the field of social choice.

Definition 3. A PVS is vulnerable to the dominated winner paradox if candidate A_p may be a winner while candidate A_q may not, despite the fact that all voters prefer A_q to A_p.

Proposition 3. *Consider Model (11). If* $\mathbf{U}$ *is an invertible matrix of order k such that* $\mathbf{U}^{-1}$ *is componentwise nonnegative and the restrictions* $\mathbf{U}\mathbf{w} \geq \mathbf{0}$ *guarantees the constraints* $w_1 \geq w_2 \geq \cdots \geq w_k$*, then the PVSs obtained from* $Z_o(0)$ *and* $\overline{Z}_o$ *are immune to the dominated winner paradox.*

Proof. Let A_p and A_q be two candidates and let $\mathbf{p}$ be a profile such that all voters prefer A_q to A_p. It is straightforward to check that $\mathbf{V}_q \geq \mathbf{V}_p$. According to expression (12) and given that $u_{ij} - u_{i+1\,j} \geq 0 \ \ (i = 1, \dots, k-1;\ j = 1, \dots, k)$ and $u_{kj} \geq 0 \ \ (j = 1, \dots, k)$, we get $Z_q(0) \geq Z_p(0)$. Therefore, if A_p is a winning candidate, A_q must be as well.

Using reasoning analogous to the previous one and expression (13), it is proven that the PVS obtained from $\overline{Z}_o$ is immune to the dominated winner paradox. □

5.4. Absolute winner paradox

The immunity to the absolute winner paradox is a weaker property than Condorcet consistency[10] and ensures that if a candidate is ranked first by an absolute majority of voters, then he/she must be the only winner (see, for instance, Llamazares & Peña, 2015a).

[10] The Condorcet consistency criterion states that if there is a candidate who wins against all others in pairwise comparisons, that candidate should be the winner.

Definition 4. Let $\mathbf{p}$ be a profile. A candidate is the absolute winner if he/she is ranked first by an absolute majority of voters.

Definition 5. A PVS is immune to the absolute winner paradox if, for every profile $\mathbf{p}$, the absolute winner, whenever he/she exists, is the only winning candidate.

Unlike the previous properties, this property is only satisfied by the PVS obtained through the $Z_o(0)$ scoring. Additionally, its proof requires an extra condition on the first column of matrix $\mathbf{U}^{-1}$. However, it is worth noting that this condition does not exclude any of the particular cases discussed in Section 4

Proposition 4. *Consider Model (11) with $\varepsilon = 0$, and where the restrictions $\mathbf{Uw} \geq \mathbf{0}$ guarantees the constraints $w_1 \geq w_2 \geq \cdots \geq w_k$. If $\mathbf{U}$ is an invertible matrix of order k such that $\mathbf{U}^{-1}$ is componentwise nonnegative and the first column of $\mathbf{U}^{-1}$ has all its elements zero except the $(1,1)$ entry, then the PVS obtained from $Z_o(0)$ is immune to the absolute winner paradox.*

Proof. According to expression (9),

$$Z_o(0) = M_o = \max_{j=1,\dots,k} \frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}.$$

Besides, since the first column of $\mathbf{U}^{-1}$ has all its elements zero except the $(1,1)$ entry, we have

$$\frac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_1}{(\mathbf{A}_1\mathbf{U}^{-1})_1} = v_{o1}.$$

Let A_p be the absolute winner of a profile $\mathbf{p}$. If n is the number of voters, then we have $Z_p(0) \geq v_{p1} > n/2$. Let A_q be any other candidate. We have $v_{q1} < n/2$, and by Lemma 3, for any $j > 1$,

$$\begin{aligned}\frac{(\mathbf{v}_q^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j} &\leq \frac{n}{2}(u_{1j} - u_{2j}) + n\sum_{i=2}^{k-1}(u_{ij} - u_{i+1j}) + nu_{kj} \\ &= \frac{n}{2}(u_{1j} - u_{2j}) + nu_{2j} = \frac{n}{2}(u_{1j} + u_{2j}) \leq \frac{n}{2}.\end{aligned}$$

Therefore $Z_q(0) \leq n/2$ and, consequently, $Z_p(0) > Z_q(0)$. □

In the field of Social Choice, some authors are strong advocates of Condorcet consistency (see, for instance, Fishburn (1977), Felsenthal (2012), Felsenthal & Tideman (2014), and references therein) and, therefore, the immunity to the absolute winner paradox is also considered an essential property by them. However, it is important to highlight that, as in the case of Condorcet consistency (Fishburn, 1977), there are profiles in which the selection of an absolute winner may be questioned. For example, consider a profile

Table 2: Ranks and scores obtained by each candidate ($n = 16$).

Candidate	v_{i1}	v_{i2}	v_{i3}	v_{i4}	$\overline{Z}_o^{\mathbf{A}_1}$	$\overline{Z}_o^{\mathbf{A}_2}$
A	4	5	5	2	4.333	1.49
B	6	1	2	1	4.25	1.57
C	1	4	5	3	3.292	1.036
D	4	1	2	3	3.25	1.037
E	1	5	2	7	3.75	1.225

of n voters (with n being odd) where a candidate A receives $(n+1)/2$ first positions and $(n-1)/2$ last positions, while another candidate B receives $(n-1)/2$ first positions and $(n+1)/2$ second positions. Would it make sense for A to always be the winner as the number of voters and candidates increases?

In the next section we will show that the PVSs obtained with the $\overline{Z}_o$ scores from Subsection 4.2 are vulnerable to the absolute winner paradox.

6. Examples

In this section, we present various examples, both academic and from Grand Prix motorcycle racing, that allow us to observe the behavior of the methods introduced in Subsection 4.2.

6.1. Academic examples

In the first example we will show that matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ may generate different rankings when $\mathbf{d} \neq \mathbf{0}$, and that in the case of matrix $\mathbf{A}_2$, the number of voters may influence the final ranking of the candidates (both facts were discussed in Subsection 4.1). Suppose Table 2 shows the results obtained in the top four positions by five candidates when 16 voters express their preferences about the candidates.

Consider now the general matrix of Subsection 4.2.1 with $\beta_j = 1,\ (j = 1, \ldots, k)$, and $\mathbf{d} = \varepsilon\mathbf{c}$, where $\mathbf{c}^{\mathrm{T}} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$. The candidates' scores obtained using $\mathbf{A}_1$ can be calculated through expression (*1c2*), while those obtained using $\mathbf{A}_2$ are given by the following expression:

$$\overline{Z}_o^{\mathbf{A}_2} = \frac{m-1}{2}\left(\max_{j=1,\ldots,k} \frac{\sum_{r=1}^{j} v_{or}}{\sum_{r=1}^{j}(n - v_{or})} + \frac{\sum_{r=1}^{k} v_{or}}{\sum_{r=1}^{k}(n - v_{or})}\right).$$

Table 3: Ranks and scores obtained by each candidate (n = 46).

Candidate	v_{i1}	v_{i2}	v_{i3}	v_{i4}	$\overline{Z}_o^{\mathbf{A}_2}$
A	4	5	5	2	0.624
B	6	1	2	1	0.622
C	8	9	8	9	1.36
D	9	7	8	6	1.314
E	7	12	5	7	1.389
F	5	4	7	13	1.123
G	7	8	11	8	1.377

As can be seen in Table 2, the rank obtained with $\mathbf{A}_1$ is A $\succ$ B $\succ$ E $\succ$ C $\succ$ D, while the rank obtained with $\mathbf{A}_2$ is B $\succ$ A $\succ$ E $\succ$ D $\succ$ C.

Regarding the fact that when matrix $\mathbf{A}_2$ is used, the number of voters may change the ranking of the candidates, consider Table 3, which shows the top 4 positions obtained by 7 candidates based on the preferences of 46 voters. As shown in Tables 2 and 3, the results obtained by candidates A and B have not changed from one table to the other. However, when there are 16 voters (Table 2), B wins over A, whereas when 46 voters are considered (Table 3), A wins over B.

The second academic example illustrates the fact that PVSs obtained with the $\overline{Z}_o$ scores from Subsection 4.2 are vulnerable to the absolute winner paradox. Table 4 shows the top four positions obtained by five candidates, and it is easy to observe that candidate A is an absolute winner. However, as shown in Table 5, the winner in all cases is candidate B (scores for the remaining candidates are omitted).

The third academic example is taken from Llamazares & Peña (2009). In Table 6 we show the number of first and second ranks obtained by three candidates,[11] while in Table 7 we present the scores obtained by the candidates using the expressions given in Subsection 4.2.

Based on the results obtained, the following conclusions can be drawn:

1. In case (*1c1*), the final ranking is A $\sim$ C $\succ$ B. However, this result could be considered unfair from the perspective of candidate B. On one hand, since candidate A defeats B, it suggests that a single

[11] We assume that there are additional candidates, not included in Table 6, such that the sum of the first ranks equals the sum of the second ranks.

Table 4: Ranks obtained by five candidates (n = 15).

Candidate	v_{i1}	v_{i2}	v_{i3}	v_{i4}
A	8	0	0	0
B	7	8	0	0
C	0	4	3	5
D	0	2	5	6
E	0	1	7	4

Table 5: Scores obtained by the candidates of Table 4 using the expressions given in Subsection 4.2.

Cand.	(*1c2*)	(*1c3*)	(*1c4*)	(*2c2*)	(*2c3*)	(*2c4*)	(*3c2*)	(*3c3*)	(*3c4*)
A	5	5.6	6.083	5.6	6.133	6.564	5.6	6	6.286
B	5.625	6.35	6.656	6.314	6.781	6.958	6.267	6.617	6.762

Table 6: Ranks of each candidate.

Candidate	v_{i1}	v_{i2}
A	101	0
B	100	101
C	0	202

Table 7: Scores obtained by the candidates of Table 6 using the expressions given in Subsection 4.2.

Cand.	(*1c1*)	(*1c2*)	(*1c3*)	(*1c4*)	(*2c1*)	(*2c2*)	(*2c3*)	(*2c4*)	(*3c1*)	(*3c2*)	(*3c3*)	(*3c4*)
A	101	75.75	84.17	88.38	101	84.17	90.9	93.79	101	84.17	88.38	90.9
B	100.5	100.5	100.42	100.38	100.33	100.33	100.27	100.24	100.33	100.33	100.29	100.27
C	101	101	84.17	75.75	67.33	67.33	53.87	48.1	67.33	67.33	58.92	53.87

first-place rank is more valuable than 101 second-place finishes. On the other hand, since candidate C also defeats B, it implies that 101 second-place finishes are more valued than 100 first-place ranks. It is also important to highlight that the expressions obtained in case (*1c1*) coincide with those obtained in the model proposed by Obata & Ishii (2003) when the discrimination intensity functions are null and the L_1-norm is used (Llamazares & Peña, 2009).

2. In case (*1c2*) the overall ranking is C $\succ$ B $\succ$ A. Notice that in this case we have

$$\overline{Z}_o = \frac{1}{2}\left(\max\left(v_{o1}, \frac{v_{o1}+v_{o2}}{2}\right) + \frac{v_{o1}+v_{o2}}{2}\right).$$

   Therefore,

   (a) If $v_{o1} \geq v_{o2}$, then $\overline{Z}_o = \frac{3}{4}v_{o1} + \frac{1}{4}v_{o2}$; that is, the scoring vector used is $(3/4, 1/4)$.

   (b) If $v_{o1} < v_{o2}$, then $\overline{Z}_o = \frac{1}{2}v_{o1} + \frac{1}{2}v_{o2}$; i.e., the scoring vector used is $(1/2, 1/2)$.

   The use of these scoring vectors makes C the winner.

3. In cases (*2c1*) and (*3c1*) the global ranking is A $\succ$ B $\succ$ C. It is easy to check that in both cases, when $v_{o1} \geq v_{o2}$, we get $\overline{Z}_o = v_{o1}$, and when $v_{o1} < v_{o2}$, we get $\overline{Z}_o = \frac{2}{3}v_{o1} + \frac{1}{3}v_{o2}$. This results in A being the winner.

4. Lastly, in the remaining cases, the final ranking is B $\succ$ A $\succ$ C, which was possibly the expected order in light of the results obtained by the candidates.

6.2. Grand Prix motorcycle racing

The 1999 Grand Prix motorcycle racing in the 125 cc category was won by the Spanish rider Emilio Alzamora without winning any races. The scoring system of that championship used a scoring rule where the first 15 positions were taken into account and $(25, 20, 16, 13, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)$ was the scoring vector used. The ranks obtained by the three riders who finished in the top positions are shown in Table 8.

The winner of the championship was Emilio Alzamora, with 227 points, followed by Marco Melandri with 226 points, and in third place Masao Azuma with 190 points. In Table 9 we show the scores obtained by the riders using the expressions given in Subsection 4.2.[12]

In light of the results obtained, some interesting conclusions can be drawn:

1. All the cases rank Alzamora in third place.

[12]The complete data of all riders are available upon explicit request to the author.

Table 8: Ranks achieved by the top 3 riders in the 1999 Grand Prix motorcycle racing (125cc).

Rider	v_{i1}	v_{i2}	v_{i3}	v_{i4}	v_{i5}	v_{i6}	v_{i7}	v_{i8}	v_{i9}	v_{i10}	v_{i11}	v_{i12}	v_{i13}	v_{i14}	v_{i15}
Alzamora	0	5	5	2	0	2	0	0	0	0	0	0	0	0	1
Melandri	5	2	2	0	1	1	0	1	0	0	0	0	0	0	0
Azuma	5	0	0	1	1	2	1	0	0	1	0	1	0	1	0

Table 9: Scores obtained by the riders using the expressions given in Subsection 4.2.

Rider	(*1c1*)	(*1c2*)	(*1c3*)	(*1c4*)	(*2c1*)	(*2c2*)	(*2c3*)	(*2c4*)	(*3c1*)	(*3c2*)	(*3c3*)	(*3c4*)
Alzamora	3.33	2.17	2.42	2.53	3.21	2.36	2.54	2.58	2.7	2.1	2.27	2.31
Melandri	5	2.9	3.16	3.43	5	3.16	3.36	3.63	5	3.16	3.35	3.47
Azuma	5	2.93	3.08	3.27	5	3.08	3.2	3.41	5	3.08	3.19	3.28

2. The cases (*1c1*), (*2c1*) and (*3c1*), where $\varepsilon = 0$, do not discriminate between Melandri and Azuma. Let's remember that in these cases $Z_o(0) = \max_{j=1,\dots,k} \dfrac{(\mathbf{v}_o^{\mathrm{T}}\mathbf{U}^{-1})_j}{(\mathbf{A}_1\mathbf{U}^{-1})_j}$, and when v_{o1} is sufficiently large, $Z_o(0) = v_{o1}$.
3. In case (*1c2*), Azuma defeats Melandri, which is quite questionable. This is due to the fact that for both drivers $M_o = 5$ (the number of first positions) and $\overline{Z}_o$ is the average of this value with the score obtained using the equal weights scoring rule, making Azuma the winner since he has scored in more races.

The results displayed in Tables 7 and 9 suggest that not all cases examined in Subsection 4.2 are suitable for practical application. Specifically, the cases (*1c1*), (*1c2*), (*2c1*), and (*3c1*) exhibit certain deficiencies that, in our opinion, render them impractical for real-world use.

7. Concluding remarks

Although scoring rules are commonly used in sports competitions and certain areas of social choice (especially the Borda rule and plurality), the selection of the weight vector remains, in a way, arbitrary. However, such a choice is crucial, as it may determine the winning candidate or the final ranking of the candidates. For this reason, and based on the pioneering model by Cook & Kress (1990), various models have emerged in the literature with the aim of avoiding the subjectivity in selecting the weight vector.

In this paper we have proposed a general model for ranking voting systems that unifies and extends some previous approaches. An important contribution of this work is the explicit formulation of candidate scores, which allows the determination of the winning candidates (or a ranking of the candidates) without the need to solve the model individually for each candidate. This fact simplifies the application of the model in real-world scenarios and allows its analysis from a theoretical point of view. In this regard, the theoretical results obtained make it possible to study the fulfillment of certain well-known properties in the field of social choice, such as monotonicity, homogeneity, and whether these PVSs are vulnerable to the dominated winner paradox and to the absolute winner paradox.

It is also important to highlight that the theoretical results obtained allow us, in some cases, to express the score achieved by the candidates as the average of two scores, further enhancing its interpretability and practical relevance. The first score represents the maximum achievable value within a specific set of weights, aligning with the original concept proposed by Cook & Kress (1990). In contrast, the second score is determined using a specific weight vector, which is a surrogate weight vector in some cases.

These analytical insights contribute directly to the overarching objectives of the paper. In particular, our main contributions are twofold: first, the formal unification of multiple PVSs into a single comprehensive framework; and second, the derivation of novel theoretical results that deepen the understanding of such systems and the social choice properties they satisfy.

It is also worth noting that the examples presented illustrate the versatility of the model while helping to identify and rule out certain PVSs that may yield questionable outcomes.

A promising direction for future research would be to extend the analysis to multiple racing competitions in order to examine how frequently the different rules yield different winners, and to investigate whether any general patterns emerge. Another possible line of research would be to analyze the concept of distortion (see, for instance, Anshelevich et al. (2021)) in the context of the SVPs presented in this work.

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