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Polar and parabolic separable extensions of the two dimensional Helmholtz equation in free space: From geometric to dynamical symmetries

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ABSTRACT

We analyze two-dimensional systems related to the Helmholtz equation that allow separation of variables in both polar and parabolic coordinates. We pay special attention to the symmetry algebras involved in the separation of variables. We show how the modification of symmetry operators can lead from purely geometric symmetries to other dynamical ones, that is, from free systems to interacting systems, with the addition of potentials, which in our case are of two types: Kepler–Coulomb and Makarov. We also calculate the spectrum and associated eigenfunctions of the corresponding quantum mechanical systems, and we present a discussion of naturally separable classical systems, including the analysis of different types of trajectories. A discussion of the global properties of polar and parabolic coordinates is included, the relevance of which is demonstrated in the spectral and classical properties of these systems.

1. Introduction

The principles of geometry and symmetry are fundamental ingredients in the description of nature, and both have been crucial in the formulation of quantum theories. While the implications, observation, and use of symmetries in quantum physics are currently the subject of debate [1–3], the rise of other fields, such as optics and photonics, offers a solid foundation for the practical investigation of various quantum phenomena. Thus, for example, recent advances at the intersection of quantum physics and optical sciences [4–7], provide a better understanding of certain phenomena and an experimental platform for testing certain theoretical models that may anticipate new applications [8–10].

One of the main challenges in quantum and optical sciences is finding exact solutions to dynamical equations. However, the repertoire of analytical models is not very broad, either in non-homogeneous structures or in free space [11,12]. Among these, some of the most prominent ones are the families of structured beams, in homogeneous and parabolic media, exhibiting rectangular, parabolic and polar symmetry [13–19]. Each family constitutes a complete set of orthonormal states that spans the entire Hilbert space of localized modes in the paraxial regime. From the point of view of their potential applications, their relevance lies in the fact that they offer available degrees of freedom of light that allow, among other things, micromanipulation, micromachining [20], and multiplexing of optical signals to increase the capacity of communication systems [21,22]. However, the symmetries and Lie algebras underlying the propagation phenomena are of fundamental importance, as they allow the construction of wave packets as coherent states [16], with definite dynamical variables such as, for example, the orbital angular momentum [17,18]. Furthermore,

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it is well known that the study of the coherence and quality properties of light beams, such as quality factors and kurtosis, requires the determination of cross-correlations and statistical moments involving integrals of the field amplitudes [23–25]. Therefore, using the wave optics operator approach [26–28], the calculation of magnitudes such as mean values and variances can be determined elegantly and simply, using the algebraic properties of basic modal fields, without needing to evaluate very complicated integrals [16,18]. A pertinent question is whether it is possible to construct more general, exactly solvable Helmholtz models, preserving particular symmetries. The answer is yes, especially in free space or homogeneous media. However, exactly solvable models with underlying algebraic structures in gradient index (GRIN) materials constitute an area of research of their own [29–31].

In this work we address the construction of Helmholtz structures in GRIN materials that exhibit defined symmetries inherited from their free-space counterparts. We first analyze the possibility of performing separation of variables for the two-dimensional (2D) Helmholtz equation simultaneously in polar and parabolic coordinates. Next, and this is the fundamental objective of this paper, we consider the possibility of adding to the free 2D Helmholtz equation some potentials that maintain the separation property in the two aforementioned coordinate systems. This will lead us directly to the Kepler–Coulomb (KC) potential and also to the 2D Makarov potential [32]. This potential, which is little known, can be interpreted in centrifugal terms and is similar to the Hartmann potential [33–36].

The procedure we apply essentially consists of transforming geometric symmetries into dynamic ones (these geometric symmetries are Killing vectors or tensors that preserve the metric of the space [37]). During this process, the separation of variables is maintained, but at the same time the character of the symmetries changes with the addition of terms to the initial potential of the system, which alters the geometric symmetries. In our case, we have two compatible extensions. In the first stage, we move from a system with geometric parabolic symmetry to a dynamic one, maintaining geometric polar symmetry. In the second extension, we obtain both parabolic and polar dynamic symmetries, so that we lose the geometric character in both. The evolution from geometric to dynamic symmetries can be applied to other types of separation of variables, but we consider the example shown here to be quite illustrative of the general technique described.

Part of the problem we want to attack was considered in the pioneering work of Jauch and Hill [38]. Subsequently, the search for symmetries for non-relativistic systems began systematically [39], and outstanding contributions have been made since then on related problems (see [32,40,41]). Our desire is to adopt an approach that can be implemented in both classical and quantum systems, which will lead us to previously known results, but at the same time we will find new ones, such as the analysis of a KC-type centrifugal potential (known as Makarov). We will show how the existence of a separation of variables in parabolic coordinates will simplify the search for both symmetries and the spectrum in the quantum case, as well as for trajectories in the classical context.

The organization of the paper is as follows. The free Helmholtz equation in two dimensions is considered in Section 2. An extension to include the KC potential is developed in Section 3, while a further extension with an additional potential is presented in Section 4. Classical considerations are included in Section 5. It should be noted that, although generally ignored, due to global properties there are important restrictions when performing changes of variables, which will be discussed in Section 6. The paper ends with the conclusions presented in Section 7.

2. The 2D Helmholtz equation in polar and parabolic variables

The two-dimensional Helmholtz equation appears very frequently in optical and/or electromagnetic phenomena. For instance, in the paraxial regime of wave optics, the enveloping amplitude $U(\mathbf{r})$ of a stationary, harmonic optical mode, propagating along the z axis in a material of refractive index n, can be cast in the form $U(\mathbf{r}) = \psi(x,y)e^{-i\kappa\gamma z}$. In this expression κ is the wave number in free space and γ is a constant, called *effective refractive index*, defining the phase velocity of the mode. In turn, ψ is a function containing information about the phase and intensity transversal distributions and satisfies the two-dimensional Helmholtz equation

$$Q\psi(x,y) := (\Delta + \omega^2)\psi(x,y) = 0, \tag{2.1}$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the 2D Laplacian and $\omega^2 = \kappa^2(n^2 - n_0^2 + 2\gamma)$, with n_0 a constant reference refractive index. In the sequel, the Helmholtz operator Q in free space will be our fundamental object of study as we are interested in constructing dynamical symmetries from purely geometric ones. In such situation $n = n_0 = 1$ and $\omega^2 = 2\kappa^2\gamma = \text{constant}$. Notice that, in this case, the above Helmholtz equation can be interpreted in a quantum mechanical context as the eigenvalue problem, in natural units, of the 2D free particle of energy $E = \omega^2$.

2.1. First order symmetries of Q

It is easy to check that the operator Q in Cartesian coordinates, denoted as $Q(x,y)$, commutes with the angular momentum operator $J_z = i(-x\partial_y + y\partial_x)$ and with the translation operators $P_x = -i\partial_x$, $P_y = -i\partial_y$. These operators close the Euclidean algebra $\mathfrak{iso}(2)$ [42], the nonzero commutators being the following

$$[J_z, P_x] = iP_y, \quad [J_z, P_y] = -iP_x. \tag{2.2}$$

The operators P_x, P_y give rise to the obvious separation in Cartesian coordinates, x, y, while the rotation operator J_z allows for the existence of separable solutions in polar coordinates (ρ, θ). The Helmholtz operator Q in polar coordinates is

$$Q(\rho,\theta) = \partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_\theta^2 + \omega^2. \tag{2.3}$$

2.2. Symmetry operators for separation of variables in polar and parabolic coordinates

From first-order symmetries we can obtain second-order symmetries that will allow us to perform the separation in more coordinate systems [42,43]. In particular, let us consider the following operator $S_x = \{J_z, P_x\}$, where $\{A, B\} = AB + BA$ denotes the anti-commutator. This symmetry operator induces separation of variables in the parabolic coordinates (u, v) defined by [42]

$$\begin{cases} u = \sqrt{\sqrt{x^2+y^2}+y}, \\ v = \sqrt{\sqrt{x^2+y^2}-y}, \end{cases} \qquad \begin{cases} x = uv, \\ y = \frac{1}{2}(u^2 - v^2), \end{cases} \tag{2.4}$$

which can be expressed in terms of polar coordinates as follows

$$\begin{cases} u = \sqrt{\rho(1+\sin\theta)}, \\ v = \sqrt{\rho(1-\sin\theta)}, \end{cases} \qquad \begin{cases} \rho = \frac{1}{2}(u^2+v^2), \\ \tan\theta = \frac{1}{2}\frac{u^2-v^2}{uv}. \end{cases} \tag{2.5}$$

Then, for negative eigenvalues $E = -\beta^2$, in parabolic coordinates the Helmholtz operator Q becomes [42]:

$$Q(u, v) = \frac{1}{u^2+v^2}(\partial_u^2 + \partial_v^2) - \beta^2, \tag{2.6}$$

and the Helmholtz equation takes the form

$$Q(u, v)\psi(u, v) = 0 \implies \left(-(\partial_u^2 + \partial_v^2) + \beta^2(u^2+v^2)\right)\psi(u, v) = 0. \tag{2.7}$$

In other words, according to (2.1) and (2.7), the negative energy $-\beta^2$ solutions of a 2D free quantum system are in correspondence with those of a 2D isotropic harmonic oscillator of frequency β and zero energy.

Separation of variables is achieved by using the eigenvalue equation for S_x, with eigenvalue λ:

$$S_x(u, v)\psi(u, v) = \lambda\psi(u, v) \implies \frac{1}{u^2+v^2}(u^2\partial_v^2 - v^2\partial_u^2)\psi(u, v) = \lambda\psi(u, v). \tag{2.8}$$

The two eigenvalue equations for Q and S_x lead us to the equations that must be satisfied by factorable solutions of the form $\psi(u, v) = \sigma(u)\mu(v)$:

$$\left(-\partial_u^2 + \beta^2 u^2\right)\sigma(u) = \lambda\,\sigma(u), \qquad \left(-\partial_v^2 + \beta^2 v^2\right)\mu(v) = -\lambda\,\mu(v). \tag{2.9}$$

If we add the rotation symmetry J_z to the initial symmetry S_x, a three-dimensional Lie algebra is generated:

$$S_x = \{J_z, P_x\}, \qquad S_y = \{J_z, P_y\}, \qquad J_z, \tag{2.10}$$

which obeys the commutation relations:

$$[J_z, S_x] = iS_y, \qquad [J_z, S_y] = -iS_x, \qquad [S_x, S_y] = -4i\omega^2 J_z. \tag{2.11}$$

Crucially, the sign of ω^2 determines the specific algebra realized:

- If $\omega^2 > 0$, we have the algebra $\mathfrak{so}(2,1)$, with generators $\{\hat{S}_x = S_x/(2\omega),\ \hat{S}_y = S_y/(2\omega),\ J_z\}$:

  $$[J_z, \hat{S}_x] = i\hat{S}_y, \qquad [J_z, \hat{S}_y] = -i\hat{S}_x, \qquad [\hat{S}_x, \hat{S}_y] = -iJ_z. \tag{2.12}$$

  The ladder operators in this case are $\hat{X}_\pm = \hat{S}_y \mp J_z$, such that $[\hat{S}_x, \hat{X}_\pm] = \pm i\hat{X}_\pm$. Now, Helmholtz equation in parabolic coordinates (2.7) is that of an inverted oscillator with imaginary frequency $i\,\omega$:

  $$\left(-(\partial_u^2 + \partial_v^2) - \omega^2(u^2+v^2)\right)\psi(u, v) = 0. \tag{2.13}$$

- If $\omega^2 = -\beta^2 < 0$, we have the algebra $\mathfrak{so}(3)$, with $\{\check{S}_x = S_x/(2\beta),\ \check{S}_y = S_y/(2\beta),\ J_z\}$:

  $$[J_z, \check{S}_x] = i\check{S}_y, \qquad [J_z, \check{S}_y] = -i\check{S}_x, \qquad [\check{S}_x, \check{S}_y] = iJ_z. \tag{2.14}$$

  The ladder operators are $\check{X}_\pm = \check{S}_y \pm iJ_z$, and satisfy $[\check{S}_x, \check{X}_\pm] = \pm\check{X}_\pm$. Now, the Helmholtz equation in parabolic coordinates is that of an isotropic oscillator with frequency β (2.7) and null energy.
- Finally, if $\omega^2 = 0$, the operators (2.11) generate an Euclidean $\mathfrak{iso}(2)$ algebra:

  $$[J_z, S_x] = iS_y, \qquad [J_z, S_y] = -iS_x, \qquad [S_x, S_y] = 0, \tag{2.15}$$

  and the Helmholtz equation simply reduces to the Laplace equation.

The solutions for the three cases above can be found using the symmetries already discussed, but we do not consider them particularly relevant, since there are no bound states; we would only obtain dispersion states. Later, we will analyze the trajectories of the corresponding classical systems.

3. First extension: the 2D Kepler–Coulomb system

After establishing the symmetry algebras associated with separating the free 2D Helmholtz equation into polar and parabolic variables, we extend our analysis by adding some non-trivial potentials to the above equation. To maintain the separation in parabolic and polar coordinate systems, we adapt the S_x and J_z symmetries by introducing "potential terms". According to (2.6) we consider separable potentials in parabolic coordinates that have the form

$$V(u,v) = \frac{f(u)+g(v)}{u^2+v^2}, \tag{3.1}$$

so that the modified Helmholtz operator is

$$\tilde{Q} = Q + V(u,v). \tag{3.2}$$

3.1. KC extension of the Helmholtz equation that allows separation into polar and parabolic coordinates

Since we want to preserve the separation in parabolic and polar coordinates along with the symmetries, we extend the free symmetry generators by three functions $\gamma_x, \gamma_y, \gamma_z$, as follows:

$$\tilde{S}_x = S_x + \gamma_x, \qquad \tilde{S}_y = S_y + \gamma_y, \qquad \tilde{J}_z = J_z + \gamma_z, \tag{3.3}$$

requiring that $\tilde{S}_x$ and $\tilde{J}_z$ commute with $\tilde{Q}$:

$$[\tilde{S}_x, \tilde{Q}] = 0, \qquad [\tilde{J}_z, \tilde{Q}] = 0. \tag{3.4}$$

After some calculations, it is found that the potential must be of the form

$$V(u,v) = v_0 + \frac{2k}{u^2+v^2}, \quad v_0, k \in \mathbb{R}, \tag{3.5}$$

while the rest of the extension functions are given by

$$\gamma_x = \frac{k(u^2-v^2)}{u^2+v^2} = \frac{ky}{\rho}, \qquad \gamma_y = \frac{-2kuv}{u^2+v^2} = \frac{-kx}{\rho}, \qquad \gamma_z = 0. \tag{3.6}$$

As usual, and without loss of generality, from now on we will choose in (3.5) $v_0 = 0$. Thus we arrive at the equation of the extended Helmholtz operator (3.2) in parabolic coordinates

$$\left\{-(\partial_u^2+\partial_v^2) + \beta^2(u^2+v^2) - 2k\right\}\psi(u,v) = 0, \qquad \beta^2 = -\omega^2, \tag{3.7}$$

which in polar coordinates is simply the 2D KC equation with the KC potential (3.5),

$$\left(-\Delta(\rho,\theta) - \frac{k}{\rho} + \beta^2\right)\psi(\rho,\theta) = 0. \tag{3.8}$$

In this context, we call $\tilde{S}_x$ and $\tilde{S}_y$ extended or dynamical symmetries of the extended Helmholtz equation or KC system. They come from symmetries of the free Helmholtz system S_x and S_y. We can interpret the extended/dynamical symmetries $\tilde{S}_x$ and $\tilde{S}_y$ in terms of the 2D Runge–Lenz vector (A_x, A_y) as follows

$$\tilde{S}_y = A_x, \qquad \tilde{S}_x = -A_y. \tag{3.9}$$

In general, we will limit our discussion to negative values of ω^2, i.e., to $\beta^2 > 0$, since this is the most interesting case in physics. Remarkably, the original algebras $\mathfrak{so}(2,1)$, $\mathfrak{so}(3)$ or $\mathfrak{iso}(2)$ which also depend on the sign ($\omega^2 > 0$, $\omega^2 = -\beta^2 < 0$ or $\omega^2 = 0$), remain intact under this deformation, demonstrating the robustness of these symmetry structures.

In summary, the equation of a 2D KC system with coefficient k and energy $-\beta^2$(3.8) transforms into an equation of a 2D isotropic oscillator with frequency β and energy $2k$ in parabolic variables (3.7).

The initial Helmholtz equation had the symmetries S_x and J_z (among others), these can be called geometric symmetries, they lead to separation of variables of the eigenvalue equation of Helmholtz operator Q. However, the new extended operators $\tilde{S}_x$ is no longer a symmetry of Q, in fact it will be a symmetry of the 2D KC operator. In other words, now the geometric symmetry S_x of a free system has changed into a dynamical symmetry $\tilde{S}_x$ of an interacting system.

If $k = 0$, we revert to the situation in Section 3, so this case includes the previous one. Note that this is a local correspondence, and therefore we must check whether the solutions we are interested in have the correct global correspondence, since changing the variables might not preserve some global properties. This point will be discussed in detail in the next section.

3.2. Separable solutions in parabolic coordinates

The new Helmholtz equation with the additional constant k term (3.7) can be separated into variables u, v, with separated solutions $\psi(u,v) = \sigma(u)\mu(v)$, in the same way as in the free case. This results in two coupled equations (with eigenvalues $k+\lambda$ and $k-\lambda$):

$$H_u\sigma(u) - (k+\lambda)\sigma(u) = 0, \qquad H_u = -\partial_u^2 + \beta^2 u^2, \tag{3.10}$$

$$H_v\mu(v) - (k-\lambda)\mu(v) = 0, \qquad H_v = -\partial_v^2 + \beta^2 v^2, \tag{3.11}$$

where the separation constant λ is the eigenvalue of the operator $\tilde{S}_x$, as before in (2.8),

$$\tilde{S}_x\ (\sigma(u)\mu(v)) = \lambda\ (\sigma(u)\mu(v))\,. \tag{3.12}$$

The Eqs. (3.10)–(3.11) describe two quantum harmonic oscillators with the same frequency β. At this point, it is convenient to consider their respective ladder operators:

$$A^{\pm} = \mp\partial_u + \beta u, \qquad B^{\pm} = \mp\partial_v + \beta v, \tag{3.13}$$

which factorize their Hamiltonians (3.10)–(3.11)

$$H_u = A^+A^- + \beta, \qquad H_v = B^+B^- + \beta, \tag{3.14}$$

and satisfy the usual commutation rules (commutators not explicitly specified are zero):

$$[A^-, A^+] = 2\beta, \qquad \left[H_u, A^{\pm}\right] = \pm 2\beta A^{\pm}, \tag{3.15}$$

$$[B^-, B^+] = 2\beta, \qquad \left[H_v, B^{\pm}\right] = \pm 2\beta B^{\pm}. \tag{3.16}$$

Since the two Eqs. (3.7) and (3.8) must have the same symmetries, we will describe the symmetries and solutions of the KC system (the Runge–Lenz vector (3.9) and the rotation generator J_z) by means of the well-known ladder operators $A^{\pm}, B^{\pm}$ of an oscillator system.

3.2.1. Symmetries of the KC extension of the Helmholtz equation

We currently know the symmetries of the KC equation $\tilde{Q}\psi = 0$: in addition to $\tilde{Q}$, these are generated by $\tilde{S}_x, \tilde{S}_y, J_z$, or in its ladder base by $\tilde{S}_x, \tilde{X}_{\pm}$. Next, we want to recover the properties of the KC equation (3.8) using the ladder operators of the 2D isotropic oscillator equation. The following symmetry operators of the oscillator equation (3.6) are well known [44]

$$\text{(i) } A^+A, \qquad \text{(ii) } B^+B, \qquad \text{(iii) } A^+B^-, \qquad \text{(iv) } A^-B^+, \tag{3.17}$$

but only three of them are independent. Since A^+A and B^+B do not modify the eigenvalues $-\beta^2$ or λ, they should be related to Q and S_x. In fact, we can manipulate the Eqs. (3.10)–(3.11) to obtain $\tilde{Q}$ and $\tilde{S}_x$:

$$\tilde{Q} = \frac{1}{u^2+v^2}(A^+A + B^+B - 2k), \tag{3.18}$$

$$\tilde{S}_x = \frac{1}{u^2+v^2}(v^2(A^+A - \beta - k) - u^2(B^+B - \beta - k)). \tag{3.19}$$

On the other hand, since A^+B^- and A^-B^+ modify λ, which is the eigenvalue of $\tilde{S}_x$(3.12), they should be related to $\tilde{X}_{\pm}$. In fact, we check that $A^{\pm}B^{\mp} = -2\beta\tilde{X}_{\pm}$. Furthermore, we have two "natural" operators of the oscillator equation, A^+B^+ and B^-A^-, which are not symmetries of (3.6), but connect systems with different values of k (see (3.15) and (3.16)). This is an interesting property that leads to some new relations of the KC system that will be discussed later.

3.2.2. Spectrum of KC extension of the Helmholtz equation in parabolic coordinates

The eigenvalue spectrum of each of the oscillators in (3.10)–(3.11) is, respectively, $E^u = k+\lambda = \beta(2n+1)$ and $E^v = k-\lambda = \beta(2m+1)$, for $n, m \in \mathbb{N}_0$, so that the spectrum of (3.7) will be the sum of them:

$$E^u + E^v = (k+\lambda) + (k-\lambda) = 2\,k = 2\beta(p+1), \quad p = n+m = 0, 1, 2, \ldots, \tag{3.20}$$

and its degeneracy is given by n, m such that $n+m = p$, that is, $p+1$. In terms of λ, this degeneracy corresponds to $\lambda = \beta\, p/2, \ldots, -\beta\, p/2$, which is associated with the representation $j = p/2$, with dimension $p+1$, of the symmetry algebra $\mathfrak{so}(3)$. This means that by fixing β^2 we determine the possible values of k, let us call them k_p, which allows us to rewrite (3.20) in the form

$$-\beta^2 = -\frac{k_p^2}{(p+1)^2}, \tag{3.21}$$

and so we have a sequence of KC systems with coefficients k_p associated with the harmonic oscillator (3.6).

If, on the other hand, in the relation (3.20) we assume that k is constant, this equality gives the energy $E_p \equiv -\beta^2$ of the 2D KC system:

$$E_p = -\frac{k^2}{(p+1)^2}, \quad p = 0, 1, 2, \ldots \tag{3.22}$$

However, it is known [45–47] that the 2D KC energy levels are given by

$$E_{\mathrm{KC}} = -\frac{k^2}{(2p+1)^2}, \quad p = 0, 1, 2, \ldots \tag{3.23}$$

This result that does not coincide with that obtained from the 2D oscillator in the variables (u, v), (3.22). The reason is that the solutions in the variables (u, v), after being transformed to polar coordinates, involve half-integer values of angular momentum when $p + 1$ is even, and therefore half of the solutions in coordinates (u, v) do not correspond to physically acceptable solutions of 2D KC: we must limit ourselves to $m + n = 2p$, as shown by (3.23). This point will be analyzed in general terms in Section 6.

Eigenfunctions are constructed as usual, acting repeatedly with the creation operators on the ground state ($m = n = 0$):

$$\psi_{m,n}(u, v) = (A^+)^m (B^+)^n \sigma_0(u)\mu_0(v), \qquad m, n \in \mathbb{N}, \tag{3.24}$$

with $\sigma_m(u)$ and $\mu_n(v)$ are eigenstates in H_u and H_v, respectively, making the $p + 1$ degeneracy explicit. Therefore, all properties of the isotropic 2D oscillator correspond to the properties of the 2D Kepler–Coulomb problem: symmetries, eigenvalues, and solutions, except for the global properties.

4. Second extension: the Makarov 2D system

Next, we want to complete the analysis of separable potentials in polar and parabolic coordinates that we have been carrying out in the previous sections. So far, we have an extended Helmholtz operator, $\tilde{Q}$ (including the KC potential from the previous section) and some extended (non-independent) symmetries:

$$\tilde{S}_x, \quad \tilde{S}_y, \quad \tilde{J}_z = J_z, \tag{4.1}$$

However, we need to examine the more general case where polar coordinates are associated with the most general second-order operator

$$\hat{J}_z^2 = J_z^2 + j(\rho, \theta), \tag{4.2}$$

while the remaining symmetries should be extended to be consistent with this new situation, keeping in mind that J_z will no longer be a symmetry while $\tilde{S}_y$ will not necessarily be an additional symmetry. The new symmetry operators replacing $\tilde{Q}$ and $\tilde{S}_x$ will be

$$\hat{Q} = \tilde{Q} + q(\rho, \theta), \quad \hat{S}_x = \tilde{S}_x + s_x(\rho, \theta). \tag{4.3}$$

The separation of the equation into polar and parabolic variables will be guaranteed if

$$[\hat{Q}, \hat{S}_x] = [\hat{Q}, \hat{J}_z^2] = 0. \tag{4.4}$$

After some calculations, in polar coordinates we find

$$q(\rho, \theta) = \frac{c_1 - c_2 \sin\theta}{\rho^2} \sec^2\theta, \tag{4.5}$$

$$j(\rho, \theta) = (c_1 - c_2 \sin\theta)\sec^2\theta, \tag{4.6}$$

$$s_x(\rho, \theta) = \frac{\sec^2\theta(c_2 - 2c_1 \sin\theta + c_2 \sin^2\theta)}{\rho}, \tag{4.7}$$

where c_1 and c_2 are arbitrary constants. With this, we can now take our next step in the search for dynamical symmetries, which will lead us to a type of KC potential with centrifugal terms, sometimes called the Makarov potential [32], which is a variant of the Hartmann potential [33]. We will therefore call this new potential the Makarov potential or extended KC potential.

Next we propose the separation of variables for the operator $\hat{Q}$ in (4.3) and obtain the following equations in each of the two parabolic coordinates:

$$\left(-\partial_{uu} + \beta^2 u^2 - \frac{c_1 + c_2}{u^2} - (k + \lambda)\right)\sigma(u) = 0, \tag{4.8}$$

$$\left(-\partial_{vv} + \beta^2 v^2 - \frac{c_1 - c_2}{v^2} - (k - \lambda)\right)\mu(v) = 0, \tag{4.9}$$

where λ is the separation constant. Comparing this second extension with the previous case (3.10)–(3.11), we see that the separated systems now correspond to a pair of one-dimensional oscillators, each with an additional centrifugal term. We can define

$$c_1 + c_2 = -l_u(l_u + 1), \qquad c_1 - c_2 = -l_v(l_v + 1), \tag{4.10}$$

and the additional term in the potential takes the following form in polar coordinates

$$q(\rho, \theta) = -l_u(l_u + 1)\,\frac{1 - \sin\theta}{2\rho^2\cos^2\theta} - l_v(l_v + 1)\,\frac{(1 + \sin\theta)}{2\rho^2\cos^2\theta}, \quad l_u, l_v \in \mathbb{R}. \tag{4.11}$$

In summary, we have found that the second modified Helmholtz equation associated to the operator $\hat{Q}$ can be expressed in parabolic coordinates as

$$(H(l_u, l_v) - 2k)\psi(u, v) \equiv \left[-(\partial_u^2 + \partial_v^2) + \beta^2(u^2 + v^2) + \frac{l_u(l_u + 1)}{u^2} + \frac{l_v(l_v + 1)}{v^2} - 2k\right]\psi(u, v) = 0, \tag{4.12}$$

which is a generalization of (3.7), or in polar coordinates as

$$\left(-\Delta(\rho,\theta) - \frac{k}{\rho} + l_u(l_u+1)\,\frac{1-\sin\theta}{2\rho^2\cos^2\theta} + l_v(l_v+1)\,\frac{(1+\sin\theta)}{2\rho^2\cos^2\theta} + \beta^2\right)\psi(\rho,\theta) = 0, \tag{4.13}$$

which is a generalization of (3.8). These two equations have two symmetries, $\hat{J}_z^2$ and $\hat{S}_x$, as indicated in (4.4), but $\hat{S}_y$ is no longer a symmetry and therefore the algebraic structure of the previous case is no longer valid. For this reason, we will now try to find the relevant algebra in the current situation, starting from the symmetries $\hat{J}_z^2$ and $\hat{S}_x$.

To better understand the symmetries and their properties in this case, we will examine the equations in parabolic coordinates (4.12) and their separation (4.8)–(4.9). These are identified with the well-known 2D isotropic oscillator plus centrifugal terms $H(l_u, l_v)$, called Smorodinsky–Winternitz systems [39,48]. Since the algebraic structure of their symmetries [44,49] is known, this will allow us to solve our problem. The details are given in Appendix, where we find that the spectrum $E(l_u, l_v) = E(l_u) + E(l_v)$ is given by

$$E(l_u, l_v) = 2\beta(3 + l_u + l_v + 2n), \quad n = 0, 1, 2 \ldots \tag{4.14}$$

The degeneracy of each level labeled by $n = 0, 1, \ldots$ is $n + 1$. Finally, we find that the spectrum of the Makarov 2D system (or KC system with a kind of centrifugal terms) is

$$E = -\beta^2 = -\frac{k^2}{(3 + l_u + l_v + 2n)^2}. \tag{4.15}$$

Thus, we have shown that this seemingly strange potential [48], can be interpreted in terms of a 2D centrifugal oscillator in parabolic coordinates. Based on this interpretation, it has been possible to identify an underlying algebraic structure of symmetries and their solutions.

5. Classical systems

In the following, we will analyze the classical analogues of those previously discussed [50]. We will focus on the first modified (or extended) Helmholtz equation, which involves the 2D KC potential, and the second modified Helmholtz equation, which involves the 2D isotropic oscillator with centrifugal terms. To find the trajectories, we perform the classical analysis of our previous quantum systems, applying canonical quantization by connecting the canonical coordinates x, p_x with the quantum operators X, P_x,

$$X \;\to\; x, P_x = -i\partial_x \;\to\; p_x, \tag{5.1}$$

while the quantum commutators $[A, B] = iC$ are replaced by Poisson brackets $\{\mathsf{A}, \mathsf{B}\} = \mathsf{C}$, where A, B, C are Hermitian operators and $\mathsf{A}, \mathsf{B}, \mathsf{C}$ are their classical analogues. It is worth noting that the resulting formalism fully corresponds to the geometric optics on phase space [51].

5.1. Classical trajectories of the KC extension of the Helmholtz equation

We consider the classical version of the quantum systems defined in (3.7) and (3.8), replacing the operators with the variables u, v and p_u, p_v or ρ, θ and p_ρ, p_θ, respectively. Based on what we did in the quantum case (3.13), we will define the classical analogue of the ladder operators as the following functions, which depend on the coordinates (u, v) and their conjugate momenta (p_u, p_v),

$$\mathsf{A}^\pm = \mp i p_u + \beta u, \qquad \mathsf{B}^\pm = \mp i p_v + \beta v, \tag{5.2}$$

and which satisfy the Poisson–Heisenberg algebra:

$$\{\mathsf{A}^-, \mathsf{A}^+\} = -2i\beta, \qquad \{\mathsf{B}^-, \mathsf{B}^+\} = -2i\beta. \tag{5.3}$$

The classical Hamiltonians corresponding to (3.10)–(3.11) are

$$\mathsf{H}_u = p_u^2 + \beta^2 u^2 = \mathsf{A}^+\mathsf{A}^-, \quad \mathsf{H}_v = p_v^2 + \beta^2 v^2 = \mathsf{B}^+\mathsf{B}^-, \quad \mathsf{H} = \mathsf{H}_u + \mathsf{H}_v, \tag{5.4}$$

and their Poisson brackets are

$$\{\mathsf{H}_u, \mathsf{A}^\pm\} = \mp 2i\beta\mathsf{A}^\pm, \qquad \{\mathsf{H}_v, \mathsf{B}^\pm\} = \mp 2i\beta\mathsf{B}^\pm. \tag{5.5}$$

The time evolution of the functions $\mathsf{A}^\pm$ and $\mathsf{B}^\pm$ are given by

$$\dot{\mathsf{A}}^\pm(t) = \{\mathsf{A}^\pm, \mathsf{H}_u\} = \pm 2i\beta\mathsf{A}^\pm, \qquad \dot{\mathsf{B}}^\pm(t) = \{\mathsf{B}^\pm, \mathsf{H}_v\} = \pm 2i\beta\mathsf{B}^\pm. \tag{5.6}$$

By solving these differential equations, we determine the time evolution of $\mathsf{A}^\pm$ and $\mathsf{B}^\pm$, and subsequently the evolution of $u(t)$ and $v(t)$. The analysis of the various trajectories in the (u, v) plane reveals the following four distinct dynamical regimes, governed by the values of the parameters β^2 and k, where the energy is $E = 2k$:

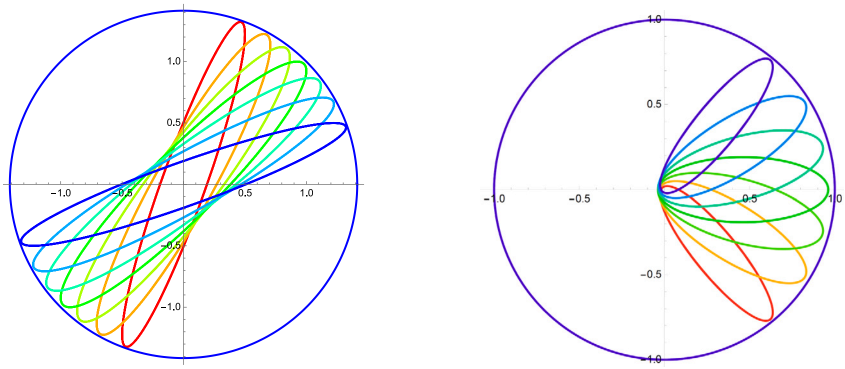

Fig. 1. Left: seven confined trajectories for the Hamiltonian H in (5.4), in parabolic coordinates, with energies $H_u = 4 + \lambda, H_v = 4 - \lambda$, taking $k = 4$, $\lambda = -3, \ldots, 3$, for $\beta = 1, \phi = \pi/8, \varphi = 0$ in (5.8). Right: image of the same seven trajectories, but in Cartesian coordinates (x, y) after doing the transformation (2.4).

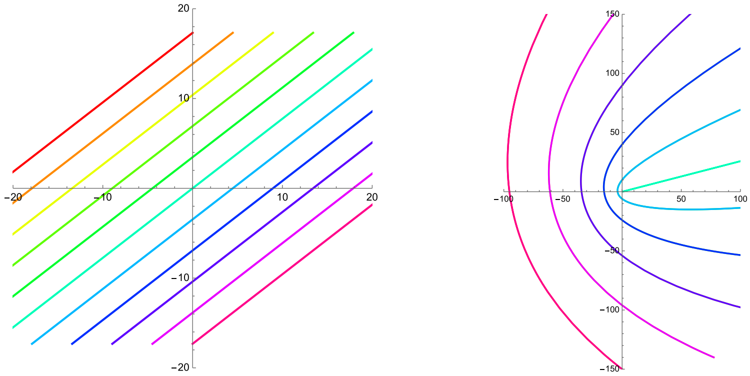

Fig. 2. Left: Free trajectories (5.10) for the Hamiltonian H in (5.4), in parabolic coordinates, for $k = 4$, $\lambda = 1$, $\varphi_u = -5, \ldots, 5$, and $\varphi_v = 0$. Right: the corresponding trajectories for the KC system, now in Cartesian coordinates.

(i) Confined trajectories ($\beta^2 > 0$, $k > 0$): the classical Hamiltonians (5.4) correspond to harmonic oscillator well potentials, leading to

$$\mathsf{A}^{\pm}(t) = \mathsf{A}_0^{\pm} e^{\pm 2i\beta t}, \qquad \mathsf{B}^{\pm}(t) = \mathsf{B}_0^{\pm} e^{\pm 2i\beta t}, \tag{5.7}$$

with amplitudes $\mathsf{A}_0^{\pm} = \sqrt{|k+\lambda|}e^{\pm i\phi}$ and $\mathsf{B}_0^{\pm} = \sqrt{|k-\lambda|}e^{\pm i\varphi}$, where the constants $\phi, \varphi \in \mathbb{R}$ are fixed by the initial conditions. Therefore, the motion in the parabolic coordinate system is:

$$u = \frac{\sqrt{|k+\lambda|}}{\beta}\cos(2\beta t + \phi), \qquad v = \frac{\sqrt{|k-\lambda|}}{\beta}\cos(2\beta t + \varphi). \tag{5.8}$$

Other equivalent expressions for the motion are

$$u = \sqrt{|k+\lambda|}\,\frac{\sin 2\beta(t+\varphi_u)}{\beta}, \qquad v = \sqrt{|k-\lambda|}\,\frac{\sin 2\beta(t+\varphi_v)}{\beta}. \tag{5.9}$$

These trajectories correspond to ellipses centered at the origin, as shown in Fig. 1. The corresponding trajectories in Cartesian coordinates (x, y) are ellipses in which one of their foci is the origin.

(ii) Free propagation ($\beta = 0$): the trajectories in the plane (u, v) are the straight lines:

$$u = 2\sqrt{|k+\lambda|}\,(t+\varphi_u), \qquad v = 2\sqrt{|k-\lambda|}\,(t+\varphi_v) \tag{5.10}$$

whose images in the plane (x, y) are parabolas, as shown in Figs. 2 and 3.

(iii) The case where $\beta^2 < 0$, $\beta = i\gamma$, and $k > 0$ corresponds to a 2D repulsive oscillator type potential, but to a 2D KC attractive potential, with positive energy. The motion is given by

$$u = \sqrt{|k+\lambda|}\,\frac{\sinh(2\gamma(t+\varphi_u))}{\gamma}, \qquad v = \sqrt{|k-\lambda|}\,\frac{\sinh(2\gamma(t+\varphi_v))}{\gamma}. \tag{5.11}$$

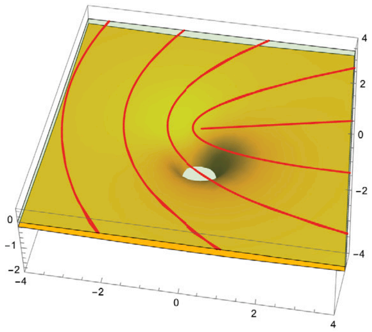

Fig. 3. The free trajectories (5.10) are plotted together with the KC potential in Cartesian coordinates.

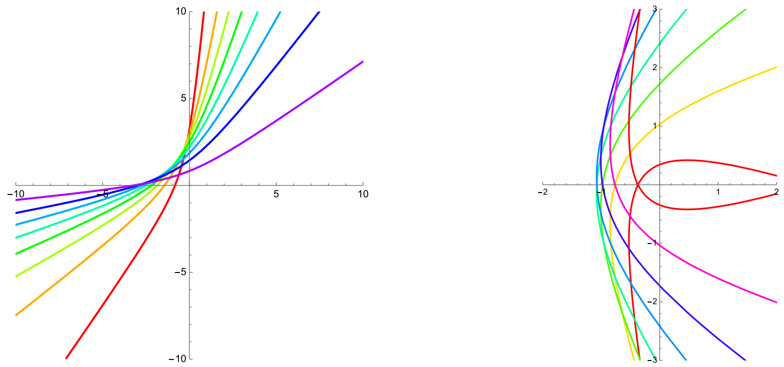

Fig. 4. Left: trajectories for (5.11) in the (u, v) plane for $k = 4$, $\lambda = -3.5, \ldots, 3.5$, $\gamma = 1$, $\varphi_u = 1$, and $\varphi_v = 2$. Right: the corresponding trajectories for the KC system in Cartesian coordinates.

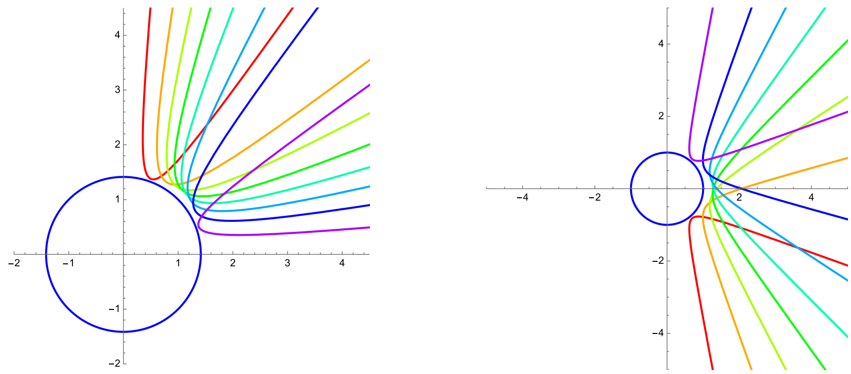

Fig. 5. Left: trajectories for (5.12) in parabolic coordinates for $k = -2$, $\gamma = 1$, $\lambda = -3.5, \ldots, 3.5$, $\varphi_u = 1$, and $\varphi_v = 2$. Right: the corresponding trajectories in Cartesian coordinates.

In both the (u, v) plane and the (x, y) plane, the trajectories are the hyperbolas shown in Fig. 4.

(iv) When $\beta^2 < 0$, $\beta = i\gamma$, and $k < 0$, we have a 2D repulsive oscillator for negative energies and repulsive KC potentials with positive energies. The solutions are given by

$$u = \sqrt{|k+\lambda|}\,\frac{\cosh(2\gamma(t+\varphi_u))}{\gamma}, \qquad v = \sqrt{|k-\lambda|}\,\frac{\cosh(2\gamma(t+\varphi_v))}{\gamma}. \tag{5.12}$$

and the trajectories are shown in Fig. 5.

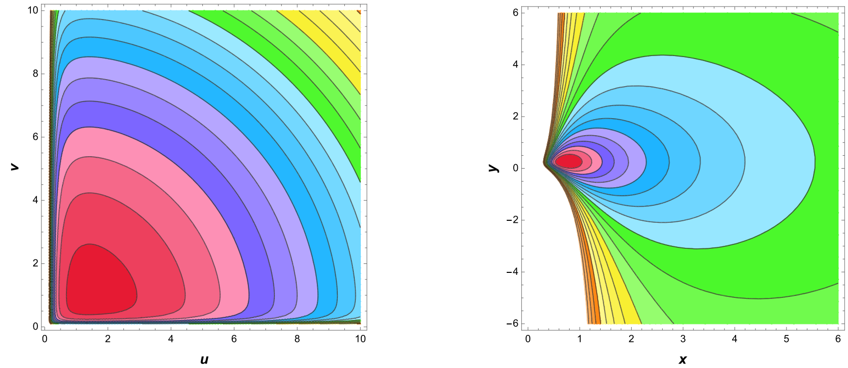

Fig. 6. Left: contour lines of the Hamiltonian potential (5.13) in the variables (u, v), for $l_u = 2, l_v = 1, \beta = 1$. Right: contour line plot of the same centrifugal KC or Makarov system in coordinates (x, y), with the same parameters and $k = 6$.

5.2. Classical trajectories of the Makarov extension of Helmholtz equation

In this case, the initial quantum equations (4.12) and (4.13) are replaced by the corresponding classical variables, but modifying $l_u(l_u+1)$ and $l_v(l_v+1)$ to l_u^2 and l_v^2, respectively. Similar considerations can be applied to determine the trajectories of the 2D isotropic oscillator with additional centrifugal terms and, therefore, for the trajectories of the 2D centrifugal KC system (or Makarov system). The classical Hamiltonian takes the form

$$h(l_u, l_v) = h_u(l_u) + h_v(l_v) = (p_u^2 + p_v^2) + \beta^2(u^2 + v^2) + \frac{l_u^2}{u^2} + \frac{l_v^2}{v^2} = \varepsilon_u + \varepsilon_v = 2k. \tag{5.13}$$

Fig. 6 shows a graph of the boundary levels of this potential. On the other hand, the ladder functions for (5.13), which mimic the ladder operators (A.1)–(A.4), are now

$$\begin{aligned} r^\pm &= c^\pm \tilde{c}^\pm = -p_u^2 - \tfrac{l_u^2}{u^2} + \beta^2 u^2 - 2i\beta u p_u = -\varepsilon_u + 2\beta^2 u^2 - 2i\beta u p_u, \\ s^\pm &= d^\pm \tilde{d}^\pm = -p_v^2 - \tfrac{l_v^2}{v^2} + \beta^2 v^2 - 2i\beta v p_v = -\varepsilon_v + 2\beta^2 v^2 - 2i\beta v p_v, \end{aligned} \tag{5.14}$$

satisfying

$$\dot{r}^\pm = \{r^\pm, h(l_u, l_v)\} = \pm 4i\beta r^\pm, \qquad \dot{s}^\pm = \{s^\pm, h(l_u, l_v)\} = \pm 4i\beta s^\pm. \tag{5.15}$$

From here it follows that

$$r^\pm(t) = r_0 e^{i(\pm 4\beta t + \phi_0)}, \qquad s^\pm(t) = s_0 e^{i(\pm 4\beta t + \psi_0)}. \tag{5.16}$$

Considering that

$$|c^\pm| = \sqrt{\varepsilon_u - 2\beta l_u}, \quad |\tilde{c}^\pm| = \sqrt{\varepsilon_u + 2\beta l_u}, \tag{5.17}$$

we have

$$-\varepsilon_u + 2\beta^2 u^2 = \sqrt{\varepsilon_u^2 - 4\beta^2 l_u^2}\cos(\pm 4\beta t + \phi_0), \tag{5.18}$$

so that

$$u(t) = \frac{1}{\sqrt{2}\beta}\sqrt{\varepsilon_u + \sqrt{\varepsilon_u^2 - 4\beta^2 l_u^2}\cos(\pm 4\beta t + \phi_0)}. \tag{5.19}$$

Similarly

$$v(t) = \frac{1}{\sqrt{2}\beta}\sqrt{\varepsilon_v + \sqrt{\varepsilon_v^2 - 4\beta^2 l_v^2}\cos(\pm 4\beta t + \psi_0)}, \tag{5.20}$$

where $\varepsilon_u + \varepsilon_v = 2k$. The Eqs. (5.19)–(5.20) provide the parametric trajectory in the parabolic plane (u, v), which allows us to obtain the trajectories in the plane (x, y), some of which are shown in Fig. 7.

6. Global restrictions

The fact that in the plane the transformation between the parabolic coordinates (u, v) and the Cartesian coordinates (x, y) is not bijective will have important consequences in the physical solutions of the problems we are analyzing.

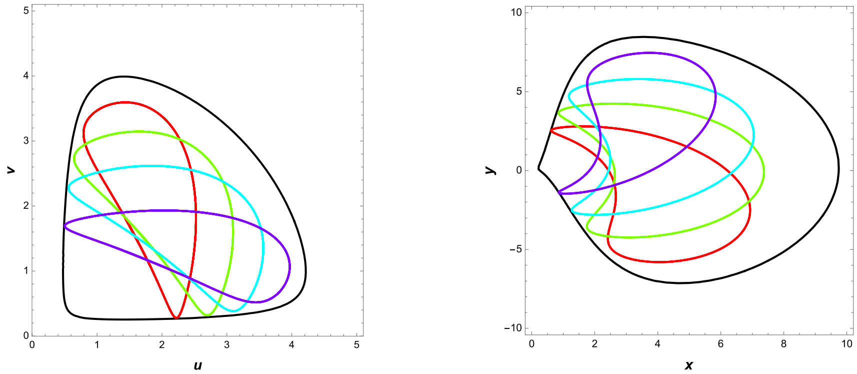

Fig. 7. Left: trajectories of the classical Makarov system (5.13) for $k = 10, l_u = 2, l_v = 1, \varphi_u = \pi/3, \varphi_v = 0$, $\lambda = -9, \dots 9(2.5)$. They are all bounded by a potential contour line (in black). Right: The corresponding trajectories in the plane (x, y), all bounded by a contour line.

In the classical context, for the first extension (5.4), the trajectories in the (u, v) plane that are closed and include the origin have a polar angle ranging from 0 to 2π. In this region, the correspondence $(u, v) \to (x, y)$ is two to one: the points (u, v) and $(-u, -v)$ will be transformed into the same point (x, y). Therefore, the physical paths $(u(t), v(t))$ that enclose the origin in the uv plane should be symmetric and will cover twice as many paths as $(x(t), y(t))$ in the (x, y) plane (see Fig. 1).

In the case of the second extension (5.4), the trajectories in the plane (u, v) are again closed, but the polar angle can vary within the interval $(0, \pi/2)$. In this region, the correspondence $(u, v) \to (x, y)$ is one-to-one (the (u, v) region will be a quadrant, while the (x, y) image will be a half-plane) and, therefore, each trajectory $(u(t), v(t)$ in the first quadrant of the plane (u, v) will lead to a trajectory $(x(t), y(t))$ in the right half-plane of (x, y) (see Figs. 6 and 7).

In the quantum context, the relationship between Cartesian and parabolic coordinates implies that the wave functions $\psi(u, v)$ defined in the (u, v) plane must be symmetric, that is, $\psi(-u, -v) = \psi(u, v)$. However, in the first extension (3.7), the basis of the solutions are functions of the form $\psi_{m,n}(u, v) = \psi_m(u)\psi_n(v)$, where m (or n) is even or odd, respectively, corresponding to an even or odd function on u (or on v). Therefore, they will only be symmetric when $m + n$ is even, $m + n = 2p$. This is why we restrict ourselves to the even sector of the spectrum of the 2D oscillator Hamiltonian in Section 3.

7. Conclusions

In this work, we have established a complete Lie-algebraic framework for the analysis of the symmetry of the two-dimensional Helmholtz equation and some related systems, separable in polar and parabolic coordinates. The motivation for this work has been the interpretation of the different systems which are separated in a set of variables. We have chosen a very interesting case in physics: systems separable simultaneously in polar and parabolic coordinates. We have shown that the different separable systems obtained have different symmetry properties, and that such properties arise from a process of transforming geometric symmetries into dynamical ones.

We begin with a free system (corresponding to the ambient space) given by the Helmholtz equation $Q = 0$, where all geometric symmetry operators, that is the Killing vector and tensor fields of the Euclidean metric, close a symmetry algebra. We have selected the symmetries of separation in polar and parabolic variables J_z and S_x. Then, we initiate the process by replacing S_x with an extended dynamical symmetry $\tilde{S}_x$, which implies an extension of the Helmholtz equation to the interacting CK system, while maintaining J_z. In this step, we have modified the initial symmetry generators: from (J_z, S_x, S_y) to those generated by $(J_z, \tilde{S}_x, \tilde{S}_y)$. The symmetry algebra is preserved (due to the fact that J_z keeps the symmetry character), but the extension of Q is a KC system characterized by a new parameter k. The extended symmetries are essentially the components of the 2D Runge–Lenz vector. Next, we continue to gradually transform the geometric symmetries. In the second step, the geometric symmetry J_z^2 is extended to $\hat{J}_z^2$ and, consequently, also to the extension $\hat{S}_x$. The extension of the KC system now contains an additional term, resulting in the Makarov system characterized by two parameters l_u, l_v. Now, the initial symmetry algebra has been reduced to that generated by these two extensions $\hat{J}_z^2, \hat{S}_x$. The initial geometric symmetries of the Helmholtz equation have been reduced or replaced by dynamic symmetries.

Therefore, we have a sequence of steps leading to separable potentials, such that at each step a symmetry changes its character from geometric to dynamic. This implies a change in the symmetry algebra at each step while new parameters appear characterizing the extension. In conclusion, we have described the sequence Helmholtz → KC → Makarov associated with separable polar-parabolic systems.

We remark some properties of this specific problem. We obtained equations for oscillators as separated systems in parabolic variables. This helps in understanding KC and Makarov symmetries in terms of the symmetries of oscillators or centrifugal oscillators (Smorodinski–Winternitz), respectively. The geometric-to-dynamic transitions we have described is implemented in both quantum and classical frames. The classical approach closely follows the quantum development. With the help of factorizations, we calculate spectra (quantum) and trajectories (classical). We have also addressed an important, often overlooked point: the global properties in the separation of variables. This can be quite relevant, as we have shown in the relationship between the oscillator and KC systems in two dimensions.

We hope this work will provide new tools for the analysis of other physical systems. This algebraic approach, which we have used throughout this work to address the simplest Helmholtz equation, can naturally be extended to anisotropic media and higher dimensions. We also expect it will soon find applications in problems related to various types of light propagation, in optics, and in waveguide design. Finally, it is worth mentioning that this method can also be implemented in relativistic systems, which relates to its application to curved systems on spheres or hyperboloids. In this work, we have limited ourselves to a simple but very important 2D Euclidean case, but extensions to the aforementioned cases are part of ongoing research.

CRediT authorship contribution statement

G. Jimenez-Trejo: Writing – review & editing, Validation, Investigation, Formal analysis, Data curation. **J. Negro:** Writing – review & editing, Writing – original draft, Validation, Supervision, Methodology, Investigation, Formal analysis, Conceptualization. **L.M. Nieto:** Writing – review & editing, Writing – original draft, Supervision, Resources, Project administration, Investigation, Funding acquisition, Formal analysis. **S. Cruz y Cruz:** Writing – review & editing, Validation, Supervision, Project administration, Investigation.

Declaration of competing interest

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Appendix

We will collect here some computations needed in Section 4. We begin by defining some basic first-order operators

$$C_1^{\pm}(l_u) = \mp\partial_u + \beta u + \frac{l_u}{u}\,, \qquad \tilde{C}_1^{\pm}(l_u) = \mp\partial_u + \beta u - \frac{l_u}{u}, \tag{A.1}$$

$$D_1^{\pm}(l_v) = \mp\partial_v + \beta v + \frac{l_v}{v}\,, \qquad \tilde{D}_1^{\pm}(l_v) = \mp\partial_v + \beta v - \frac{l_v}{v}, \tag{A.2}$$

from which the following second-order operators are constructed:

$$C^{+}(l_u) = C_1^{+}(l_u)\tilde{C}_1^{+}(l_u)\,, \qquad C^{-}(l_u) = C_1^{-}(l_u+1)\tilde{C}_1^{-}(l_u+1), \tag{A.3}$$

$$D^{+}(l_v) = D_1^{+}(l_v)\tilde{D}_1^{+}(l_v)\,, \qquad D^{-}(l_v) = D_1^{-}(l_v+1)\tilde{D}_1^{-}(l_v+1). \tag{A.4}$$

Next we check that these second-order operators $C^{\pm}(l_u)$ and $D^{\pm}(l_u)$ act as ladder operators for the equations in u and v, respectively, similar to those in the previous section:

$$[H(l_u,l_v), C^{\pm}(l_u)] = \pm 4\beta\, C^{\pm}(l_u)\,, \qquad [H(l_u,l_v), D^{\pm}(l_v)] = \pm 4\beta\, D^{\pm}(l_u), \tag{A.5}$$

so we can obtain fourth order symmetry operators $\hat{S}(l_u,l_v)^{\pm}$ in the form

$$\hat{S}^{\pm}(l_u,l_v) := C^{\pm}(l_u)D^{\mp}(l_v)\,, \tag{A.6}$$

such that

$$[H(l_u,l_v), \hat{S}^{\pm}(l_u,l_v)] = 0. \tag{A.7}$$

Using these operators, we can find the eigenfunctions and eigenvalues. The ground states $\phi_0(u)$ and $\psi_0(v)$ for each separate oscillator in (4.8)–(4.9) are given, respectively, by

$$\tilde{C}_1^{-}(l_u+1)\phi_0(u) = 0, \quad E_u^0(l_u) = 2\beta(3/2+l_u), \tag{A.8}$$

$$\tilde{D}_1^-(l_v+1)\psi_0(v)=0,\quad E_v^0(l_v)=2\beta(3/2+l_v). \tag{A.9}$$

From the relation $E(l_u,l_v)=E(l_u)+E(l_v)$, we finally obtain

$$E(l_u,l_v)=2\beta(3+l_u+l_v+2n),\quad n=0,1,2\ldots,\quad l_u,l_v\in\mathbb{R}. \tag{A.10}$$

Data availability

No data was used for the research described in the article.

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