



## Duals of multiplicity codes

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Received: 22 May 2025 / Revised: 9 January 2026 / Accepted: 29 January 2026  
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### Abstract

Multivariate multiplicity codes have been recently explored because of their importance for list decoding and local decoding. Given a multivariate multiplicity code, in this paper, we compute its dimension using Gröbner basis tools, its dual in terms of indicator functions, and explicitly describe a parity-check matrix. In contrast with Reed–Muller, Reed–Solomon, univariate multiplicity, and other evaluation codes, the dual of a multivariate multiplicity code is not equivalent or isometric to a multiplicity code (i.e., this code family is not closed under duality). We use our explicit description to provide a lower bound on the minimum distance for the dual of a multiplicity code.

**Keywords** Evaluation codes · Footprint bound · Multiplicity codes · Polynomial codes · Polynomial ideal codes · Reed–Muller codes · Reed–Solomon codes · Schwartz-Zippel bound

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Communicated by R. Jurrius.

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## 1 Introduction

Reed-Solomon and Reed–Muller codes are well-known families of linear error-correcting codes obtained by evaluating polynomials over a finite field. They are the most widely used families of algebraic codes, and many generalizations have been proposed over the years, such as folded Reed-Solomon codes [18] or polynomial ideal codes [19]. An interesting generalization is given by multiplicity codes, which were introduced in [27] and generalize both Reed-Solomon and Reed–Muller codes by also evaluating the derivatives of the polynomials. The first motivation to study these codes was that they provide a family that can be locally decodable efficiently with an asymptotic rate of 1 [27]. Moreover, they can also be list decoded, achieving the optimal rate versus list-decodability tradeoff [20, 25]. See also [5, 9, 33, 35] for more on univariate multiplicity codes and polynomial ideal codes and [4–6, 26, 36] for more on multivariate multiplicity codes.

Duals and parity-check matrices of any family of codes are ubiquitous in many applications of coding theory: they are essential for syndrome-based decoding algorithms [12, 14, 34] and are required to construct secret sharing schemes [8] or quantum codes [2, 24], among others. Furthermore, in the case of Reed–Muller codes (multiplicity 1), their duals are used to estimate their weight enumerator and in one of the proofs that they achieve the Shannon capacity on different channels [1, Sec. 4.3].

Reed-Solomon codes are obtained by evaluating univariate polynomials of degree lower than or equal to  $d$ , and their duals are also Reed-Solomon codes, up to an isometry. Similarly, Reed-Muller codes and Cartesian codes are generalizations of Reed-Solomon codes, where multivariate polynomials are evaluated instead. Again, the duals of these codes belong to the same family [30]. A different generalization is given by univariate multiplicity codes, in which we evaluate univariate polynomials and their derivatives. In [33], it was proven that the duals are also univariate multiplicity codes. Finally, (multivariate) multiplicity codes generalize all the previous families by evaluating multivariate polynomials and their derivatives.

In this work, we explicitly compute the duals (and parity-check matrices) of general multiplicity codes (Theorems 5.10 and 5.13). Surprisingly, we show that this “closedness under duality” no longer holds for multiplicity greater than one and more than one variable (see, e.g., Example 5.1 and Theorem 5.10), revealing an unexpected behavior of duals of multiplicity codes, compared to previous evaluation codes. To this end, we start by considering general multiplicity codes as in [16], i.e., where we evaluate any decreasing set of derivatives. Then we show that duals of “box” multiplicity codes (those whose orders of derivatives correspond to a box) are again “box” multiplicity codes (Theorem 5.6). We then use this result to compute duals of classical multiplicity codes (as defined in [27]) in Theorem 5.10, since they are obtained by puncturing “box” multiplicity codes. We also use our explicit description of duals of multiplicity codes to lower bound their minimum distance (see Proposition 5.14).

The organization is as follows. In Sect. 2, we give preliminaries on Hasse derivatives and multiplicities. In Sect. 3, we construct Hermite interpolation polynomials on one and several variables. These polynomials are used to express duals. In Sect. 4, we compute the dimensions of multiplicity codes, which are important for duality. In Sect. 5, we explicitly compute duals of multiplicity codes. For the convenience of the reader, we include Appendix B and Appendix C, where we explicitly compute duals for the univariate case (Proposition B.3) and the case of 2 variables and multiplicity 2 (Corollary C.2 and Proposition C.3), respectively. The general case is presented in Theorems 5.10 and 5.13. We end Sect. 5 with a lower bound on the minimum distance of duals of multivariate multiplicity codes (Proposition 5.14).

## 2 Preliminaries

In this section, we introduce notation and the main relevant results from the literature.

**Basic notation.** Throughout this manuscript,  $\mathbb{F}$  denotes an arbitrary field and  $\mathbb{F}_q$  the finite field of size  $q$ , which is a power of a prime. We denote by  $\mathbb{F}[\mathbf{x}] = \mathbb{F}[x_1, \dots, x_m]$  the ring of polynomials in the  $m$  variables  $x_1, \dots, x_m$  with coefficients in  $\mathbb{F}$ , and we denote by  $\mathbb{F}[\mathbf{x}]_{<k}$  and  $\mathbb{F}[\mathbf{x}]_{\leq k}$  the sets of polynomials in  $\mathbb{F}[\mathbf{x}]$  of degree less than  $k$  and at most  $k$ , respectively. We set  $\mathbb{N} = \{0, 1, 2, \dots\}$ , and for  $\mathbf{i} = (i_1, \dots, i_m) \in \mathbb{N}^m$ , we denote  $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} \dots x_m^{i_m}$  and  $|\mathbf{i}| = i_1 + \dots + i_m$ . Note that  $\deg(\mathbf{x}^{\mathbf{i}}) = |\mathbf{i}|$ . For a monomial ordering  $<$ , we denote by  $\text{in}_<(f)$  (or just  $\text{in}(f)$ ) the initial or leading monomial of  $f \in \mathbb{F}[\mathbf{x}]$ . We denote by  $\text{Coeff}(f, \mathbf{x}^{\mathbf{i}}) \in \mathbb{F}$  the coefficient of  $f \in \mathbb{F}[\mathbf{x}]$  in the monomial  $\mathbf{x}^{\mathbf{i}}$ . We denote by  $\langle A \rangle$  the ideal generated by  $A$  in a ring, and  $\langle A \rangle_{\mathbb{F}}$  the vector space over  $\mathbb{F}$  generated by  $A$ . For integers  $m \leq n$ , we denote  $[m, n] = \{m, m + 1, \dots, n\}$  and  $[n] = [1, n]$ . We also denote by  $\mathbf{1} = (1, 1, \dots, 1) \in \mathbb{N}^m$  the vector of all ones, and by  $\delta_{i,j}$  the Kronecker delta (i.e.,  $\delta_{i,j} = 1$  if  $i = j$  and  $\delta_{i,j} = 0$  otherwise).

For a code  $\mathcal{C} \subseteq \mathbb{F}^{tn}$ , we consider the  $t$ -folded Hamming metric in  $\mathbb{F}^{tn}$ , i.e., the Hamming metric of length  $n$  over the alphabet  $\mathbb{F}^t$ . More precisely, we define

$$w(\mathbf{c}_1, \dots, \mathbf{c}_n) = |\{i \in [n] : \mathbf{c}_i \neq \mathbf{0}\}|,$$

for  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{F}^t$ . We then define the folded distance as  $d(\mathbf{c}, \mathbf{c}') = w(\mathbf{c} - \mathbf{c}')$ , for  $\mathbf{c}, \mathbf{c}' \in \mathbb{F}^{tn}$ , and similarly for the minimum folded distance  $d(\mathcal{C})$  of a code  $\mathcal{C} \subseteq \mathbb{F}^{tn}$ .

For a code  $\mathcal{C} \subseteq \mathbb{F}^{tn}$ , we define the classical Euclidean dual as

$$\mathcal{C}^{\perp} = \{\mathbf{c} \in \mathbb{F}^{tn} : \mathbf{c} \cdot \mathbf{c}' = 0, \text{ for all } \mathbf{c}' \in \mathcal{C}\},$$

where  $\mathbf{c} \cdot \mathbf{c}' = (c_1, \dots, c_{tn}) \cdot (c'_1, \dots, c'_{tn}) = \sum_{i=1}^{tn} c_i c'_i$ .

### 2.1 Multiplicity codes

In this work, we consider Hasse derivatives to count multiplicities.

**Definition 2.1** (Hasse derivative [21]) Take  $f(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ . Given another family of independent variables  $\mathbf{y} = (y_1, \dots, y_m)$ , the polynomial  $f(\mathbf{x} + \mathbf{y}) \in \mathbb{F}[\mathbf{x}, \mathbf{y}]$  can be written uniquely as

$$f(\mathbf{x} + \mathbf{y}) = \sum_{\mathbf{i} \in \mathbb{N}^m} f^{(\mathbf{i})}(\mathbf{x}) \mathbf{y}^{\mathbf{i}},$$

for some polynomials  $f^{(\mathbf{i})}(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ , for  $\mathbf{i} \in \mathbb{N}^m$ . For  $\mathbf{i} \in \mathbb{N}^m$ , we define the  $\mathbf{i}$ th Hasse derivative of  $f(\mathbf{x})$  as the polynomial  $f^{(\mathbf{i})}(\mathbf{x}) \in \mathbb{F}[\mathbf{x}]$ .

Using the Taylor expansion of a polynomial, it is immediate to see that the classic  $\mathbf{i}$ th derivative of  $f(\mathbf{x})$  is given in terms of the Hasse derivative by

$$\frac{\partial^{|\mathbf{i}|}}{\partial x_1^{i_1} \dots \partial x_m^{i_m}} f(\mathbf{x}) = \mathbf{i}! f^{(\mathbf{i})}(\mathbf{x}),$$

where  $\mathbf{i}! = i_1! \dots i_m!$ . Therefore, in positive characteristic, classical derivatives can be recovered from Hasse derivatives, but not vice versa. Hence, Hasse derivatives contain (strictly) more information about the polynomial (in characteristic 0, they contain the same information). If  $\mathbf{i} \in [0, 1]^m$ , then both derivatives coincide over any field since  $\mathbf{i}! = 1$ . We now

present the Leibniz rule for Hasse derivatives. This result will be used in Definition 4.2 and Theorem 5.6.

**Proposition 2.2** (Leibniz rule) For  $f, g \in \mathbb{F}[\mathbf{x}]$ , we have

$$(fg)^{(i)}(\mathbf{x}) = \sum_{\mathbf{j}+\mathbf{k}=\mathbf{i}} f^{(j)}(\mathbf{x})g^{(k)}(\mathbf{x}).$$

**Proof** By the definition of Hasse derivatives,

$$(fg)(\mathbf{x} + \mathbf{y}) = \left( \sum_{\mathbf{j} \in \mathbb{N}^m} f^{(j)}(\mathbf{x})\mathbf{y}^{\mathbf{j}} \right) \left( \sum_{\mathbf{k} \in \mathbb{N}^m} g^{(k)}(\mathbf{x})\mathbf{y}^{\mathbf{k}} \right) = \sum_{\mathbf{i} \in \mathbb{N}^m} \left( \sum_{\mathbf{j}+\mathbf{k}=\mathbf{i}} f^{(j)}(\mathbf{x})g^{(k)}(\mathbf{x}) \right) \mathbf{y}^{\mathbf{i}}.$$

Thus, we obtain the result. □

Multiplicity codes were introduced in [27, Def. 3.4]. We now consider the case when the evaluation points form a grid or a Cartesian product  $S = S_1 \times \dots \times S_m$ , where every  $S_i \subseteq \mathbb{F}$  is finite. In this work, all codes are linear (over  $\mathbb{F}$ ) unless otherwise stated.

**Definition 2.3** (Multiplicity codes [27]) Take  $n = |S|$  and  $t = \binom{m+r-1}{m}$ . We define the multiplicity code on  $S$  of multiplicity  $r$  and degree  $k$  as

$$\mathcal{M}(S, r, k) = \left\{ \left( \left( f^{(i)}(\mathbf{a}) \right)_{|\mathbf{i}| < r} \right)_{\mathbf{a} \in S} : f \in \mathbb{F}[\mathbf{x}]_{<k} \right\} \subseteq (\mathbb{F}^t)^n.$$

For easy notation, we sometimes omit the inner parentheses. The parameter  $r$  bounds the derivatives, and  $k$  bounds the degrees of the evaluating polynomials. When the number of variables is  $m = 1$  and  $k < |S_1|$ , then  $k$  is the dimension of the multiplicity code. But in general,  $k$  does not represent the dimension.

**Example 2.4** Assume  $\mathbb{F}_q = \{a_1, \dots, a_q\}$ . The multiplicity code  $\mathcal{M}(\mathbb{F}_q, 1, k)$  is the classical Reed-Solomon code

$$\mathcal{M}(\mathbb{F}_q, 1, k) = \{(f(a_1), \dots, f(a_q)) : f \in \mathbb{F}_q[x]_{<k}\} \subseteq \mathbb{F}_q^q.$$

We can also see that

$$\mathcal{M}(\mathbb{F}_q, 2, k) = \left\{ \left( (f(a_1), f^{(1)}(a_1)), \dots, (f(a_q), f^{(1)}(a_q)) \right) : f \in \mathbb{F}_q[x]_{<k} \right\} \subseteq (\mathbb{F}_q^2)^q.$$

A method to lower bound the minimum distance of multiplicity codes over grids is the Schwartz-Zippel bound with multiplicities given in [13, Lemma 8].

**Definition 2.5** (Multiplicity) We define the multiplicity of  $\mathbf{a} \in \mathbb{F}^m$  in  $f \in \mathbb{F}[\mathbf{x}]$ , denoted  $m(f, \mathbf{a})$ , as the largest  $r \in \mathbb{N}$  such that  $f^{(i)}(\mathbf{a}) = 0$ , for all  $\mathbf{i} \in \mathbb{N}^m$  such that  $|\mathbf{i}| < r$ .

**Lemma 2.6** (Schwartz-Zippel [13]) If  $s = |S_1| = \dots = |S_m|$ , then for any  $f \in \mathbb{F}[\mathbf{x}]$ ,

$$\sum_{\mathbf{a} \in S} m(f, \mathbf{a}) \leq \deg(f)s^{m-1}.$$

As a consequence, we obtain the following result, which is essentially [27, Lemma 3.5]. We will also use Lemma 2.6 in Proposition 5.14.

**Proposition 2.7** ([27]) If  $s = |S_1| = \dots = |S_m|$  and  $n = s^m = |S|$ , then

$$d(\mathcal{M}(S, r, k)) \geq s^m - \frac{(k-1)s^{m-1}}{r} = \left( 1 - \frac{k-1}{rs} \right) n.$$

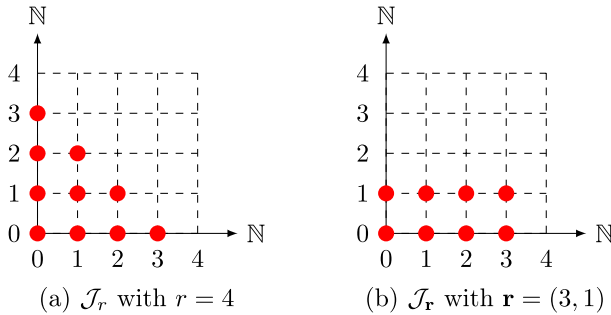


Fig. 1 (a) The multiplicity set  $\mathcal{J}_r$  with  $r = 4$ . (b) The box multiplicity set  $\mathcal{J}_r$  with  $\mathbf{r} = (3, 1)$

### 2.2 General multiplicity codes

In order to compute duals of multiplicity codes, we consider general multiplicity codes for decreasing or down sets of Hasse derivatives, as in [16].

**Definition 2.8** ([16]) The set  $\mathcal{J} \subseteq \mathbb{N}^m$  is decreasing if whenever  $\mathbf{i} \in \mathcal{J}$  and  $\mathbf{j} \in \mathbb{N}^m$  are such that  $\mathbf{j} \leq \mathbf{i}$ , it holds that  $\mathbf{j} \in \mathcal{J}$ , where  $\leq$  is the coordinatewise partial order in  $\mathbb{N}^m$ . We typically consider two families of decreasing sets: The classical multiplicity set  $\mathcal{J}_r = \{\mathbf{i} \in \mathbb{N}^m : |\mathbf{i}| < r\}$  for  $r \in \mathbb{N}$ , and the coordinatewise or box multiplicity set  $\mathcal{J}_r = \{\mathbf{i} \in \mathbb{N}^m : \mathbf{i} \leq \mathbf{r}\}$  for  $\mathbf{r} \in \mathbb{N}^m$ . Figure 1 shows an example of a multiplicity set and a box multiplicity set.

We now define general multiplicity codes, an extension of Definition 2.5 that has already been considered in [16, Def. 4.16].

**Definition 2.9** (General multiplicity codes [16]) Given a decreasing set  $\mathcal{J} \subseteq \mathbb{N}^m$ , we define the multiplicity code on  $S$  with multiplicity set  $\mathcal{J}$  and degree  $k$  as

$$\mathcal{M}(S, \mathcal{J}, k) = \left\{ \left( \left( f^{(\mathbf{i})}(\mathbf{a}) \right)_{\mathbf{i} \in \mathcal{J}} \right)_{\mathbf{a} \in S} : f \in \mathbb{F}[\mathbf{x}]_{<k} \right\} \subseteq \left( \mathbb{F}^{|\mathcal{J}|} \right)^{|S|}.$$

For easy notation, we sometimes omit the inner parentheses. Clearly,  $\mathcal{M}(S, r, k) = \mathcal{M}(S, \mathcal{J}_r, k)$ , hence Definition 2.9 generalizes Definition 2.5. Furthermore, if  $\mathbf{r} = r\mathbf{1} = (r, \dots, r)$  and  $\mathbf{1} = (1, \dots, 1)$ , then  $\mathcal{M}(S, r, k)$  is a punctured code of  $\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-\mathbf{1}}, k)$ . However, the dual of the latter is easier to compute in the multivariate case (note they coincide in one variable). This is our approach in Sect. 5 (and also in Appendix C). In the following, it is useful to remember at every moment that

$$\mathcal{J}_{\mathbf{r}-\mathbf{1}} = [0, r - 1]^m,$$

when  $\mathbf{r} = r\mathbf{1}$ . We write  $\mathcal{J}_{\mathbf{r}-\mathbf{1}}$  or  $[0, r - 1]^m$  interchangeably in the remainder of the manuscript.

### 2.3 Isometries for the folded metric

The linear isometries for the folded Hamming metric are known since the folded Hamming metric is a particular case of the sum-rank metric, see [31, Th. 2].

**Proposition 2.10** ([31]) Let  $\phi : \mathbb{F}^m \rightarrow \mathbb{F}^m$  be a linear vector space isomorphism. Then  $\phi$  is an isometry for the  $t$ -folded Hamming metric (i.e.,  $w(\phi(\mathbf{c})) = w(\mathbf{c})$  for all  $\mathbf{c} \in \mathbb{F}^m$ )

if, and only if, there exist invertible matrices  $A_1, \dots, A_n \in \text{GL}_t(\mathbb{F})$  and a permutation  $\sigma : [n] \rightarrow [n]$  such that, for all  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{F}^t$ ,

$$\phi(\mathbf{c}_1, \dots, \mathbf{c}_n) = (\mathbf{c}_{\sigma(1)}A_1, \dots, \mathbf{c}_{\sigma(n)}A_n).$$

**Definition 2.11** The codes  $\mathcal{C}, \mathcal{C}' \subseteq \mathbb{F}^{tn}$  are equivalent if there exists a linear isometry  $\phi : \mathbb{F}^{tn} \rightarrow \mathbb{F}^{tn}$  for the folded metric such that  $\mathcal{C}' = \phi(\mathcal{C})$ . We also say that  $\phi$  is an equivalence.

In particular, two codes are equivalent if, and only if, so are their duals. More concretely, we have the following. The proof is straightforward.

**Proposition 2.12** Given invertible matrices  $A_1, \dots, A_n \in \text{GL}_t(\mathbb{F})$  and a permutation  $\sigma : [n] \rightarrow [n]$ , define the equivalences  $\phi, \psi : \mathbb{F}^{tn} \rightarrow \mathbb{F}^{tn}$  by

$$\begin{aligned} \phi(\mathbf{c}_1, \dots, \mathbf{c}_n) &= (\mathbf{c}_{\sigma(1)}A_1, \dots, \mathbf{c}_{\sigma(n)}A_n), \\ \psi(\mathbf{c}_1, \dots, \mathbf{c}_n) &= (\mathbf{c}_{\sigma(1)}(A_1^T)^{-1}, \dots, \mathbf{c}_{\sigma(n)}(A_n^T)^{-1}), \end{aligned}$$

for  $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbb{F}^t$ . Then for any linear code  $\mathcal{C} \subseteq \mathbb{F}^{tn}$ ,

$$\phi(\mathcal{C})^\perp = \psi(\mathcal{C}^\perp).$$

**Remark 2.13** Any family  $\Gamma$  of codes  $\mathcal{C} \subseteq \mathbb{F}^{tn}$  can be generalized to include all equivalent codes  $\bar{\Gamma} = \{\phi(\mathcal{C}) : \mathcal{C} \in \Gamma, \phi \text{ is an equivalence}\}$ , as is done with generalized Reed–Solomon codes. However, by Proposition 2.12, if we are able to compute the family  $\Gamma^\perp$  of codes that are dual to some code in  $\Gamma$ , then we have also computed the family  $\bar{\Gamma}^\perp$  of codes that are dual to some code in  $\bar{\Gamma}$ . In Sect. 5, we compute the duals of multiplicity codes as in Definition 2.5, and by this remark, we have computed the duals of “generalized multiplicity codes”, i.e., codes equivalent to a multiplicity code.

### 3 Hermite interpolation

In this section, we study Hermitian interpolator polynomials, whose coefficients are important to compute the duals of multiplicity codes. We see how a Hermitian interpolation basis for the set  $\mathcal{J}_{\mathbf{r}-1}$  is formed by polynomials with separated variables, when  $\mathbf{r} = r\mathbf{1}$ . Such a Hermitian interpolation basis is easier than for the set  $\mathcal{J}_r$ . For convenience of the reader, we include the cases of univariate multiplicity 1 and 2 in Appendix A.

#### 3.1 Single-variable Hermite interpolation

Let  $T = \{a_1, \dots, a_s\}$  be a set of (distinct) elements of  $\mathbb{F}$ . Let  $b_{10}, b_{11}, \dots, b_{1r}, \dots, b_{s0}, b_{s1}, \dots, b_{sr}$  be any  $s(r + 1)$  elements in  $\mathbb{F}$ . Define the map

$$\begin{aligned} ev^{(r)} : \mathbb{F}[x] &\rightarrow \mathbb{F}^{(r+1)s} \\ f &\mapsto ev^{(r)}(f) = (f(a_1), f^{(1)}(a_1), \dots, f^{(r)}(a_1), \dots, f(a_s), f^{(1)}(a_s), \dots, f^{(r)}(a_s)). \end{aligned}$$

**Theorem 3.1** (Single-variable Hermite interpolation) Let  $G(x) = (x - a_1) \cdots (x - a_s)$  be the vanishing polynomial and  $h_1, \dots, h_s$  the indicator functions of  $T$  (see Appendix A). There exists a unique polynomial  $f \in \mathbb{F}[x]_{\leq (r+1)s-1}$  such that

$$ev^{(r)}(f) = (b_{10}, b_{11}, \dots, b_{1r}, \dots, b_{s0}, b_{s1}, \dots, b_{sr}).$$

Moreover,  $f = (h_{1,0}, \dots, h_{1,r}, \dots, h_{s,0}, \dots, h_{s,r}) \cdot (b_{10}, b_{11}, \dots, b_{1r}, \dots, b_{s0}, b_{s1}, \dots, b_{sr})$ , where we define, recursively in  $n = r, r - 1, \dots, 1, 0$ , in decreasing order,  $h_{i,r}(x) = (x - a_i)^r h_i^{r+1}(x)$  and

$$h_{i,n}(x) = (x - a_i)^n h_i^{r+1}(x) - \sum_{k=n+1}^r h_i^{r+1(k-n)}(a_i) h_{i,k}(x),$$

for  $i \in [s]$ . Here,  $h_i^{r+1(j)}(a_i)$  is the value of the  $j$ th derivative of  $h_i^{r+1}(x)$  at  $x = a_i$ .

**Proof** Let  $(e_{1,0}, \dots, e_{1,r}, \dots, e_{s,0}, \dots, e_{s,r})^\top$  be the  $(r + 1)s \times (r + 1)s$  identity matrix. In other words, the  $e_{i,n}$ 's represent the standard vectors for  $i \in [1, s]$  and  $n \in [0, r]$ . We show that  $ev^{(r)}(h_{i,n}) = e_{i,n}$ , for all  $i \in [s]$ , by induction in  $n = r, \dots, 1, 0$  (in decreasing order).

(Case  $n = r$ ) The polynomial  $h_{i,r}(x) = (x - a_i)^r h_i^{r+1}(x)$  and its first  $r - 1$  derivatives vanish on every element of  $T$ . By Proposition 2.2,

$$h_{i,r}^{(r)}(x) = (x - a_i)h_i(x)g(x) + h_i^{r+1}(x),$$

for some  $g(x) \in \mathbb{F}[x]$ . Thus,  $h_{i,r}^{(r)}(a_j) = h_i^{r+1}(a_j) = \delta_{i,j}$ , meaning  $ev^{(r)}(h_{i,r}) = e_{i,r}$  for  $i \in [s]$ .

(Case  $n \leq r - 1$ ) Fix  $i \in [s]$ . We assume by induction in  $n$  that  $ev^{(r)}(h_{i,t}) = e_{i,t}$ , for all  $t \in [n + 1, r]$ . Given  $h_{i,n}(x)$  as in the proposition, we now show that  $ev^{(r)}(h_{i,n}) = e_{i,n}$ . We consider different cases in order to compute  $h_{i,n}^{(\ell)}(a_j)$ .

If  $0 \leq \ell \leq n - 1$ , then we have  $((x - a_i)^n h_i^{r+1}(x))^{(\ell)}(a_j) = 0$  and  $h_{i,n+1}^{(\ell)}(a_j) = \dots = h_{i,r}^{(\ell)}(a_j) = 0$ , thus  $h_{i,n}^{(\ell)}(a_j) = 0$ , for all  $j \in [s]$ .

If  $\ell = n$ , then  $((x - a_i)^n h_i^{r+1}(x))^{(n)}(a_j) = h_i^{r+1}(a_j)$  by Proposition 2.2, and  $h_{i,n+1}^{(n)}(a_j) = \dots = h_{i,r}^{(n)}(a_j) = 0$ , thus  $h_{i,n}^{(n)}(a_j) = h_i^{r+1}(a_j) = \delta_{i,j}$ , for all  $j \in [s]$ .

If  $n + 1 \leq \ell \leq r$ , then  $((x - a_i)^n h_i^{r+1}(x))^{(\ell)}(a_j) = h_i^{r+1(\ell-n)}(a_j)$  by Proposition 2.2, which is 0 if  $j \neq i$ . Now, by the induction hypothesis, we have

$$\begin{aligned} h_{i,n}^{(\ell)}(a_j) &= ((x - a_i)^n h_i^{r+1}(x))^{(\ell)}(a_j) - \sum_{k=n+1}^r h_i^{r+1(k-n)}(a_i) h_{i,k}^{(\ell)}(a_j) \\ &= h_i^{r+1(\ell-n)}(a_j) - h_i^{r+1(\ell-n)}(a_j) \delta_{i,j} \\ &= 0, \end{aligned}$$

for all  $j \in [s]$ , and we are done. □

**Remark 3.2** Notice that, since  $h_{i,r}(x)$  only depends on  $h_i(x)$ , then by induction all of the polynomials  $h_{i,n}(x)$ , for  $n \in [0, r]$ , can be expressed as a function of only  $h_i(x)$  (and its derivatives at  $x = a_i$ ), as in Propositions A.1. However, we omit this formula in general for brevity.

**Remark 3.3** In case the set  $T$  and the element  $a_i$  are relevant, we also denote  $h_{i,j}$  from Theorem 3.1 by  $h_{T,a_i,j}$ . Note that the polynomials  $h_{i,j}$ 's are playing the role of indicator functions with derivatives.

**Remark 3.4** From Theorem 3.1, we have that

$$h_{i,r}(x) = (x - a_i)^r h_i^{r+1}(x) = (x - a_i)^r \frac{G(x)^{r+1}}{(G^{(1)}(a_i))^{r+1} (x - a_i)^{r+1}}.$$

In particular,  $\text{Coeff}(h_{i,r}, x^{(r+1)s-1}) = (G^{(1)}(a_i))^{-r-1} \neq 0$ . See also Appendix A.

**Remark 3.5** In the case  $T = \mathbb{F}_q$ , we have  $G(x) = x^q - x$ . Hence  $G^{(1)}(x) = -1$ . Therefore,

$$\text{Coeff}(h_{i,r}, x^{(r+1)q-1}) = (-1)^{r+1}.$$

Moreover, by Remark A.2, for  $r = 1$ , we have that  $h_{i,0}(x) = h_i^2(x)$ , since  $2G^{(2)}(a_i) = 0$ , for all  $i \in [q]$  in that case. In particular,  $\text{deg}(h_{i,0}) = 2q - 2$ , thus for  $r = 1$ ,

$$\text{Coeff}(h_{i,0}, x^{2q-1}) = 0.$$

In the case  $T = \mathbb{F}_q^*$ , we have  $G(x) = x^{q-1} - 1$ . Hence  $G^{(1)}(x) = (q - 1)x^{q-2} = -x^{q-2} = -x^{-1}$  and  $G^{(1)}(a_i) = (-a_i)^{-1}$ . Therefore,

$$\text{Coeff}(h_{i,r}, x^{(r+1)(q-1)-1}) = (-a_i)^{r+1}.$$

### 3.2 Multivariate Hermite interpolation

Recall that  $S = S_1 \times \dots \times S_m$  represents a Cartesian set, where every  $S_i \subseteq \mathbb{F}$  is finite. Fix a positive integer  $r$ , set  $\mathbf{r} = r\mathbf{1}$  and recall that  $\mathcal{J}_{\mathbf{r}-\mathbf{1}} = [0, r - 1]^m$ . Let  $\{b_{\mathbf{a},\mathbf{i}}\}_{\mathbf{a} \in S, \mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}}$  be a multiset of  $|S|^{r^m}$  elements in  $\mathbb{F}$ . Similarly to the single-variable case, we define the evaluation map

$$ev^{(\mathcal{J}_{\mathbf{r}-\mathbf{1}})} : \mathbb{F}[\mathbf{x}] \rightarrow \mathbb{F}^{|S|^{r^m}}$$

$$f \mapsto ev^{(\mathcal{J}_{\mathbf{r}-\mathbf{1}})}(h) = \left( \left( h^{(\mathbf{i})}(\mathbf{a}) \right)_{\mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}} \right)_{\mathbf{a} \in S}.$$

**Corollary 3.6** (Rectangular Hermite interpolation) Assume  $|S_i| = s_i$ . There exists a unique  $h \in \mathbb{F}[\mathbf{x}]$  with  $\text{deg}_{x_j}(h) \leq rs_j - 1$  for  $j \in [m]$  such that

$$ev^{(\mathcal{J}_{\mathbf{r}-\mathbf{1}})}(h) = \left( (b_{\mathbf{a},\mathbf{i}})_{\mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}} \right)_{\mathbf{a} \in S}.$$

Moreover,  $h = (h_{\mathbf{a},\mathbf{i}})_{\mathbf{a} \in S, \mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}} \cdot (b_{\mathbf{a},\mathbf{i}})_{\mathbf{a} \in S, \mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}}$ , where

$$h_{\mathbf{a},\mathbf{i}}(\mathbf{x}) = \prod_{j=1}^m h_{S_j, a_j, i_j}(x_j) \in \mathbb{F}[\mathbf{x}],$$

and  $h_{S_j, a_j, i_j}$  is defined in Remark 3.3 (and Theorem 3.1) for  $\mathbf{a} = (a_1, \dots, a_m) \in S$  and  $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}$ .

**Proof** The map that sends a polynomial  $h \in \mathbb{F}[\mathbf{x}]$  with  $\text{deg}_{x_j}(h) \leq rs_j - 1$ , for  $j \in [m]$ , to the evaluations  $((h^{(\mathbf{i})}(\mathbf{a}))_{\mathbf{i} \in \mathcal{J}_{\mathbf{r}-\mathbf{1}}})_{\mathbf{a} \in S}$  is a vector space isomorphism from the space of such polynomials to  $(\mathbb{F}^{|\mathcal{J}_{\mathbf{r}-\mathbf{1}}|})^{|S|}$ . The map is surjective by using Theorem 3.1. Thus, it is an isomorphism since the dimensions of both spaces are the same over  $\mathbb{F}$ .  $\square$

**Remark 3.7** We also denote  $h_{\mathbf{a},\mathbf{i}}$  by  $h_{S,\mathbf{a},\mathbf{i}}$  in case the set  $S$  is relevant.

**Remark 3.8** Notice that

$$\text{Coeff}(h_{\mathbf{a}, \mathbf{r}-1}, x_1^{rs_1-1} \cdots x_m^{rs_m-1}) = \prod_{j=1}^m \text{Coeff}(h_{S_j, a_j, i_j}(x_j), x_j^{rs_j-1}).$$

Therefore, by Remark 3.4, we have that

$$\text{Coeff}(h_{\mathbf{a}, \mathbf{r}-1}, x_1^{rs_1-1} \cdots x_m^{rs_m-1}) = \left(G_1^{(1)}(a_1) \cdots G_m^{(1)}(a_m)\right)^{-r} \neq 0, \tag{1}$$

for  $\mathbf{a} = (a_1, \dots, a_m) \in S$ , where  $G_i(x)$  is the vanishing polynomial of  $S_i$ .

### 4 Dimension

In this section, we compute the dimensions of general multiplicity codes (Definition 2.9) using Gröbner basis tools (the reader is referred to [10] for an introduction to the theory of Gröbner bases). The dimension plays an important role in computing the dual.

**Definition 4.1** For any finite decreasing set  $\mathcal{J} \subseteq \mathbb{N}^m$ , we define

$$\mathcal{B}_{\mathcal{J}} = \{\mathbf{i} \in \mathbb{N}^m \setminus \mathcal{J} : \mathbf{j} \in \mathbb{N}^m \setminus \mathcal{J} \text{ and } \mathbf{j} \leq \mathbf{i} \implies \mathbf{i} = \mathbf{j}\}.$$

We next consider the polynomials whose derivatives of orders in  $\mathcal{J}$  vanish in  $S$ . The following definition is [16, Def. 3.18].

**Definition 4.2** ([16]) Given a finite decreasing set  $\mathcal{J} \subseteq \mathbb{N}^m$ , we define

$$I(S; \mathcal{J}) = \left\{f \in \mathbb{F}[\mathbf{x}] : f^{(\mathbf{i})}(\mathbf{a}) = 0, \text{ for all } \mathbf{a} \in S \text{ and all } \mathbf{i} \in \mathcal{J}\right\}.$$

Using that  $\mathcal{J}$  is decreasing and the Leibniz rule (Proposition 2.2), it is easy to see that  $I(S; \mathcal{J})$  is an ideal in  $\mathbb{F}[\mathbf{x}]$ . In [16, Th. 4.7], a Gröbner basis for such an ideal is computed for any monomial ordering.

**Theorem 4.3** ([16]) Take  $G_j(x_j) = \prod_{\alpha \in S_j} (x_j - \alpha)$ , for  $j \in [m]$ . For any finite decreasing set  $\mathcal{J} \subseteq \mathbb{N}^m$ , the family

$$\mathcal{F} = \left\{ \prod_{j=1}^m G_j(x_j)^{u_j} : (u_1, u_2, \dots, u_m) \in \mathcal{B}_{\mathcal{J}} \right\}$$

is a reduced Gröbner basis for the ideal  $I(S; \mathcal{J})$  with respect to any monomial ordering (see [10, Def. 2.5.5 & Def. 2.7.4] for the definitions).

We also need the concept of footprint [15], which is well known in the commutative algebra literature.

**Definition 4.4** ([15]) Given a monomial ordering  $\prec$  and an ideal  $I \subseteq \mathbb{F}[\mathbf{x}]$ , we define its footprint with respect to  $\prec$  as

$$\Delta_{\prec}(I) = \{\mathbf{x}^{\mathbf{i}} : \mathbf{x}^{\mathbf{i}} \notin \langle \text{in}_{\prec}(I) \rangle\},$$

where  $\text{in}_{\prec}(I) = \{\text{in}_{\prec}(f) : f \in I\}$ , i.e.,  $\langle \text{in}_{\prec}(I) \rangle$  is the initial ideal of  $I$  with respect to  $\prec$ .

**Remark 4.5** For  $\mathbf{r} = r\mathbf{1}$  and any monomial ordering  $\prec$ , using the Gröbner basis from Theorem 4.3, it is straightforward to verify that

$$\Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-\mathbf{1}})) = \left\{ \mathbf{x}^{\mathbf{i}} : \mathbf{i} \in \prod_{j=1}^m [0, rs_j - 1] \right\}.$$

In particular, any  $f \in \langle \Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-\mathbf{1}})) \rangle_{\mathbb{F}}$  satisfies  $\deg_{x_j}(f) \leq rs_j - 1$ , for all  $j \in [m]$ .

Footprints are useful for reducing polynomials while preserving their evaluations, as we show now. For instance, they play a major role when computing the dimension of a general multiplicity code in Theorem 4.9.

**Lemma 4.6** For any monomial ordering  $\prec$  and any finite decreasing set  $\mathcal{J} \subseteq \mathbb{N}^m$ , the maps

$$\begin{array}{ccc} \langle \Delta_{\prec}(I(S; \mathcal{J})) \rangle_{\mathbb{F}} & \xrightarrow{\rho} & \mathbb{F}[\mathbf{x}]/I(S; \mathcal{J}) & \xrightarrow{\eta} & (\mathbb{F}^t)^n \\ f & \mapsto & f + I(S; \mathcal{J}) & \mapsto & \left( (f^{(\mathbf{i})}(\mathbf{a}))_{\mathbf{i} \in \mathcal{J}} \right)_{\mathbf{a} \in S} \end{array}$$

are isomorphisms of vector spaces over  $\mathbb{F}$ , where  $t = |\mathcal{J}|$  and  $n = |S|$ . In particular, the images of the monomials in  $\Delta_{\prec}(I(S; \mathcal{J}))$  by  $\rho$  (i.e., modulo  $I(S; \mathcal{J})$ ) and  $\eta \circ \rho$  (i.e., their Hasse derivatives) are  $\mathbb{F}$ -linearly independent, and

$$|\Delta_{\prec}(I(S; \mathcal{J}))| = |\mathcal{J}| \cdot |S| = tn. \tag{2}$$

**Proof** The fact that  $\rho$  is an isomorphism of vector spaces over  $\mathbb{F}$  follows from [10, Prop. 5.3.1]. Finally, the fact that  $\eta$  is an isomorphism of vector spaces over  $\mathbb{F}$  follows from the  $\mathbb{F}$ -linearity of Hasse derivatives, the definition of  $I(S; \mathcal{J})$  (for injectivity) and Corollary 3.6 (for surjectivity).  $\square$

**Definition 4.7** With notation as in Lemma 4.6, given  $f \in \mathbb{F}[\mathbf{x}]$ , we define its reduced form in  $S$  and  $\mathcal{J}$  as

$$\bar{f} = \rho^{-1}(f + I(S; \mathcal{J})) \in \langle \Delta_{\prec}(I(S; \mathcal{J})) \rangle_{\mathbb{F}}.$$

We deduce the following properties, which will be used in Theorems 4.9 and 5.6.

**Corollary 4.8** With notation as in Lemma 4.6, given  $f \in \mathbb{F}[\mathbf{x}]$ , we have

1.  $\deg(\bar{f}) \leq \deg(f)$  if  $\prec$  is a graded monomial ordering.
2.  $\bar{f}^{(\mathbf{i})}(\mathbf{a}) = f^{(\mathbf{i})}(\mathbf{a})$ , for all  $\mathbf{a} \in S$  and all  $\mathbf{i} \in \mathcal{J}$ .
3.  $\bar{f} = f$  if  $f \in \langle \Delta_{\prec}(I(S; \mathcal{J})) \rangle_{\mathbb{F}}$ .

**Proof** 1. By the division algorithm [10, Th. 2.3.3], we have that

$$f = q_1g_1 + \dots + q_Mg_M + r,$$

for some  $q_1, \dots, q_M \in \mathbb{F}[\mathbf{x}]$ , where  $\mathcal{F} = \{g_1, g_2, \dots, g_M\}$  is the Gröbner basis of  $I(S; \mathcal{J})$  from Theorem 4.3, and where  $r$  is an  $\mathbb{F}$ -linear combination of monomials in  $\Delta_{\prec}(I(S; \mathcal{J}))$ . Since we are using a graded monomial ordering, it must hold  $\deg(r) \leq \deg(f)$ , given how the division algorithm works. Finally, since  $\mathcal{F}$  is a Gröbner basis of  $I(S; \mathcal{J})$ , then we must have  $r = \bar{f}$  by uniqueness of remainders [10, Prop. 2.6.1], and we are done.

2. It follows from

$$\left( \left( \bar{f}^{(i)}(\mathbf{a}) \right)_{i \in \mathcal{J}} \right)_{\mathbf{a} \in S} = \eta(\rho(\bar{f})) = \eta(f + I(S; \mathcal{J})) = \left( \left( f^{(i)}(\mathbf{a}) \right)_{i \in \mathcal{J}} \right)_{\mathbf{a} \in S}.$$

3. It follows by the uniqueness of remainders since  $\mathcal{F}$  is a Gröbner basis, as in Item 1. □

From the previous results, we deduce the following theorem, which includes a first formula for the dimension of general multiplicity codes as in Definition 2.9.

**Theorem 4.9** *Let  $\mathcal{J} \subseteq \mathbb{N}^m$  be a finite decreasing set and  $\prec$  any graded monomial ordering. Then*

$$\mathcal{M}(S, \mathcal{J}, k) = \left\{ \left( \left( f^{(i)}(\mathbf{a}) \right)_{i \in \mathcal{J}} \right)_{\mathbf{a} \in S} : f \in \langle \Delta_{\prec}(I(S; \mathcal{J})) \rangle_{\mathbb{F}} \cap \mathbb{F}[\mathbf{x}]_{\prec k} \right\}.$$

In particular, if  $M^k = \{\mathbf{x}^{\mathbf{i}} : |\mathbf{i}| < k\}$ , we have

$$\dim_{\mathbb{F}}(\mathcal{M}(S, \mathcal{J}, k)) = |M^k \cap \Delta_{\prec}(I(S; \mathcal{J}))|.$$

Moreover, if  $\mathcal{J} = \mathcal{J}_{\mathbf{r}}$  or  $\mathcal{J} = \mathcal{J}_{\mathbf{r}-1}$  (or in general  $\mathcal{J} \subseteq \mathcal{J}_{\mathbf{r}-1}$ ), where  $\mathbf{r} = r\mathbf{1}$ , and  $k > \sum_{j=1}^m (rs_j - 1)$ , then  $\mathcal{M}(S, \mathcal{J}, k) = \mathbb{F}^n$ , i.e., it is the total space.

**Proof** The first equation follows by combining the three items in Corollary 4.8 with Definition 2.9. From there, the formula for the dimension follows by the fact that the set of Hasse derivatives  $\eta(\rho(M^k \cap \Delta_{\prec}(I(S; \mathcal{J})))$  is an  $\mathbb{F}$ -linearly independent set by Lemma 4.6.

Finally, let  $\mathcal{J} = \mathcal{J}_{\mathbf{r}-1}$ , with  $\mathbf{r} = r\mathbf{1}$ , and  $k = \sum_{j=1}^m (rs_j - 1) + 1$ . Then one can easily see that  $M^k \cap \Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-1})) = \Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-1}))$ , since  $\Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-1}))$  corresponds to the box  $\prod_{j=1}^m [0, rs_j - 1]$  (see Remark 4.5). Hence  $\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)$  is the total space by Lemma 4.6. Finally, if  $\mathcal{J} \subseteq \mathcal{J}_{\mathbf{r}-1}$ , we also have  $M^k \cap \Delta_{\prec}(I(S; \mathcal{J})) = \Delta_{\prec}(I(S; \mathcal{J}))$  and the same result holds. □

We may also obtain the following reduced form of multiplicity codes, meaning that degrees in each variable may be upper bounded without leaving out any codeword.

**Corollary 4.10** *If  $\mathcal{J} \subseteq \mathbb{N}^m$  is a decreasing set such that  $\mathcal{J} \subseteq \mathcal{J}_{\mathbf{r}-1}$  (for instance  $\mathcal{J} = \mathcal{J}_{\mathbf{r}}$  or  $\mathcal{J} = \mathcal{J}_{\mathbf{r}-1}$ ), where  $\mathbf{r} = r\mathbf{1}$ , then*

$$\mathcal{M}(S, \mathcal{J}, k) = \left\{ \left( \left( f^{(i)}(\mathbf{a}) \right)_{i \in \mathcal{J}} \right)_{\mathbf{a} \in S} : f \in \mathbb{F}[\mathbf{x}]_{\prec k}, \deg_{x_j}(f) \leq rs_j - 1, \forall j \in [m] \right\}.$$

**Proof** It follows from Theorem 4.9, Remark 4.5 and  $\Delta_{\prec}(I(S; \mathcal{J})) \subseteq \Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-1}))$ . □

In some cases, explicit formulas for the dimension are possible, as we show next. This formula is also new to the best of our knowledge.

**Corollary 4.11** *Take  $r \in \mathbb{N}, k \leq m(rs - 1)$ , and  $S_1, \dots, S_m \subseteq \mathbb{F}$  such that  $s_1 = \dots = s_m = s$ . Then, for  $\mathbf{r} = r\mathbf{1}$ ,*

$$\dim_{\mathbb{F}}(\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)) = \sum_{t=0}^{k-1} \sum_{i=0}^m (-1)^i \binom{m}{i} \binom{t - irs + m - 1}{t - irs}.$$

**Proof** By Theorem 4.9, we just need to count the monomials in the footprint with degree  $< k$ . By Remark 4.5, if we look at the exponents of the monomials in the footprint, we have a hypercube of size  $rs \times \dots \times rs = (rs)^m$ . Therefore, this is the same problem as computing the dimension of a Reed–Muller code over the entire affine space  $\mathbb{F}_q^m$ , changing  $q$  with  $rs$ , which gives the stated formula (see [23] and [32, Prop. 5.4.7]).  $\square$

**Remark 4.12** A similar result follows for Cartesian sets with different sizes by substituting in the formula for the dimension of affine Cartesian codes (see [29, Thm. 3.1]) the sizes of the constituent sets by  $rs_i$ , for  $i \in [m]$ .

### 5 Dual

The dual of a multiplicity code is equivalent (isometric for the folded Hamming metric) to a multiplicity code in the univariate case  $m = 1$  (see [33, Sec. 3]). The same holds for Reed–Muller codes, i.e., multiplicity codes for several variables but multiplicity  $r = 1$ , see [11, 23]. However, this is no longer the case in general for multiplicity  $r > 1$  and  $m > 1$  variables, as we now show.

**Example 5.1** Take  $q = m = r = 2$  and  $S_1 = S_2 = \mathbb{F}_2$ , i.e.,  $S = \mathbb{F}_2^2$ . For  $k = 0, 1, 2, 3, 4, 5$ , the dimensions of the multiplicity codes  $\mathcal{M}(S, r, k)$  are 0, 1, 3, 6, 10, 12, respectively. Since the code length over the alphabet  $\mathbb{F}_2$  is  $\binom{m+r-1}{m}|S| = 12$ , the corresponding duals have dimensions 12, 11, 9, 6, 2, 0, respectively. In other words, only those of dimensions 0, 6 and 12 can have a dual that is equivalent to a multiplicity code for  $q = m = r = 2$  and  $S = \mathbb{F}_2^2$ . We come back to the multiplicity code for  $k = 4$  in Example C.5.

In Appendices B and C, we obtain the duals for the case  $m = 1$  and  $m = r = 2$ , respectively. We refer the interested reader to those appendices to see how our techniques apply to simpler cases, which can help to understand the general case. Even though the dual is no longer equivalent to a multiplicity code, in this section, we give explicit expressions for the dual of a multiplicity code in general. We use this expression in Proposition 5.14 to lower bound the minimum folded distance of such duals.

As in [33], we consider duality with respect to the usual inner product in  $\mathbb{F}^{rn}$  (see Sect. 2). We first compute the dual of  $\mathcal{M}(S, \mathcal{J}_{r-1}, k)$ , where we fix  $\mathbf{r} = r\mathbf{1}$ , for a positive integer  $r$ , throughout this subsection. This will be an auxiliary step to computing the dual of  $\mathcal{M}(S, r, k)$  (see also Appendix C). Recall that  $S = S_1 \times \dots \times S_m$ , where  $S_1, \dots, S_m \subseteq \mathbb{F}$  are non-empty and of finite sizes  $s_i = |S_i|$ , for  $i \in [m]$ .

**Definition 5.2** We order  $\mathcal{J}_{r-1} = \{\mathbf{i}_0, \mathbf{i}_1, \dots, \mathbf{i}_{r^m-1}\}$  according to the graded lexicographic order. In particular,  $\mathbf{i}_0 = \mathbf{0}$  and  $\mathbf{i}_{r^m-1} = \mathbf{r} - \mathbf{1}$ .

The following technical result is helpful when describing the duals in Theorem 5.10.

**Lemma 5.3** Ordering  $\mathcal{J}_{r-1}$  as in Definition 5.2, the first  $\binom{m+r-1}{m}$  elements of  $\mathcal{J}_{r-1}$  are exactly the  $\mathbf{i} \in [0, r - 1]^m$  such that  $|\mathbf{i}| < r$ , and the first  $r^m - \binom{m+r-1}{m}$  elements of  $\mathcal{J}_{r-1}$  are exactly the  $\mathbf{i} \in [0, r - 1]^m$  such that  $|\mathbf{i}| < (m - 1)(r - 1)$ . In other words,

$$\left\{ \mathbf{i}_j : j \in \left[ 0, \binom{m+r-1}{m} - 1 \right] \right\} = \mathcal{J}_r,$$

$$\left\{ \mathbf{i}_j : j \in \left[ 0, r^m - \binom{m+r-1}{m} - 1 \right] \right\} = \mathcal{J}_{(m-1)(r-1)} \cap [0, r - 1]^m.$$

**Proof** The first equality is obvious. Now we prove the second one. Consider the map  $\varphi : [0, r - 1]^m \rightarrow [0, r - 1]^m$  defined as

$$\varphi(a_1, \dots, a_m) = (r - 1 - a_1, \dots, r - 1 - a_m).$$

It is straightforward to check that this map is a bijection. Now if  $(a_1, \dots, a_m) \in [0, r - 1]^m$ , then  $\sum_{i=1}^m a_i \geq r$  if, and only if,  $\sum_{i=1}^m (r - 1 - a_i) = m(r - 1) - \sum_{i=1}^m a_i \leq m(r - 1) - r = (m - 1)(r - 1) - 1$ . Hence  $\varphi([0, r - 1]^m \setminus \mathcal{J}_r) = \mathcal{J}_{(m-1)(r-1)} \cap [0, r - 1]^m$ , and in particular,

$$r^m - \binom{m+r-1}{m} = r^m - |\mathcal{J}_r| = |[0, r - 1]^m \setminus \mathcal{J}_r| = |\mathcal{J}_{(m-1)(r-1)} \cap [0, r - 1]^m|,$$

since  $\varphi$  is a bijection. Taking into account that we chose a graded ordering for  $\mathcal{J}_{r-1}$ , we have obtained that the first  $r^m - \binom{m+r-1}{m}$  elements of  $\mathcal{J}_{r-1}$  are the elements of  $\mathcal{J}_{(m-1)(r-1)} \cap [0, r - 1]^m$ , which concludes the proof.  $\square$

We also need to introduce the invertible matrices that give us the equivalence (as in Proposition 2.10) for the dual code.

Recall that  $S = S_1 \times \dots \times S_m$  with  $s_i = |S_i|$  for  $i \in [m]$ . Let  $\{h_{S_j, a_j}\}_{s_j \in S_j}$  be the indicator functions and  $G_j(x) = \prod_{a \in S_j} (x - a)$  the vanishing polynomial of  $S_j$  (see Theorem 3.1 or Appendix A). By Corollary 3.6, the rectangular Hermite interpolation is given by the polynomials

$$\left\{ h_{S, \mathbf{a}, \mathbf{i}}(x) = \prod_{j=1}^m h_{S_j, a_j, i_j}(x_j) \right\}_{\mathbf{a} \in S, \mathbf{i} \in \mathcal{J}_{r-1}},$$

where  $h_{S_j, a_j, i_j}(x_j)$  is defined in Remark 3.3 (and Theorem 3.1) for  $\mathbf{a} = (a_1, \dots, a_m) \in S$  and  $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{J}_{r-1}$ .

**Definition 5.4** For each  $\mathbf{a} \in S$  and  $\mathbf{i} \in \mathcal{J}_{r-1}$ , define

$$\lambda_{\mathbf{i}}^{\mathbf{a}} = \text{Coeff}(h_{S, \mathbf{a}, \mathbf{i}}, x_1^{rs_1-1} \dots x_m^{rs_m-1}) \in \mathbb{F}.$$

We also set  $\lambda_{\mathbf{i}}^{\mathbf{a}} = 0$  if  $\mathbf{i} \in \mathbb{N}^m \setminus \mathcal{J}_{r-1}$  and define the  $r^m \times r^m$  matrix  $M_{\mathbf{a}} = (\lambda_{\mathbf{i}_u + \mathbf{i}_v}^{\mathbf{a}})_{u=0, v=0}^{r^m-1, r^m-1}$ .

We need the following properties of such matrices, which will be used in Theorems 5.6 and 5.10.

**Lemma 5.5**  $M_{\mathbf{a}}$  is symmetric, upper anti-triangular ( $\lambda_{\mathbf{i}_u + \mathbf{i}_v}^{\mathbf{a}} = 0$  if  $u + v > r^m - 1$ ) and every entry in its main anti-diagonal is

$$\lambda_{\mathbf{i}_u + \mathbf{i}_v}^{\mathbf{a}} = \lambda_{\mathbf{r}-\mathbf{1}}^{\mathbf{a}} = \text{Coeff}(h_{S, \mathbf{a}, \mathbf{r}-\mathbf{1}}, x_1^{rs_1-1} \dots x_m^{rs_m-1}) \neq 0,$$

for  $u + v = r^m - 1$  by (1). In particular,  $M_{\mathbf{a}}$  is invertible for all  $\mathbf{a} \in S$ .

**Proof** First, notice that  $u + v = r^m - 1$  is equivalent to  $\mathbf{i}_u + \mathbf{i}_v = \mathbf{r} - \mathbf{1}$  due to the graded lexicographic order in Definition 5.2. For the same reason, if  $u + v > r^m - 1$ , then  $\mathbf{i}_u + \mathbf{i}_v \in \mathbb{N}^m \setminus \mathcal{J}_{r-1}$ , hence  $\lambda_{\mathbf{i}_u + \mathbf{i}_v}^{\mathbf{a}} = 0$  by Definition 5.4.  $\square$

We may now describe the dual of  $\mathcal{M}(S, \mathcal{J}_{r-1}, k)$  as desired.

**Theorem 5.6** Fix a positive integer  $k \leq N$ , where  $N = \sum_{j=1}^m (rs_j - 1)$ . Then

$$\begin{aligned} \mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)^\perp &= \left\{ \left( (g^{(i)}(\mathbf{a}))_{i \in \mathcal{J}_{\mathbf{r}-1}} \cdot M_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in \mathbb{F}[\mathbf{x}]_{\leq N-k} \right\} \\ &= \left\{ \left( (g^{(i)}(\mathbf{a}))_{i \in \mathcal{J}_{\mathbf{r}-1}} \cdot M_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}, \deg_{x_j}(g) \leq rs_j - 1, \forall j \in [m] \right\}, \end{aligned}$$

which is equivalent to  $\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, N - k + 1)$ , where  $\mathcal{J}_{\mathbf{r}-1}$  is ordered according to the graded lexicographic ordering.

**Proof** Let  $f, g \in \mathbb{F}[\mathbf{x}]$  be such that  $\deg(f) \leq k - 1$  and  $\deg(g) \leq N - k$ . Using Definition 5.2, we may see the codewords associated to  $f$  and  $g$  as  $(\mathbf{f}(\mathbf{a}))_{\mathbf{a} \in S}$  and  $(\mathbf{g}(\mathbf{a}))_{\mathbf{a} \in S}$ , respectively, where

$$\begin{aligned} \mathbf{f}(\mathbf{a}) &= (f^{(i_0)}(\mathbf{a}), \dots, f^{(i_{m-1})}(\mathbf{a})), \\ \mathbf{g}(\mathbf{a}) &= (g^{(i_0)}(\mathbf{a}), \dots, g^{(i_{m-1})}(\mathbf{a})), \end{aligned}$$

for  $\mathbf{a} \in S$ . By Corollary 4.8,  $\overline{fg}$  has the same Hasse derivatives as  $fg$ , and moreover  $\overline{fg} \in \langle \Delta_{\prec}(I(S; \mathcal{J}_{\mathbf{r}-1})) \rangle_{\mathbb{F}}$ , i.e.,  $\deg_{x_j}(\overline{fg}) \leq rs_j - 1$ , for all  $j \in [m]$  (see Remark 4.5). Thus, by Corollary 3.6, we have that

$$\overline{fg} = \sum_{\mathbf{a} \in S} \sum_{i \in \mathcal{J}_{\mathbf{r}-1}} (fg)^{(i)}(\mathbf{a}) h_{S, \mathbf{a}, i}. \tag{3}$$

Therefore, we deduce that

$$\begin{aligned} 0 &\stackrel{(a)}{=} \text{Coeff}(\overline{fg}, x_1^{rs_1-1} \dots x_m^{rs_m-1}) \\ &\stackrel{(b)}{=} \text{Coeff} \left( \sum_{\mathbf{a} \in S} \sum_{i \in \mathcal{J}_{\mathbf{r}-1}} (fg)^{(i)}(\mathbf{a}) h_{S, \mathbf{a}, i}, x_1^{rs_1-1} \dots x_m^{rs_m-1} \right) \\ &= \sum_{\mathbf{a} \in S} \sum_{i \in \mathcal{J}_{\mathbf{r}-1}} (fg)^{(i)}(\mathbf{a}) \text{Coeff}(h_{S, \mathbf{a}, i}, x_1^{rs_1-1} \dots x_m^{rs_m-1}) \\ &\stackrel{(c)}{=} \sum_{\mathbf{a} \in S} \sum_{i \in \mathcal{J}_{\mathbf{r}-1}} \left( \sum_{\mathbf{j}+\mathbf{k}=i} f^{(\mathbf{j})}(\mathbf{a}) g^{(\mathbf{k})}(\mathbf{a}) \right) \lambda_{\mathbf{i}}^{\mathbf{a}} \\ &= \sum_{\mathbf{a} \in S} \sum_{i \in \mathcal{J}_{\mathbf{r}-1}} \left( \sum_{\mathbf{j}+\mathbf{k}=i} f^{(\mathbf{j})}(\mathbf{a}) g^{(\mathbf{k})}(\mathbf{a}) \lambda_{\mathbf{j}+\mathbf{k}}^{\mathbf{a}} \right) \\ &= \sum_{\mathbf{a} \in S} \sum_{\mathbf{j} \in \mathcal{J}_{\mathbf{r}-1}} \sum_{\mathbf{k} \in \mathcal{J}_{\mathbf{r}-1}} f^{(\mathbf{j})}(\mathbf{a}) g^{(\mathbf{k})}(\mathbf{a}) \lambda_{\mathbf{j}+\mathbf{k}}^{\mathbf{a}} = \sum_{\mathbf{a} \in S} \mathbf{f}(\mathbf{a}) \cdot M_{\mathbf{a}} \cdot \mathbf{g}(\mathbf{a})^\top, \end{aligned}$$

where we use  $\deg(\overline{fg}) \leq \deg(fg) \leq N - 1 < \sum_{i=1}^m (rs_i - 1)$  (by Corollary 4.8) in (a), Eq. (3) in (b), and Leibniz rule (Proposition 2.2) in (c), together with the definition of  $\lambda_{\mathbf{i}}^{\mathbf{a}}$  and  $M_{\mathbf{a}}$ . Finally, the last expression is simply

$$0 = \sum_{\mathbf{a} \in S} \mathbf{f}(\mathbf{a}) \cdot M_{\mathbf{a}} \cdot \mathbf{g}(\mathbf{a})^\top = \sum_{\mathbf{a} \in S} \mathbf{f}(\mathbf{a}) \cdot (\mathbf{g}(\mathbf{a}) M_{\mathbf{a}})^\top,$$

by the symmetry of  $M_{\mathbf{a}}$ . This means that

$$\left\{ \left( (g^{(i)}(\mathbf{a}))_{i \in \mathcal{J}_{\mathbf{r}-1}} \cdot M_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in \mathbb{F}[\mathbf{x}]_{\leq N-k} \right\} \subseteq \mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)^\perp,$$

and they are equal since both have the same dimension by Theorem 4.9 or by Corollary 4.11 (similar calculation as the classical Reed–Muller codes over  $\mathbb{F}_q^m$  but replacing  $q$  by  $rs$ ).

Finally, for each  $g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}$ , consider its reduced form  $\bar{g} \in \langle \Delta_{<}(I(S; \mathcal{J}_{\mathbf{r}-1}) \rangle_{\mathbb{F}}$  with respect to any graded monomial ordering. By Remark 4.5, we have that  $\deg_{g, x_j}(\bar{g}) \leq rs_j - 1$ , for  $j \in [m]$ . Moreover, by Corollary 4.8, we have that  $\deg(\bar{g}) \leq \deg(g) \leq N - k$  and  $\bar{g}^{(i)}(\mathbf{a}) = g^{(i)}(\mathbf{a})$  for all  $\mathbf{i} \in \mathcal{J}_{\mathbf{r}-1}$  and all  $\mathbf{a} \in S$ . Hence, we conclude the last equality in the proposition.  $\square$

In order to compute the dual of  $\mathcal{M}(S, r, k)$ , we need to see this code as a restriction of  $\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)$ .

**Proposition 5.7** *Let  $\pi_{r,m} : \mathbb{F}^{(r^m)} \rightarrow \mathbb{F}^{\binom{m+r-1}{m}}$  be given by projecting onto the first  $\binom{m+r-1}{m}$  coordinates, and extend it to  $\pi_{r,m} : (\mathbb{F}^{(r^m)})^{|S|} \rightarrow (\mathbb{F}^{\binom{m+r-1}{m}})^{|S|}$  coordinatewise. Then*

$$\mathcal{M}(S, r, k) = \pi_{r,m}(\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)).$$

**Proof** Straightforward using the ordering in Definition 5.2.  $\square$

The following lemma is well known, see [22, Th. 1.5.7]. We will use it in the proof of Theorem 5.10.

**Lemma 5.8** *For any code  $C \subseteq \mathbb{F}^n$  and any  $I \subseteq [n]$ , we have*

$$(\pi_I(C))^\perp = \pi_I(C^\perp \cap \ker(\pi_{[n] \setminus I})),$$

where  $\pi_I : \mathbb{F}^n \rightarrow \mathbb{F}^{|I|}$  is the projection onto the coordinates in  $I$ .

**Definition 5.9** Given the matrix  $M_{\mathbf{a}}$  from Definition 5.4, we define the matrix  $N_{\mathbf{a}}$  as the (invertible) square submatrix of  $M_{\mathbf{a}}$  formed by the entries in its last  $\binom{m+r-1}{m}$  rows and first  $\binom{m+r-1}{m}$  columns.

Therefore, we conclude the following.

**Theorem 5.10** *For positive integers  $r$  and  $k \leq N$ , where  $N = \sum_{j=1}^m (rs_j - 1)$ , define*

$$\mathcal{J}_r^\perp = \mathcal{J}_{(m-1)(r-1)} \cap [0, r - 1]^m = \{\mathbf{i} \in [0, r - 1]^m : |\mathbf{i}| < (m - 1)(r - 1)\}.$$

Then, we have that

$$\begin{aligned} \mathcal{M}(S, r, k)^\perp &= \left\{ \left( (g^{(i)}(\mathbf{a}))_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} \right)_{\mathbf{a} \in S} : \right. \\ &\quad \left. g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}, g^{(i)}(\mathbf{a}) = 0, \forall \mathbf{i} \in \mathcal{J}_r^\perp, \forall \mathbf{a} \in S \right\} \\ &= \left\{ \left( (g^{(i)}(\mathbf{a}))_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} \right)_{\mathbf{a} \in S} : \right. \\ &\quad \left. g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}, \deg_{x_j}(g) \leq rs_j - 1, \forall j \in [m], g^{(i)}(\mathbf{a}) = 0, \forall \mathbf{i} \in \mathcal{J}_r^\perp, \forall \mathbf{a} \in S \right\}, \end{aligned}$$

where  $[0, r - 1]^m \setminus \mathcal{J}_r^\perp$  is ordered according to the graded lexicographic order.

**Proof** Define  $\bar{\pi}_{r,m} : \mathbb{F}^{(r^m)} \rightarrow \mathbb{F}^{(r^m - \binom{m+r-1}{m})}$  as the projection onto the last  $r^m - \binom{m+r-1}{m}$  coordinates and extend it blockwise  $\bar{\pi}_{r,m} : (\mathbb{F}^{(r^m)})^{|S|} \rightarrow (\mathbb{F}^{(r^m - \binom{m+r-1}{m})})^{|S|}$ . We have

$$\mathcal{M}(S, r, k)^\perp \stackrel{(a)}{=} \pi_{r,m}(\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)^\perp) \stackrel{(b)}{=} \pi_{r,m}(\mathcal{M}(S, \mathcal{J}_{\mathbf{r}-1}, k)^\perp \cap \ker(\bar{\pi}_{r,m})),$$

where we use Proposition 5.7 in (a) and Lemma 5.8 in (b). Hence we only need to compute  $\pi_{r,m}(\mathcal{M}(S, \mathcal{J}_{r-1}, k)^\perp \cap \ker(\overline{\pi}_{r,m}))$ .

Take  $g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}$  (respectively,  $\deg_{x_j}(g) \leq rs_j - 1$ , for  $j \in [m]$ ). For each  $\mathbf{a} \in S$ , define the square submatrix  $P_{\mathbf{a}}$  of  $M_{\mathbf{a}}$  formed by the entries in the first  $r^m - \binom{m+r-1}{m}$  rows and last  $r^m - \binom{m+r-1}{m}$  columns. Due to the anti-triangular shape of  $M_{\mathbf{a}}$ , we have that

$$\left( \left( g^{(i_u)}(\mathbf{a}) \right)_{u=0}^{r^m-1} \cdot M_{\mathbf{a}} \right)_{\mathbf{a} \in S} \in \ker(\overline{\pi}_{r,m}) \text{ if and only if } \left( g^{(i_u)}(\mathbf{a}) \right)_{u=0}^{r^m - \binom{m+r-1}{m} - 1} \cdot P_{\mathbf{a}} = \mathbf{0},$$

for all  $\mathbf{a} \in S$ . However,  $P_{\mathbf{a}}$  is also anti-triangular with non-zero entries in its main anti-diagonal, hence it is also invertible, and the last equation is equivalent to  $g^{(i_u)}(\mathbf{a}) = 0$ , for all  $u \in [0, r^m - \binom{m+r-1}{m} - 1]$  and all  $\mathbf{a} \in S$ . Now, if this condition is satisfied, then

$$\pi_{r,m} \left( \left( g^{(i_u)}(\mathbf{a}) \right)_{u=0}^{r^m-1} \cdot M_{\mathbf{a}} \right) = \left( g^{(i_u)}(\mathbf{a}) \right)_{u=r^m - \binom{m+r-1}{m}}^{r^m-1} \cdot N_{\mathbf{a}},$$

for all  $\mathbf{a} \in S$ . To conclude the proof, we notice that, by Lemma 5.3, the sets  $\mathcal{J}_r^\perp$  and  $[0, r-1]^m \setminus \mathcal{J}_r^\perp$  are formed, respectively, by the first  $r^m - \binom{m+r-1}{m}$  elements and last  $\binom{m+r-1}{m}$  elements of  $\mathcal{J}_{r-1}$  according to the graded lexicographic order.  $\square$

**Remark 5.11** By Remark 3.5, if  $r = 2$  and  $S_i = \mathbb{F}_q$  for all  $i \in [m]$ , i.e.,  $S = \mathbb{F}_q^m$ , then for all  $\mathbf{a} \in \mathbb{F}_q^m$ , we may assume that  $M_{\mathbf{a}}$  and  $N_{\mathbf{a}}$  are anti-diagonal matrices whose main anti-diagonals are formed by ones, as in Remark C.4. This only means that the codewords of the dual  $\mathcal{M}(S, r, k)^\perp$  are of the form  $\left( (g^{(i)}(\mathbf{a}))_{i \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \right)_{\mathbf{a} \in S}$ , but where we order the derivatives in  $[0, r-1]^m \setminus \mathcal{J}_r^\perp$  in reversed order than the graded lexicographic order. See also Corollary B.2 and Example C.5.

In the case  $r = 1$ , that is, the case of Reed–Muller codes over the Cartesian product  $S = S_1 \times \dots \times S_m$ , we obtain the following well-known expression for the dual (see [28, Theorem 2.3]). We only need to note that, for  $r = 1$ ,  $r^m = \binom{m+r-1}{m} = 1$ .

**Corollary 5.12** (Cartesian codes) *For a set  $S = S_1 \times \dots \times S_m$ , where  $S_i \subseteq \mathbb{F}$  is of size  $s_i$ , for  $i \in [m]$ , and for positive integers  $k \leq N$ , where  $N = \sum_{j=1}^m (s_j - 1)$ , we have*

$$\begin{aligned} \mathcal{M}(S, 1, k)^\perp &= \{(g(\mathbf{a}) \cdot C_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}\} \\ &= \{(g(\mathbf{a}) \cdot C_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}, \deg_{x_j}(g) \leq s_j - 1, \forall j \in [m]\}, \end{aligned}$$

where  $C_{\mathbf{a}}$  can be explicitly computed in terms of the indicator functions of  $S$ .

Similarly to Proposition C.3, we can regard the dual of a multiplicity code described in Theorem 5.10 as an evaluation code. In the next theorem, we give a basis of the dual of a multiplicity code (i.e., one of its generator matrices, or equivalently, a parity-check matrix of the original multiplicity code).

**Theorem 5.13** *Let  $r$  and  $k \leq N$  be positive integers, where  $N = \sum_{j=1}^m (rs_j - 1)$ . We define the sets*

$$\begin{aligned} \mathcal{M}_{\mathbf{u}}^{N-k} &= \left\{ \mathbf{x}^{\mathbf{i}} : \mathbf{i} \in \prod_{j=1}^m [0, (r - u_j)s_j - 1], |\mathbf{i}| \leq N - k - \sum_{j=1}^m u_j s_j \right\}, \\ A_{\mathbf{u}}^k &= \prod_{j=1}^m G_j(x_j)^{u_j} \cdot \mathcal{M}_{\mathbf{u}}^{N-k}, \end{aligned}$$

for  $\mathbf{u} = (u_1, \dots, u_m) \in \mathcal{B}_{\mathcal{J}_r^\perp}$ , where  $G_j(x_j) = \prod_{\alpha \in S_j} (x_j - \alpha) \in \mathbb{F}[x_j]$  for  $j \in [m]$ .

Setting  $A^k = \bigcup_{\mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}} A_{\mathbf{u}}^k$ , we have

$$\mathcal{M}(S, r, k)^\perp = \left\{ \left( (g^{(\mathbf{i})}(\mathbf{a}))_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in \langle A^k \rangle_{\mathbb{F}} \right\}, \tag{4}$$

where  $[0, r - 1]^m \setminus \mathcal{J}_r^\perp$  is ordered according to the graded lexicographic order. Moreover, for any graded monomial ordering  $\prec$  and any subset  $B^k \subseteq A^k$  with  $\text{in}_\prec(B^k) = \text{in}_\prec(A^k)$  and such that  $\text{in}_\prec(f) \neq \text{in}_\prec(g)$  for any two distinct  $f, g \in B^k$ , then  $|B^k| = \dim(\mathcal{M}(S, r, k)^\perp)$ , and a basis of  $\mathcal{M}(S, r, k)^\perp$  is given by

$$\left\{ \left( (g^{(\mathbf{i})}(\mathbf{a}))_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in B^k \right\}. \tag{5}$$

**Proof** We follow the idea of the proof of Proposition C.3.

First, the inclusion  $\supseteq$  in (4) is straightforward, since any  $g \in \langle A^k \rangle_{\mathbb{F}}$  satisfies that  $\deg(g) \leq N - k$  and  $g^{(\mathbf{i})}(\mathbf{a}) = 0$ , for all  $\mathbf{i} \in \mathcal{J}_r^\perp$  and all  $\mathbf{a} \in S$ , by definition of Hasse derivatives.

We now prove the inclusion  $\subseteq$  in (4). Let  $g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}$  be such that  $\deg_{x_j}(g) \leq rs_j - 1$ , for  $j \in [m]$ , and  $g^{(\mathbf{i})}(\mathbf{a}) = 0$ , for all  $\mathbf{i} \in \mathcal{J}_r^\perp$  and all  $\mathbf{a} \in S$  (recall Theorem 5.10). In other words,  $g \in I(S; \mathcal{J}_r^\perp)$ . Since the polynomials  $\prod_{j=1}^m G_j(x_j)^{u_j}$ , for  $\mathbf{u} = (u_1, \dots, u_m) \in \mathcal{B}_{\mathcal{J}_r^\perp}$ , form a Gröbner basis of  $I(S; \mathcal{J}_r^\perp)$  for the ordering  $\prec$  by Theorem 4.3, then by the division algorithm [10, Th. 2.3.3], we can write

$$g = \sum_{\mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}} \left( f_{\mathbf{u}} \prod_{j=1}^m G_j(x_j)^{u_j} \right),$$

for polynomials  $f_{\mathbf{u}} \in \mathbb{F}[\mathbf{x}]$  such that

$$\begin{aligned} \deg(f_{\mathbf{u}}) + \sum_{j=1}^m u_j s_j &= \deg \left( f_{\mathbf{u}} \prod_{j=1}^m G_j(x_j)^{u_j} \right) \leq \deg(g) \leq N - k, \text{ and} \\ \deg_{x_j}(f_{\mathbf{u}}) + \sum_{j=1}^m u_j s_j &= \deg_{x_j} \left( f_{\mathbf{u}} \prod_{j=1}^m G_j(x_j)^{u_j} \right) \leq \deg_{x_j}(g) \leq rs_j - 1, \end{aligned}$$

for  $j \in [m]$  (recall that  $\prec$  is graded). In other words, the monomials  $\mathbf{x}^{\mathbf{i}}$  appearing in the polynomials  $f_{\mathbf{u}}$  satisfy

$$|\mathbf{i}| \leq N - k - \sum_{j=1}^m u_j s_j \quad \text{and} \quad 0 \leq i_j \leq (r - u_j)s_j - 1,$$

for  $j \in [m]$ . Therefore,  $g \in \langle A^k \rangle_{\mathbb{F}}$  and we have proved Eq. (4). To prove that Eq. (5) gives a basis of  $\mathcal{M}(S, r, k)^\perp$ , we first show that  $|\text{in}_\prec(A^k)| = \dim(\mathcal{M}(S, r, k)^\perp)$ . Define

$$\begin{aligned} T_{\mathbf{u}}^k &= \{ \mathbf{x}^{\mathbf{i}} : |\mathbf{i}| \leq N - k \text{ and } u_j s_j \leq i_j \leq rs_j - 1, \text{ for } j \in [m] \}, \quad \text{for } \mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}, \\ T^k &= \bigcup_{\mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}} T_{\mathbf{u}}^k. \end{aligned}$$

By construction,  $\text{in}_{<}(A^k) = T^k$ , so we only need to show  $|T^k| = \dim(\mathcal{M}(S, r, k)^\perp)$ . Clearly, the map

$$\psi : \Delta_{<}(I(S; \mathcal{J}_{r-1})) \longrightarrow \Delta_{<}(I(S; \mathcal{J}_{r-1}))$$

$$\mathbf{x}^{\mathbf{i}} \longmapsto \mathbf{x}^{(rs_1-1, \dots, rs_m-1)-\mathbf{i}}$$

is a bijection, since  $\Delta_{<}(I(S; \mathcal{J}_{r-1})) = \{\mathbf{x}^{\mathbf{i}} : 0 \leq i_j \leq rs_j - 1, \forall j \in [m]\}$  by Remark 4.5.

We next show that  $\psi$  restricts to a bijection between  $\Delta_{<}(I(S; \mathcal{J}_{r-1})) \setminus \Delta_{<}(I(S; \mathcal{J}_r^\perp))$  and  $\Delta_{<}(I(S; \mathcal{J}_r))$ . Take  $\mathbf{x}^{\mathbf{i}} \in \Delta_{<}(I(S; \mathcal{J}_{r-1})) \setminus \Delta_{<}(I(S; \mathcal{J}_r^\perp))$ . By definition, we have  $\mathbf{x}^{\mathbf{i}} \in \text{in}_{<}(I(S; \mathcal{J}_r^\perp))$ , and therefore there exists  $\mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}$  such that  $\text{in}_{<}(G_1(x_1)^{u_1} \cdots G_m(x_m)^{u_m})$  divides  $\mathbf{x}^{\mathbf{i}}$ , by Theorem 4.3. In other words,  $u_j s_j \leq i_j \leq rs_j - 1$ , for all  $j \in [m]$ . If  $\mathbf{k} = (rs_1 - 1, \dots, rs_m - 1) - \mathbf{i}$ , then  $0 \leq k_j \leq (r - u_j)s_j - 1$ , for all  $j \in [m]$ . Now, since  $\mathbf{u} \in \mathcal{B}_{\mathcal{J}_r^\perp}$ , then by definition,  $\mathbf{u} \in \mathcal{J}_{r-1} \setminus \mathcal{J}_r^\perp$ , and therefore we have that  $\mathbf{r} - \mathbf{u} - \mathbf{1} \in \mathcal{J}_r$ , as in the proof of Lemma 5.3. Assume that  $\mathbf{x}^{\mathbf{k}} \notin \Delta_{<}(I(S; \mathcal{J}_r))$ . Then by definition,  $\mathbf{x}^{\mathbf{k}} \in \text{in}_{<}(I(S; \mathcal{J}_r))$ . Hence there exists  $\mathbf{v} \in \mathcal{B}_{\mathcal{J}_r}$  such that  $\text{in}_{<}(G_1(x_1)^{v_1} \cdots G_m(x_m)^{v_m})$  divides  $\mathbf{x}^{\mathbf{k}}$ , by Theorem 4.3. Thus  $v_j s_j \leq k_j \leq (r - u_j)s_j - 1$ , which implies  $v_j \leq r - u_j - 1$ , for all  $j \in [m]$ . However this cannot happen since  $\mathbf{r} - \mathbf{u} - \mathbf{1} \in \mathcal{J}_r$ ,  $\mathbf{v} \notin \mathcal{J}_r$  (because  $\mathbf{v} \in \mathcal{B}_{\mathcal{J}_r}$ ) and  $\mathcal{J}_r$  is decreasing. Thus we conclude that  $\mathbf{x}^{\mathbf{k}} = \psi(\mathbf{x}^{\mathbf{i}}) \in \Delta_{<}(I(S; \mathcal{J}_r))$ . Finally, we have that

$$|\Delta_{<}(I(S; \mathcal{J}_{r-1}))| = |\mathcal{J}_{r-1}| \cdot |S| = (|\mathcal{J}_r| + |\mathcal{J}_r^\perp|)|S| = |\Delta_{<}(I(S; \mathcal{J}_r))| + |\Delta_{<}(I(S; \mathcal{J}_r^\perp))|,$$

by Eq. (2). Since we have shown that  $\psi(\Delta_{<}(I(S; \mathcal{J}_{r-1})) \setminus \Delta_{<}(I(S; \mathcal{J}_r^\perp))) \subseteq \Delta_{<}(I(S; \mathcal{J}_r))$ , and they both have the same size, we have proven that

$$\psi : \Delta_{<}(I(S; \mathcal{J}_{r-1})) \setminus \Delta_{<}(I(S; \mathcal{J}_r^\perp)) \longrightarrow \Delta_{<}(I(S; \mathcal{J}_r))$$

$$\mathbf{x}^{\mathbf{i}} \longmapsto \mathbf{x}^{(rs_1-1, \dots, rs_m-1)-\mathbf{i}}$$

is a bijection. Similarly to the computations above, we have that  $T^k = \{\mathbf{x}^{\mathbf{i}} \in \Delta_{<}(I(S; \mathcal{J}_{r-1})) \setminus \Delta_{<}(I(S; \mathcal{J}_r^\perp)) : |\mathbf{i}| \leq N - k\}$ , and therefore it is clear that

$$\psi(T^k) = B_{\geq k} = \{\mathbf{x}^{\mathbf{i}} \in \Delta_{<}(I(S; \mathcal{J}_r)) : |\mathbf{i}| \geq k\}.$$

As a consequence, we can finally conclude that

$$|\text{in}_{<}(A^k)| = |T^k| = |B_{\geq k}| = |\mathcal{J}_r| \cdot |S| - \dim(\mathcal{M}(S, r, k)) = \dim(\mathcal{M}(S, r, k)^\perp),$$

where the length of the codes is  $tn = |\mathcal{J}_r| \cdot |S| = |\Delta_{<}(I(S; \mathcal{J}_r))|$  (see Eq. (2)).

Coming back to the set in Eq. (5), we have shown that  $|B^k| = |\text{in}_{<}(A^k)| = \dim(\mathcal{M}(S, r, k)^\perp)$ . Thus, we only need to prove that the vectors

$$\left\{ \left( \left( g^{(\mathbf{i})}(\mathbf{a}) \right)_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} \right)_{\mathbf{a} \in S} : g \in B^k \right\}$$

are  $\mathbb{F}$ -linearly independent. Set  $B^k = \{g_1, \dots, g_M\}$ , with  $M = |B^k|$  and assume that there exist  $\lambda_1, \dots, \lambda_M \in \mathbb{F}$  such that

$$\left( g^{(\mathbf{i})}(\mathbf{a}) \right)_{\mathbf{i} \in [0, r-1]^m \setminus \mathcal{J}_r^\perp} \cdot N_{\mathbf{a}} = \mathbf{0},$$

for all  $\mathbf{a} \in S$ , where  $g = \sum_{i=1}^M \lambda_i g_i$ . Since  $N_{\mathbf{a}}$  is invertible (see Theorem 5.10), then  $g^{(\mathbf{i})}(\mathbf{a}) = 0$ , for all  $\mathbf{a} \in S$  and all  $\mathbf{i} \in [0, r - 1]^m \setminus \mathcal{J}_r^\perp$ . Since the same holds for all  $\mathbf{i} \in \mathcal{J}_r^\perp$ ,

then  $g \in I(S; \mathcal{J}_{r-1})$ . However, if  $g \neq 0$ , then  $\text{in}_{<}(g) = \text{in}_{<}(g_j)$ , for some  $j \in [M]$  (since the initials of  $g_1, \dots, g_M$  are all distinct by hypothesis). This implies that

$$\text{in}_{<}(g) = \text{in}_{<}(g_j) \in \text{in}_{<}(B^k) \subseteq \text{in}_{<}(A^k) \subseteq \Delta_{<}(I(S; \mathcal{J}_{r-1})),$$

which contradicts that  $\text{in}_{<}(g) \in \text{in}_{<}(I(S; \mathcal{J}_{r-1}))$ . Hence,  $g = \sum_{i=1}^M \lambda_i g_i = 0$ , but then it must hold that  $\lambda_1 = \dots = \lambda_M = 0$  since  $g_1, \dots, g_M$  all have distinct initial monomials.  $\square$

To conclude, we observe that we may give the following lower bound on the minimum distance of the dual when  $|S_1| = \dots = |S_m|$ .

**Proposition 5.14** *If  $S = S_1 \times \dots \times S_m$ , where  $S_i \subseteq \mathbb{F}$ ,  $s = |S_1| = \dots = |S_m|$ ,  $n = s^m = |S|$ . Let  $N = \sum_{j=1}^m (r|S_j| - 1) = m(rs - 1)$ ,  $0 \leq k \leq N$  and  $r$  be a positive integer, then*

$$d(\mathcal{M}(S, r, k)^\perp) \geq \left(1 - \frac{m(rs - 1) - k}{rs}\right)n.$$

**Proof** Let  $g \in \mathbb{F}[\mathbf{x}]_{\leq N-k}$  be such that  $g^{(i)}(\mathbf{a}) = 0$ , for all  $\mathbf{i} \in \mathcal{J}_r^\perp$  and all  $\mathbf{a} \in S$ . Now, if  $\mathbf{a} \in S$  is such that  $g^{(i)}(\mathbf{a}) = 0$ , for all  $\mathbf{i} \in [0, r - 1]^m \setminus \mathcal{J}_r^\perp$ , then  $m(g, \mathbf{a}) \geq r$ . Thus we deduce that the codeword associated to  $g$  in  $\mathcal{M}(S, r, k)^\perp$  (see Theorem 5.10) has folded weight at least  $|S| - \frac{(N-k)s^{m-1}}{r}$  by Lemma 2.6, and the result follows.  $\square$

## Appendix A Hermite interpolation for multiplicity 1 and 2

**Multiplicity 1. Lagrange interpolation.** Define the map  $ev: \mathbb{F}[x] \rightarrow \mathbb{F}^s$ ,  $f \mapsto ev(f) = (f(a_1), \dots, f(a_s))$ . For any  $s$  elements  $b_1, \dots, b_s$  in  $\mathbb{F}$ , the classic interpolation asks for a polynomial  $f \in \mathbb{F}[x]$  such that  $\deg(f) < s$  and

$$ev(f) = (b_1, \dots, b_s). \tag{6}$$

Lagrange interpolation easily solves this problem. Indeed, let  $G(x) = \prod_{a \in T} (x - a)$  be the vanishing polynomial of  $T$ . For  $i \in [s]$ , the polynomial  $h_i(x) = \frac{G(x)}{G^{(1)}(a_i)(x - a_i)}$  satisfies  $\deg(h_i) = s - 1$  and  $h_i(a_j) = \delta_{i,j}$ . Thus,

$$f(x) = \sum_{i=1}^s h_i(x)b_i = (h_1(x), \dots, h_s(x)) \cdot (b_1, \dots, b_s)$$

satisfies Eq. (6) and  $\deg(f) < s$ . We call the polynomials  $h_1, \dots, h_s \in \mathbb{F}[x]$  the **indicator functions** of  $T$ .

**Multiplicity 2.** Let  $b_1, b'_1, \dots, b_s, b'_s$  be any  $2s$  elements in  $\mathbb{F}$ . Define the map

$$ev^{(1)}: \mathbb{F}[x] \rightarrow \mathbb{F}^{2s}$$

$$f \mapsto ev^{(1)}(f) = (f(a_1), f^{(1)}(a_1), \dots, f(a_s), f^{(1)}(a_s)).$$

For easy notation, we are omitting the inner parentheses. The goal of the Hermite interpolation of order 1 is to find a polynomial  $f \in \mathbb{F}[x]$  such that  $\deg(f) \leq 2s - 1$  and

$$ev^{(1)}(f) = (b_1, b'_1, \dots, b_s, b'_s). \tag{7}$$

Lagrange interpolation (order 0) relies on  $h_1, \dots, h_s$ . To find  $f$  that satisfies Eq. (7), we now look for  $2s$  polynomials  $h_{1,0}, h_{1,1}, \dots, h_{s,0}, h_{s,1}$  of degree  $\leq 2s - 1$  whose images under the map  $ev^{(1)}$  are the standard vectors.

$$\begin{aligned} ev^{(1)}(h_{1,0}) &= (1, 0, \dots, 0, 0) \\ ev^{(1)}(h_{1,1}) &= (0, 1, \dots, 0, 0) \\ &\vdots \\ ev^{(1)}(h_{s,0}) &= (0, 0, \dots, 1, 0) \\ ev^{(1)}(h_{s,1}) &= (0, 0, \dots, 0, 1). \end{aligned}$$

In other words,  $h_{i,0}(a_j)$  is the Kronecker delta, but its derivative is 0 on every element of  $T$ . The polynomial  $h_{i,1}(a_j)$  is 0 on every element of  $T$ , but its derivative is the Kronecker delta.

The following result shows how to find  $h_{i,1}(x)$  and  $h_{i,0}(x)$ .

**Proposition A.1** (Single-variable Hermite interpolation of multiplicity 2) Let  $G(x)$  be the vanishing polynomial and  $h_1, \dots, h_s$  the indicator functions of  $T$ . There exists a unique polynomial  $f \in \mathbb{F}[x]_{\leq 2s-1}$  such that

$$ev^{(1)}(f) = (b_1, b'_1, \dots, b_s, b'_s).$$

Moreover,  $f = (h_{1,0}, h_{1,1}, \dots, h_{s,0}, h_{s,1}) \cdot (b_1, b'_1, \dots, b_s, b'_s)$ , where

$$(h_{i,0}(x)h_{i,1}(x)) = \left(1 - h_i^{2(1)}(a_i) 0 \ 1\right) (1x - a_i) h_i^2(x),$$

for  $i \in [s]$ , and  $h_i^{2(1)}(a_i)$  is the evaluation of the derivative of  $h_i^2(x)$  at  $x = a_i$ .

**Proof** By definition, we can see that

$$h_{i,1}(x) = (x - a_i)h_i^2(x) \quad \text{and} \tag{8}$$

$$h_{i,0}(x) = h_i^2(x) - h_i^{2(1)}(a_i)h_{i,1}(x). \tag{9}$$

We have that  $h_{i,0}(a_j) = \delta_{i,j}$  and  $h_{i,1}(a_j) = 0$  for  $j \in [s]$ . By taking derivatives and using Proposition 2.2, we obtain

$$\begin{aligned} h_{i,1}^{(1)}(x) &= (x - a_i)^{(1)}h_i^2(x) + (x - a_i)h_i^{2(1)}(x) \\ &= h_i^2(x) + (x - a_i)2h_i(x)h_i^{(1)}(x) \\ h_{i,0}^{(1)}(x) &= h_i^{2(1)}(x) - h_i^{2(1)}(a_i)h_{i,1}^{(1)}(x). \end{aligned}$$

It is straightforward to see that  $h_{i,1}^{(1)}(a_j) = \delta_{i,j}$  and  $h_{i,0}^{(1)}(a_j) = 0$  for  $j \in [s]$ . Note the polynomial

$$f = (h_{1,0}, h_{1,1}, \dots, h_{s,0}, h_{s,1}) \cdot (b_1, b'_1, \dots, b_s, b'_s)$$

satisfies Eq. (7) and has degree  $\leq 2s - 1$ . The uniqueness of  $f \in \mathbb{F}[x]_{\leq 2s-1}$  is due to the fact that  $f$  is the solution of a system of  $2s$  equations and  $2s$  indeterminates.  $\square$

**Remark A.2** We have

$$(h_{i,0}(x)h_{i,1}(x)) = \left(1 - 2\frac{G^{(2)}(a_i)}{G^{(1)}(a_i)} 0 \ 1\right) (1x - a_i) h_i^2(x).$$

Indeed, by Proposition A.1, we only need to compute  $h_i^{2(1)}(a_i)$ . Note  $(x - a_i)h_i(x) = \frac{G(x)}{G^{(1)}(a_i)}$ . By Proposition 2.2, we have

$$(x - a_i)h_i^{(2)}(x) + h_i^{(1)}(x) = \frac{G^{(2)}(x)}{G^{(1)}(a_i)} \quad \text{and} \quad h_i^{2(1)}(x) = 2h_i(x)h_i^{(1)}(x).$$

Thus,

$$h_i^{2(1)}(a_i) = 2h_i(a_i)h_i^{(1)}(a_i) = 2h_i^{(1)}(a_i) = 2\frac{G^{(2)}(a_i)}{G^{(1)}(a_i)}.$$

### Appendix B Dual in the case $m = 1$

In this appendix, we give an explicit expression for the dual of single-variable multiplicity codes. Even though this case was solved in [33], we include it for convenience of the reader. This case is significantly simpler since  $\mathcal{J}_{r-1} = \mathcal{J}_r$  (where  $\mathbf{r} = r\mathbf{1}$ ) in  $m = 1$  variable.

Recall  $S = \{a_1, \dots, a_s\} \subseteq \mathbb{F}$ ,  $G(x) = \prod_{a \in S}(x - a)$  is its vanishing polynomial, and  $\{h_i(x)\}_{i=1}^s$  are the indicator functions of  $S$ . We denote by  $h_i^{r+1(j)}(x)$  the  $j$ th derivative of  $h_i^{r+1}(x)$ . We will assume in this appendix that  $k < s$ .

**Proposition B.1** *The dual of*

$$\mathcal{M}(S, 2, k) = \left\{ \left( (f(a_1), f^{(1)}(a_1)), \dots, (f(a_s), f^{(1)}(a_s)) \right) : f \in \mathbb{F}_q[x]_{<k} \right\}$$

is given by

$$\mathcal{M}(S, 2, k)^\perp = \left\{ \left( (g(a_1), g^{(1)}(a_1))M_1, \dots, (g(a_s), g^{(1)}(a_s))M_s \right) : g \in \mathbb{F}_q[x]_{<2s-k} \right\},$$

where  $M_i = (G^{(1)}(a_i))^{-3} (-2G^{(2)}(a_i) G^{(1)}(a_i)G^{(1)}(a_i) 0)$  (which is invertible since  $G^{(1)}(a_i) \neq 0$ , for all  $i \in [s]$ ).

**Proof** Take  $f \in \mathbb{F}_q[x]_{<k}$  and  $g \in \mathbb{F}_q[x]_{<2s-k}$ . By Propositions 2.2 and A.1,

$$\begin{aligned} fg &= \sum_{i=1}^s \left( (fg)(a_i), (fg)^{(1)}(a_i) \right) \cdot (h_{i,0}(x), h_{i,1}(x)) \\ &= \sum_{i=1}^s \left( (fg)(a_i), (f^{(1)}g)(a_i) + (fg^{(1)})(a_i) \right) \cdot (h_{i,0}(x), h_{i,1}(x)) \\ &= \sum_{i=1}^s \left( f(a_i), f^{(1)}(a_i) \right) \cdot \left( g(a_i)h_{i,0}(x) + g^{(1)}(a_i)h_{i,1}(x), g(a_i)h_{i,1}(x) \right) \\ &= \sum_{i=1}^s \left( f(a_i), f^{(1)}(a_i) \right) \begin{pmatrix} h_{i,0}(x) & h_{i,1}(x) \\ h_{i,1}(x) & 0 \end{pmatrix} \begin{pmatrix} g(a_i) \\ g^{(1)}(a_i) \end{pmatrix}, \end{aligned}$$

where, by Proposition A.1,

$$(h_{i,0}(x)h_{i,1}(x)) = \begin{pmatrix} 1 & -h_i^{2(1)}(a_i)0 & 1 \end{pmatrix} (1x - a_i) h_i^2(x).$$

We also have that  $\text{Coeff}(h_{i,0}(x), x^{2s-1}) = -h_i^{2(1)}(a_i) (G^{(1)}(a_i))^{-2}$  and  $\text{Coeff}(h_{i,1}(x), x^{2s-1}) = (G^{(1)}(a_i))^{-2}$ . Finally, as  $\text{deg}(fg) \leq 2s - 2$  and  $\text{deg}(h_{i,0}) = \text{deg}(h_{i,1}) = 2s - 1$ , the addition of the coefficients of  $fg$  in  $x^{2s-1}$  is 0. Therefore,

$$\sum_{i=1}^s (f(a_i), f^{(1)}(a_i)) (G^{(1)}(a_i))^{-2} (-h_i^{2(1)}(a_i) \ 11 \ 0) (g(a_i)g^{(1)}(a_i)) = 0.$$

Thus, the result follows using Remark A.2 in order to express  $h_i^{2(1)}(a_i)$  in terms of  $G(x)$ .  $\square$

**Corollary B.2** Assume  $\mathbb{F}_q = \{a_1, \dots, a_q\}$ . The dual of

$$\mathcal{M}(\mathbb{F}_q, 2, k) = \left\{ (f(a_1), f^{(1)}(a_1), \dots, f(a_q), f^{(1)}(a_q)) : f \in \mathbb{F}_q[x]_{<k} \right\}$$

is given by

$$\mathcal{M}(\mathbb{F}_q, 2, k)^\perp = \left\{ (g^{(1)}(a_1), g(a_1), \dots, g^{(1)}(a_q), g(a_q)) : g \in \mathbb{F}_q[x]_{<2q-k} \right\}.$$

**Proof** The vanishing polynomial of  $\mathbb{F}_q$  is  $G(x) = x^q - x$ . Then,  $G^{(1)}(x) = -1$  and  $2G^{(2)}(x) = 0$  (independently of  $q$ ). Thus, the result follows from Remark A.2 and Proposition B.1.  $\square$

**Proposition B.3** The dual of

$$\mathcal{M}(S, r, k) = \left\{ ((f(a_1), \dots, f^{(r-1)}(a_1)), \dots, (f(a_s), \dots, f^{(r-1)}(a_s))) : f \in \mathbb{F}_q[x]_{<rs-k} \right\}$$

is given by

$$\mathcal{M}(S, r, k)^\perp = \left\{ ((g(a_1), \dots, g^{(r-1)}(a_1))M_1, \dots, (g(a_s), \dots, g^{(r-1)}(a_s))M_s) : g \in \mathbb{F}_q[x]_{<rs-k} \right\},$$

where  $M_i$  is the invertible matrix given by

$$M_i = \begin{pmatrix} \lambda_{i,0} & \lambda_{i,1} & \cdots & \lambda_{i,r-2} & \lambda_{i,r-1} \\ \lambda_{i,1} & \lambda_{i,2} & \cdots & \lambda_{i,r-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_{i,r-2} & \lambda_{i,r-1} & \cdots & 0 & 0 \\ \lambda_{i,r-1} & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where  $\lambda_{i,r-1} = \text{Coeff}(h_i^r(x), x^{rs-1}) = (G^{(1)}(a_i))^{-r}$  and, recursively in  $n = r - 2, r - 3, \dots, 1, 0$ , (in decreasing order), we define  $\lambda_{i,n} = -\sum_{k=n+1}^{r-1} h_i^{r(k-n)}(a_i)\lambda_{i,k}$ .

**Proof** Take  $f \in \mathbb{F}_q[x]_{<k}$  and  $g \in \mathbb{F}_q[x]_{<rs-k}$ . By Proposition 2.2 and Theorem 3.1,

$$\begin{aligned} fg &= \sum_{i=1}^s ((fg)(a_i), \dots, (fg)^{(r-1)}(a_i)) \cdot (h_{i,0}(x), \dots, h_{i,r-1}(x)) \\ &= \sum_{i=1}^s ((fg)(a_i), \dots, ((fg^{(r-1)})(a_i) + \cdots + (f^{(r-1)}g)(a_i)) \cdot (h_{i,0}(x), \dots, h_{i,r-1}(x)) \\ &= \sum_{i=1}^s (f(a_i), \dots, f^{(r-1)}(a_i)) \begin{pmatrix} h_{i,0}(x) & h_{i,1}(x) & \cdots & h_{i,r-1}(x) \\ h_{i,1}(x) & h_{i,2}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ h_{i,r-1}(x) & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} g(a_i) \\ g^{(1)}(a_i) \\ \vdots \\ g^{(r-1)}(a_i) \end{pmatrix}, \end{aligned}$$

where  $h_{i,0}(x), \dots, h_{i,r-1}(x)$  are given in Theorem 3.1. Since  $\deg(fg) \leq rs - 2$ , the addition of the coefficients of  $fg$  in  $x^{rs-1}$  is 0. Therefore,

$$\sum_{i=1}^s \left( f(a_i), \dots, f^{(r-1)}(a_i) \right) M_i \left( g(a_i), \dots, g^{(r-1)}(a_i) \right)^T = 0,$$

where  $M_i$  is given in the proposition statement, given that  $\text{Coeff}(h_{i,n}(x), x^{rs-1}) = \lambda_{i,n}$ , for all  $n \in [0, r - 1]$  and  $i \in [s]$ . By a dimensional argument, we get the result.  $\square$

**Example B.4** By Proposition B.3, and using the Coding Theory Package [3] of *Macaulay2* [17], or *Magma* [7], we obtain that the dual of

$$\mathcal{M}(\mathbb{F}_2, 3, 3) = \left\{ \left( f(0), f^{(1)}(0), f^{(2)}(0), f(1), f^{(1)}(1), f^{(2)}(1) \right) : f \in \mathbb{F}_2[x]_{<3} \right\}$$

is given by

$$\left\{ \left( (g^{(2)} + g^{(1)})(0), (g^{(1)} + g)(0), g(0), (g^{(2)} + g^{(1)})(1), (g^{(1)} + g)(1), g(1) \right) : g \in \mathbb{F}_2[x]_{<3} \right\}.$$

In contrast with the case of multiplicity  $r = 1$  (see Corollary B.2), in this case of multiplicity  $r = 2$ , the codewords of the form

$$\left( g^{(2)}(0), g^{(1)}(0), g(0), g^{(2)}(1), g^{(1)}(1), g(1) \right),$$

for  $g \in \mathbb{F}_2[x]_{<3}$ , are not all in the dual  $\mathcal{M}(\mathbb{F}_2, 3, 3)^\perp$  even for  $S = \mathbb{F}_2$ , the whole field. For instance, if  $g = x^2$ , then

$$\left( g^{(2)}(0), g^{(1)}(0), g(0), g^{(2)}(1), g^{(1)}(1), g(1) \right) = (1, 0, 0, 1, 0, 1) \notin \mathcal{M}(\mathbb{F}_2, 3, 3)^\perp,$$

since this word is not orthogonal to

$$\left( f(0), f^{(1)}(0), f^{(2)}(0), f(1), f^{(1)}(1), f^{(2)}(1) \right) = (0, 1, 0, 1, 1, 0) \in \mathcal{M}(\mathbb{F}_2, 3, 3),$$

for  $f = x$ . This is because, for  $r = 2$ , it is not true that the matrix  $M_i$  from Proposition B.3 is anti-diagonal with ones in its main anti-diagonal, even if  $S = \mathbb{F}_q$  is the whole field (whereas this holds for  $r = 1$  by Corollary B.2, see also Remark C.4, Example C.5 and Remark 5.11).

### Appendix C Dual in the case $m = r = 2$

In this appendix,  $S = S_1 \times S_2$  represents a Cartesian product where every  $S_i \subseteq \mathbb{F}$  is finite. In this appendix, we compute the dual of multiplicity codes  $\mathcal{M}(S, 2, k)$  for  $m = r = 2$  and  $k \leq N = (2s_1 - 1) + (2s_2 - 1)$  (recall that for larger  $k$  such codes become the total space, see Theorem 4.9). For simplicity, we use the notation

$$\partial_{x^i y^j} f = f^{(i,j)}(x, y)$$

for the  $(i, j)$ th Hasse derivative of the bivariate polynomial  $f = f(x, y) \in \mathbb{F}[x, y]$ . We may order the Hasse derivatives of order  $(i, j) \leq (1, 1)$  of  $f$  in  $\mathbf{a} \in S$  as

$$\left( f(\mathbf{a}), \partial_x f(\mathbf{a}), \partial_y f(\mathbf{a}), \partial_{xy} f(\mathbf{a}) \right) \in \mathbb{F}^4,$$

i.e., according to the graded lexicographic order with  $x < y$ . Thus, the derivatives of total order  $i + j < 2$  are the first three elements, i.e.,

$$\left( f(\mathbf{a}), \partial_x f(\mathbf{a}), \partial_y f(\mathbf{a}) \right) \in \mathbb{F}^3.$$

Let  $\pi : \mathbb{F}^4 \rightarrow \mathbb{F}^3$  and  $\bar{\pi} : \mathbb{F}^4 \rightarrow \mathbb{F}$  be the projections onto the first 3 coordinates and the last 1 coordinate, respectively, and extend them coordinatewise to  $\pi : (\mathbb{F}^4)^{|S|} \rightarrow (\mathbb{F}^3)^{|S|}$  and  $\bar{\pi} : (\mathbb{F}^4)^{|S|} \rightarrow (\mathbb{F}^1)^{|S|}$ , respectively. Then,

$$\mathcal{M}(S, 2, k) = \pi(\mathcal{M}(S, \mathcal{J}_{(1,1)}, k)).$$

In particular, by [22, Th. 1.5.7], we have

$$\mathcal{M}(S, 2, k)^\perp = \pi(\mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp) = \pi(\mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp \cap \ker(\bar{\pi})), \tag{10}$$

which is usually known as a shortened code. Hence, we first compute the dual  $\mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp$ .

Similarly to the evaluation function, given  $f \in \mathbb{F}[x, y]$  and  $\mathbf{a} \in S$ , we define the element  $\mathbf{f}(\mathbf{a}) = (f(\mathbf{a}), \partial_x f(\mathbf{a}), \partial_y f(\mathbf{a}), \partial_{xy} f(\mathbf{a})) \in \mathbb{F}^4$ .

**Proposition C.1** *Let  $h_1, \dots, h_{s_1}$  (resp.  $\ell_1, \dots, \ell_{s_2}$ ) be the indicator functions and  $G_1$  (resp.  $G_2$ ) the vanishing polynomials of  $S_1$  (resp.  $S_2$ ). The dual of  $\mathcal{M}(S, \mathcal{J}_{(1,1)}, k) = \{(\mathbf{f}(\mathbf{a}))_{\mathbf{a} \in S} : f \in \mathbb{F}[x, y]_{\leq k}\}$  is*

$$\begin{aligned} \mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp &= \{(\mathbf{g}(\mathbf{a})M_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[x, y]_{\leq N-k}\} \\ &= \{(\mathbf{g}(\mathbf{a})M_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[x, y]_{\leq N-k}, \deg_x(g) \leq 2s_1 - 1, \deg_y(g) \leq 2s_2 - 1\}, \end{aligned}$$

where  $M_{\mathbf{a}} = (G_1^{(1)}(a_i)G_2^{(1)}(a_j))^{-2} \begin{pmatrix} -\ell_j^{2(1)}(a_j) & 11 & 0 \end{pmatrix} \otimes \begin{pmatrix} -h_i^{2(1)}(a_i) & 11 & 0 \end{pmatrix}$  for  $\mathbf{a} = (a_i, a_j) \in S$ . In particular, we have that  $\mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp$  is equivalent to  $\mathcal{M}(S, \mathcal{J}_{(1,1)}, N - k + 1)$ .

**Proof** Let  $f, g \in \mathbb{F}[x, y]$  be such that  $\deg(f) \leq k - 1$  and  $\deg(g) \leq N - k$ . By Corollary 4.8,  $\overline{fg}$  has the same Hasse derivatives as  $fg$ . Therefore, using the Leibniz rule (Proposition 2.2) and the interpolation of Corollary 3.6, we get

$$\begin{aligned} \overline{fg} &= \sum_{\mathbf{a} \in S} \mathbf{f}(\mathbf{a}) \cdot (h_{\mathbf{a},(0,0)}, h_{\mathbf{a},(1,0)}, h_{\mathbf{a},(0,1)}, h_{\mathbf{a},(1,1)}) \\ &= \sum_{\mathbf{a} \in S} ((fg)(\mathbf{a})h_{\mathbf{a},(0,0)} + \partial_x(fg)(\mathbf{a})h_{\mathbf{a},(1,0)} + \partial_y(fg)(\mathbf{a})h_{\mathbf{a},(0,1)} + \partial_{xy}(fg)(\mathbf{a})h_{\mathbf{a},(1,1)}). \\ &= \sum_{\mathbf{a} \in S} (f(\mathbf{a})g(\mathbf{a})h_{\mathbf{a},(0,0)} + (\partial_x f(\mathbf{a})g(\mathbf{a}) + f(\mathbf{a})\partial_x g(\mathbf{a}))h_{\mathbf{a},(1,0)} + (\partial_y f(\mathbf{a})g(\mathbf{a}) \\ &\quad + f(\mathbf{a})\partial_y g(\mathbf{a}))h_{\mathbf{a},(0,1)} + (\partial_{xy} f(\mathbf{a})g(\mathbf{a}) + \partial_x f(\mathbf{a})\partial_y g(\mathbf{a}) \\ &\quad + \partial_y f(\mathbf{a})\partial_x g(\mathbf{a}) + f(\mathbf{a})\partial_{xy} g(\mathbf{a}))h_{\mathbf{a},(1,1)}) \\ &= \sum_{\mathbf{a} \in S} (f(\mathbf{a}), \partial_x f(\mathbf{a}), \partial_y f(\mathbf{a}), \partial_{xy} f(\mathbf{a})) \begin{pmatrix} h_{\mathbf{a},(0,0)} & h_{\mathbf{a},(1,0)} & h_{\mathbf{a},(0,1)} & h_{\mathbf{a},(1,1)} \\ h_{\mathbf{a},(1,0)} & 0 & h_{\mathbf{a},(1,1)} & 0 \\ h_{\mathbf{a},(0,1)} & h_{\mathbf{a},(1,1)} & 0 & 0 \\ h_{\mathbf{a},(1,1)} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} g(\mathbf{a}) \\ \partial_x g(\mathbf{a}) \\ \partial_y g(\mathbf{a}) \\ \partial_{xy} g(\mathbf{a}) \end{pmatrix} \\ &= \sum_{\mathbf{a}=(a_i, a_j) \in S} \mathbf{f}(\mathbf{a}) \begin{pmatrix} h_{i,0}\ell_{j,0} & h_{i,1}\ell_{j,0} & h_{i,0}\ell_{j,1} & h_{i,1}\ell_{j,1} \\ h_{i,1}\ell_{j,0} & 0 & h_{i,1}\ell_{j,1} & 0 \\ h_{i,0}\ell_{j,1} & h_{i,1}\ell_{j,1} & 0 & 0 \\ h_{i,1}\ell_{j,1} & 0 & 0 & 0 \end{pmatrix} \mathbf{g}(\mathbf{a})^\top \quad (\text{by Corollary 3.6}) \\ &= \sum_{\mathbf{a}=(a_i, a_j) \in S} \mathbf{f}(\mathbf{a}) \begin{pmatrix} \ell_{j,0} & \ell_{j,1} \\ \ell_{j,1} & 0 \end{pmatrix} \otimes \begin{pmatrix} h_{i,0} & h_{i,1} \\ h_{i,1} & 0 \end{pmatrix} \mathbf{g}(\mathbf{a})^\top. \end{aligned}$$

Note  $\overline{fg} \in \langle \Delta_{\prec}(I(S; \mathcal{J}_{(1,1)})) \rangle_{\mathbb{F}}$ , i.e.,  $\deg_x(\overline{fg}) \leq 2s_1 - 1$  and  $\deg_y(\overline{fg}) \leq 2s_2 - 1$  (see Remark 4.5). By Corollary 4.8,  $\deg(\overline{fg}) \leq \deg(fg) \leq N - 1 < (2s_1 - 1) + (2s_2 - 1)$ . Thus,  $\text{Coeff}(\overline{fg}, x_1^{2s_1-1} x_2^{2s_2-1}) = 0$ . Using Proposition B.3 to compute the coefficients of  $h_{i,0}, h_{i,1}$  and  $\ell_{j,0}, \ell_{j,1}$  in  $x_1^{2s_1-1}$  and  $x_2^{2s_2-1}$ , respectively, we get  $0 = \sum_{\mathbf{a} \in S} \mathbf{f}(\mathbf{a}) M_{\mathbf{a}} \mathbf{g}(\mathbf{a})^T$ , where  $M_{\mathbf{a}}$  is given in the proposition statement for  $\mathbf{a} = (a_i, a_j) \in S$ . By the symmetry of  $M_{\mathbf{a}}$ , this means that

$$\left\{ ((g(\mathbf{a}), \partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) M_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[x, y]_{\leq N-k} \right\} \subseteq \mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp,$$

and they are equal since both have the same dimension by Theorem 4.9 or by Corollary 4.11 (it is the same calculation as with classical Reed–Muller codes over the whole space  $\mathbb{F}_q^2$  but replacing  $q$  by  $2s$ ).

Finally, for each  $g \in \mathbb{F}[x, y]_{\leq N-k}$ , consider its reduced form  $\overline{g} \in \langle \Delta_{\prec}(I(S; \mathcal{J}_{(1,1)})) \rangle_{\mathbb{F}}$  with respect to any graded monomial ordering. By Remark 4.5, we have that  $\deg_x(\overline{g}) \leq 2s_1 - 1$  and  $\deg_y(\overline{g}) \leq 2s_2 - 1$ . By Corollary 4.8,  $\deg(\overline{g}) \leq \deg(g) \leq N - k$  and  $\partial_{x^i y^j} \overline{g}(\mathbf{a}) = \partial_{x^i y^j} g(\mathbf{a})$  for all  $i, j \in \{0, 1\}$  and all  $\mathbf{a} \in S$ . Hence, we conclude the second equality for the dual is valid. □

We now compute the dual of classical multiplicity codes for  $m = r = 2$  by combining Proposition C.1 and Eq. (10).

**Corollary C.2** *Let  $h_1, \dots, h_{s_1}$  (resp.  $\ell_1, \dots, \ell_{s_2}$ ) be the indicator functions and  $G_1$  (resp.  $G_2$ ) the vanishing polynomials of  $S_1$  (resp.  $S_2$ ). The dual of  $\mathcal{M}(S, 2, k)$  is given by*

$$\begin{aligned} \mathcal{M}(S, 2, k)^\perp &= \{ ((\partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) N_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \mathbb{F}[x, y]_{\leq N-k}, g(\mathbf{a}) = 0, \forall \mathbf{a} \in S \} \\ &= \{ ((\partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) N_{\mathbf{a}})_{\mathbf{a} \in S} : \\ &\quad g \in \mathbb{F}[x, y]_{\leq N-k}, \deg_x(g) \leq 2s_1 - 1, \deg_y(g) \leq 2s_2 - 1, g(\mathbf{a}) = 0, \forall \mathbf{a} \in S \}, \end{aligned}$$

where  $N_{\mathbf{a}} = \left( G_1^{(1)}(a_i) G_2^{(1)}(a_j) \right)^{-2} \begin{pmatrix} -\ell_j^{2(1)}(a_j) & 0 & 1 - h_j^{2(1)}(a_i) & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$  for  $\mathbf{a} = (a_i, a_j) \in S$ .

**Proof** By Eq. (10), we need to compute  $\mathcal{C} = \mathcal{M}(S, \mathcal{J}_{(1,1)}, k)^\perp \cap \ker(\overline{\pi})$  and project it onto the first 3 coordinates in each block of 4 coordinates. By Proposition C.1,  $\mathcal{C}$  is formed by the codewords

$$((g(\mathbf{a}), \partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) M_{\mathbf{a}})_{\mathbf{a} \in S}, \tag{11}$$

for  $g \in \mathbb{F}[x, y]_{\leq N-k}$  (respectively,  $\deg_x(g) \leq 2s_1 - 1$  and  $\deg_y(g) \leq 2s_2 - 1$ ), where the last component in every block of 4 coordinates is zero. That is, where  $g(\mathbf{a}) = 0$  for all  $\mathbf{a} \in S$  (by the anti-triangular shape of  $M_{\mathbf{a}}$ ). Hence, the codewords in (11) have the form

$$((0, \partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) M_{\mathbf{a}})_{\mathbf{a} \in S} = (((\partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) N_{\mathbf{a}}, 0))_{\mathbf{a} \in S},$$

for  $g \in \mathbb{F}[x, y]_{\leq N-k}$  (respectively,  $\deg_x(g) \leq 2s_1 - 1$  and  $\deg_y(g) \leq 2s_2 - 1$ ) and  $g(\mathbf{a}) = 0$  for all  $\mathbf{a} \in S$ . We obtain the result by projecting onto the first 3 coordinates in each block. □

In Corollary C.2, we determined the dual of a multiplicity code as a vector space with certain equations, that is, it is determined implicitly. In the following result, we show how to regard this dual also as an evaluation code, and we explicitly compute one of its bases (i.e., one of its generator matrices or, equivalently, a parity-check matrix of the original multiplicity code).

**Proposition C.3** *Using the notation from Theorem 4.3, let*

$$A_1 = G_1(x) \cdot \{x^\alpha y^\beta : 0 \leq \alpha < s_1, 0 \leq \beta < 2s_2, \alpha + \beta \leq N - k - s_1\},$$

$$A_2 = G_2(y) \cdot \{x^\alpha y^\beta : 0 \leq \alpha < 2s_1, 0 \leq \beta < s_2, \alpha + \beta \leq N - k - s_2\}.$$

Then,

$$\mathcal{M}(S, 2, k)^\perp = \{((\partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) \cdot N_{\mathbf{a}})_{\mathbf{a} \in S} : g \in \langle A_1 \cup A_2 \rangle_{\mathbb{F}}\}.$$

Finally, let  $\prec$  be a graded monomial ordering and let  $A \subseteq A_1 \cup A_2$  be such that its initial monomials with respect to  $\prec$  are all different and are exactly those in

$$T = \{x^\alpha y^\beta : \alpha < 2s_1, \beta < 2s_2, \alpha + \beta \leq N - k, \text{ and } (s_1 \leq \alpha \text{ or } s_2 \leq \beta)\}.$$

Then a basis of  $\mathcal{M}(S, 2, k)^\perp$  is given by

$$\{((\partial_x g(\mathbf{a}), \partial_y g(\mathbf{a}), \partial_{xy} g(\mathbf{a})) \cdot N_{\mathbf{a}})_{\mathbf{a} \in S} : g \in A\}.$$

**Proof** Let  $g \in \mathbb{F}[x, y]_{\leq N-k}$  be such that  $\deg_x(g) \leq 2s_1 - 1$ ,  $\deg_y(g) \leq 2s_2 - 1$ , and  $g(\mathbf{a}) = 0$ , for all  $\mathbf{a} \in S$ . By definition, we have  $g \in I(S; \mathcal{J}_1)$ . Since  $\{G_1(x), G_2(y)\}$  forms a Gröbner basis of  $I(S; \mathcal{J}_1)$  with respect to  $\prec$  (Theorem 4.3), then by the division algorithm [10, Th. 2.3.3], we can write

$$g(x, y) = f_1(x, y)G_1(x) + f_2(x, y)G_2(y),$$

where  $\deg_x(f_i G_i) < 2s_1$ ,  $\deg_y(f_i G_i) < 2s_2$  and  $\deg(f_i G_i) \leq N - k$ , for  $i = 1, 2$ . Hence  $g \in \langle A_1 \cup A_2 \rangle_{\mathbb{F}}$ , and the first part on the generators follows.

Next, since  $\text{in}_\prec(G_1(x)) = x^{s_1}$  and  $\text{in}_\prec(G_2(y)) = y^{s_2}$ , the set of initial monomials of all the polynomials in  $A_1 \cup A_2$  is precisely  $T$ , hence we may choose  $A \subseteq A_1 \cup A_2$  as in the proposition (notice that  $\text{in}_\prec(A_1) \cap \text{in}_\prec(A_2) \neq \emptyset$  thus necessarily  $A \subsetneq A_1 \cup A_2$ ).

Since the initial monomials of the polynomials in  $A$  are all distinct, then such polynomials are  $\mathbb{F}$ -linearly independent, and by Lemma 4.6, so are their Hasse derivatives. Since  $|A| = |T|$ , then the only thing left to prove is  $|T| = \dim(\mathcal{M}(S, 2, k)^\perp)$ .

The map  $x^\alpha y^\beta \mapsto x^{2s_1-\alpha-1} y^{2s_2-\beta-1}$  gives a bijection between  $T$  and the monomials in  $\Delta_\prec(I(S; \mathcal{J}_2))$  of degree  $\geq k$ . The number of such monomials is  $tn - \dim(\mathcal{M}(S, 2, k))$  by Theorem 4.9 (recall that  $t = |\mathcal{J}_2|$ ,  $n = |S|$  and  $tn = |\Delta_\prec(I(S; \mathcal{J}_2))|$  by Lemma 4.6). Therefore,

$$|T| = tn - \dim(\mathcal{M}(S, 2, k)) = \dim(\mathcal{M}(S, 2, k)^\perp),$$

and we obtain the result. □

When we evaluate over the whole affine plane  $S = \mathbb{F}_q^2$ , we have the following simple formula for the matrices  $M_{\mathbf{a}}$  and  $N_{\mathbf{a}}$ .

**Remark C.4** By Remark 3.5, if  $S_1 = S_2 = \mathbb{F}_q$ , i.e.,  $S = \mathbb{F}_q^2$ , then for all  $(a, b) \in \mathbb{F}_q^2$ ,

$$M_{(a,b)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N_{(a,b)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We illustrate the previous results with a worked example for  $q = m = r = 2$ .

**Example C.5** Consider  $q = m = r = 2$  and  $S_1 = S_2 = \mathbb{F}_2$ , i.e.,  $S = \mathbb{F}_2^2$ , as in Example 5.1. Set  $N = m(rs - 1) = 6$  and  $k = 4$ . The multiplicity code  $\mathcal{M}(S, r, k) = \mathcal{M}(\mathbb{F}_2^2, 2, 4)$  has dimension 10 and one of its bases, by Corollary 4.10, is given by the Hasse derivatives

$$(f(a, b), \partial_x f(a, b), \partial_y f(a, b))_{(a,b) \in \mathbb{F}_2^2} \in (\mathbb{F}_2^3)^4$$

of the monomials  $1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3$ . For instance, if  $f_1 = x$ , then  $\partial_x f_1 = 1$  and  $\partial_y f_1 = 0$ , since such Hasse derivatives coincide with classical derivatives (see Sect. 2). If we order  $S$  as  $(0, 0), (1, 0), (0, 1)$  and  $(1, 1)$ , then the corresponding codeword is

$$\mathbf{c}_1 = ((0, 1, 0), (1, 1, 0), (0, 1, 0), (1, 1, 0)) \in (\mathbb{F}_2^3)^4.$$

Similarly, if  $f_2 = xy$ , we have  $\partial_x f_2 = y$  and  $\partial_y f_2 = x$ , thus we obtain the codeword

$$\mathbf{c}_2 = ((0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 1, 1)) \in (\mathbb{F}_2^3)^4.$$

Now we look at the dual code  $\mathcal{M}(\mathbb{F}_2^2, 2, 4)^\perp$ , which has dimension  $12 - 10 = 2$ . By Proposition C.3, a basis of  $\mathcal{M}(\mathbb{F}_2^2, 2, 4)^\perp$  may be given by the Hasse derivatives

$$\begin{aligned} & ((\partial_x g(a, b), \partial_y g(a, b), \partial_{xy} g(a, b)) \cdot N_{(a,b)})_{(a,b) \in \mathbb{F}_2^2} \\ &= (\partial_{xy} g(a, b), \partial_y g(a, b), \partial_x g(a, b))_{(a,b) \in \mathbb{F}_2^2} \in (\mathbb{F}_2^3)^4 \end{aligned}$$

(where we have used Remark C.4 for  $N_{(a,b)}$ ) of the polynomials

$$g_1 = G_1(x) = x^2 + x \quad \text{and} \quad g_2 = G_2(y) = y^2 + y.$$

Since  $\partial_x g_1 = 1$  and  $\partial_y g_1 = \partial_{xy} g_1 = 0$ , the corresponding codeword is

$$\mathbf{d}_1 = ((0, 0, 1), (0, 0, 1), (0, 0, 1), (0, 0, 1)) \in (\mathbb{F}_2^3)^4,$$

and we can easily verify that  $\mathbf{c}_1 \cdot \mathbf{d}_1 = \mathbf{c}_2 \cdot \mathbf{d}_1 = 0$ . Similarly for  $g_2$ . Using  $g_1$  and  $g_2$ , a generator matrix of  $\mathcal{M}(\mathbb{F}_2^2, 2, 4)^\perp$ , i.e., a parity-check matrix of  $\mathcal{M}(\mathbb{F}_2^2, 2, 4)$  is given by

$$H = \left( \begin{array}{ccc|ccc|ccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array} \right) \in \mathbb{F}_2^{2 \times 12}.$$

**Acknowledgements** This work was started while H. H. López and U. Martínez-Peñas were visiting the Simons Institute for the Theory of Computing (University of California, Berkeley), and while H. H. López and E. Camps Moreno were visiting the IMUVa-Mathematics Research Institute, University of Valladolid, Spain.

**Author contributions** All authors contributed equally to the manuscript.

**Funding** Open access funding provided by FEDER European Funds and the Junta de Castilla y León under the Research and Innovation Strategy for Smart Specialization (RIS3) of Castilla y León 2021-2027. This work has been partially supported by Grant PID2022-138906NB-C21 funded by MICIU/AEI/10.13039/501100011033 and by ERDF/EU. Hiram H. López was partially supported by the NSF Grants DMS-2401558 and DMS-2502705. Rodrigo San-José was also partially supported by Grant FPU20/01311 funded by the Spanish Ministry of Universities.

**Data availability** No datasets were generated or analysed during the current study.

## Declarations

**Conflict of interest** The authors declare no conflict of interest.

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