



Full length article

The rational rank of the support of generalized power series solutions of differential and q -difference equations

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Abstract

Given a differential or q -difference equation P of order n , we prove that the set of exponents of a generalized power series solution has its rational rank bounded by the rational rank of the support of P plus n . We also prove that when the support of the solution has maximum rational rank, it is convergent. Using the Newton polygon technique, we show also that in the maximum rational rank case, an initial segment can always be completed to a true solution. The techniques are the same for the differential and the q -difference case.

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1. Introduction

An important class of solutions of differential and q -difference equations are formal power series. Their relevance comes from algebraic, geometric, analytical, combinatorial and logical point of views. Denef and Lipschitz's paper [9] deals with the logical questions; Singer's and Grigoriev's works study the possibility of finding these solutions, and some of their convergence properties [15,23]; the excellent book on combinatorics by Flajolet and Sedgewick [12] contains a complete section devoted to the relation between solutions of polynomial differential equations and combinatorics; and one cannot forget the relation between power series, derivations, Hardy fields, \mathfrak{o} -minimal structures and related topics [8,19,20].

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Finally, any survey of the bibliography would be incomplete without citing van der Hoeven’s works on transseries [16] and the *magnum opus* [1]. This latter work provides an extensive study of the algebraic-differential properties of the H -field of well-based transseries. Our work deals with the classical differential subfield Ω of generalized power series introduced by Grigoriev and Singer [15], which is also an H -field.

We work in the field Ω of generalized power series with complex coefficients and real exponents is the set of series $\sum_{i=0}^{\infty} a_i x^{v_i}$, where $a_i \in \mathbb{C}$, $v_i \in \mathbb{R}$, $v_0 < v_1 < \dots$ and $\lim v_i = \infty$. Notice that other authors (see for instance [10,21]) call generalized power series those with well ordered set of exponents.

Grigoriev and Singer in [15] prove that if $y(x) = \sum_{i=0}^{\infty} a_i x^{v_i} \in \Omega$ is a solution of a non trivial polynomial ordinary differential equation $P(y, y', \dots, y^{(n)}) = 0$ with coefficients in the field of formal Laurent series $\mathbb{C}((x))$, then its support $\text{supp } y(x) = \{v_i \mid a_i \neq 0\}$ (the set of exponents with non-zero coefficient), generates a finite \mathbb{Z} -module but they provide no information on its rational rank. Our first aim (Theorem 1) is to prove in an elementary way that, modulo the support of the equation, this rational rank is at most the order of the equation. Our arguments work almost word by word for q -difference equations. Our proof of Theorem 1 allows us to show that in both cases, when the rational rank of the solution reaches its maximum possible value, then the solution is necessarily convergent (assuming P is a polynomial), which is Theorem 2. We make use of the convergence results of Gontsov, Goryuchkina and Lastra [13,14]. Theorem 3 addresses the possibility of completing an initial segment $s_0(x) = \sum_{1 \leq i \leq k} c_i x^{\mu_i}$ to an actual power series $s(x)$ solution of the differential or q -difference equation $P = 0$. Finally, Theorem 4 is the consequence of applying the previous results to the case of autonomous first order differential equations, providing new proofs about the existence and convergence of Puiseux solutions of these equations given in [7].

Other authors have proved related finiteness results in other contexts [5,6,8,18–20,22], but among these, none gives effective bounds for the rational rank of the support of the solutions. The unique effective bound we have knowledge of is by van der Hoeven, in [16] (Corollary 8.38), who shows in the language of transseries that if $P(y, y', \dots, y^{(n)}) = 0$ is an algebraic differential equation and f is a transseries solution, then *essentially* the rational rank of the semigroup of monomials of f can only increase by the order n of P . This is the same result as our Theorem 1 in the case of differential equations, and over a field which neither contains nor is contained in Ω . Our arguments have the advantage of simplicity and conciseness.

In this paper we work indistinctly with ordinary differential equations and q -difference equations. Given a polynomial $P \in \Omega[y_0, y_1, \dots, y_n]$, the equation $P = 0$ means $P(y(x), y'(x), \dots, y^{(n)}(x)) = 0$, with the assumption that $y'(x)$ refers to either the usual derivation operator $y'(x) = \frac{dy(x)}{dx}$, the Euler derivation $y'(x) = x \frac{dy(x)}{dx}$ or $y'(x) = y(qx)$, for some $q \in \mathbb{C}$, with $|q| \neq 1$. We do not consider mixed equations.

Given $P \in \Omega[y_0, y_1, \dots, y_n]$ we denote by $\text{supp } P$ the union of the supports of the coefficients of P , and if E is a subset of \mathbb{R} then $\langle E \rangle$ is the \mathbb{Q} -linear subspace of \mathbb{R} generated by E . Our first result is Theorem 1 which states that if $s(x) \in \Omega$ is a solution of $P = 0$ then

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } s(x) \cup \text{supp } P \rangle}{\langle \text{supp } P \rangle} \leq n. \tag{1}$$

In particular, if the coefficients of P are formal power series with integral exponents, then the maximal number of rationally independent irrational exponents that can appear in $\text{supp } s(x)$ is the order n of the equation $P = 0$.

The second result of this paper addresses the question of the convergence of the solution $s(x)$ provided P is a polynomial in x, y_0, \dots, y_n . It is well known that the convergence of a solution

is not guaranteed: Euler's example $y - x^2 y' - x = 0$ has as solution the formal power series $\varepsilon(x) = \sum_{n \geq 0} n! x^{n+1}$. B. Malgrange in [17] gives a sufficient condition for a formal power series solution $s(x)$ of a polynomial differential equations $P = 0$ to be convergent in terms of the linearized differential operator along the solution, $L = \sum_{i=0}^n \frac{\partial P}{\partial y_i}(s(x), \dots, s^{(n)}(x)) \delta^i$: if L has a regular singularity at the origin then the solution $s(x)$ is convergent. This criterion has been extended in the differential case by R. Gontsov and I. Goryunchkina in [13] and in the q -difference case by R. Gontsov and I. Goryunchkina and A. Lastra in [14] for power series solutions with complex exponents. In [Theorem 2](#) of this paper we provide a new sufficient condition for the convergence of the solution $s(x)$ which only depends on the order n of the differential or q -difference equation $P = 0$, and on the support of the solution. Specifically, if $\langle \text{supp } s(x) \rangle$ has the maximal possible rational rank (Eq. (1) is an equality) then $s(x)$ is necessarily convergent. In particular, if P is of order one and with constant coefficients, any solution of the equation $P = 0$ is convergent. [Theorem 2](#) provides a negative criterion for a series to be a solution of a differential or q -difference equation. For instance, neither the series $x^\pi + \varepsilon(x)$ nor $x^\pi \varepsilon(x)$ (where $\varepsilon(x)$ is as above) is a solution of a nonlinear first order differential (or q -difference) equation with convergent coefficients.

Our third main result addresses the question of the possibility of completing an initial segment $s_0(x) = \sum_{1 \leq i \leq k} c_i x^{\mu_i}$ to an actual power series solution $s(x)$ solution of the equation $P = 0$. Given an equation $P = 0$, the Newton polygon method [5,6,15], provides, for any positive integer k , a finite family of necessary initial conditions $\text{NIC}_k(P)$ given by Eq. (10), such that the first k coefficients and exponents of any solution $\sum_{i=1}^{\infty} a_i x^{\nu_i}$ of $P = 0$ satisfy $\text{NIC}_k(P)$. We say that $s_0(x)$ is an *admissible initial segment* for $P = 0$ if its coefficients and exponents satisfy $\text{NIC}_k(P)$.

In the algebraic case (i.e. $n = 0$), Puiseux's Theorem shows that any admissible initial segment for $P = 0$ is the truncation of an actual solution. This is no longer true for differential and q -difference equations: even in order and degree one, there are examples [2,3] of admissible initial segments which cannot be completed to a solution. Section 3 of [15] introduces the concept of *stabilization* as a criterion which guarantees the existence (and uniqueness) of an actual solution with initial segment $s_0(x)$. In [Theorem 3](#) we introduce a criterion of a different nature: if the support of an admissible polynomial $s_0(x)$ for $P = 0$ has n linearly independent irrational numbers over \mathbb{Q} (modulo $\langle \text{supp } P \rangle$) then $s_0(x)$ is in fact the truncation of an actual solution of $P = 0$. Indeed, there can be more than one solution $s(x)$ with the same truncation $s_0(x)$, unlike the stabilization criterion, for which the solution is unique (see example in Section 5). In particular, if $P = P(x, y) = 0$ is a polynomial in x, y (i.e. the equation of an algebraic curve), then the hypothesis of [Theorem 3](#) always holds, so that it may be understood as a kind of generalization of Puiseux's Theorem to differential and q -difference equations.

Section 2 introduces the context and a technical result from which the two main theorems follow in Section 3. Then, in Section 4 we recall the essential notions and results on the Newton polygon process (see [2–6,15]), which we use to prove the technical [Proposition 1](#), from which [Theorem 3](#) follows straightforwardly. In Section 4.3 we apply the previous results to the case of autonomous ordinary differential equations. In Section 5 we give a detailed example to illustrate our main results.

Notice that instead of generalized power series in x , we could have worked with generalized power series in x^{-1} . The results and proves are the same with the obvious modifications.

2. Notation and a technical lemma

Following [15], we denote by Ω the field of formal power series $s(x) = \sum c_i x^{\mu_i}$ with complex coefficients and real exponents, such that $\lim_{i \rightarrow \infty} \mu_i = \infty$, where we always assume

that $\mu_1 < \mu_2 < \dots$. The *support* of a power series $s(x)$ is the set of exponents with non-zero coefficient: $\text{supp } s(x) = \{\mu_i : c_i \neq 0\}$. Given an additive subgroup Γ of \mathbb{R} we denote in this work

$$\mathbb{C}((x^\Gamma)) = \{s(x) \in \Omega : \text{supp } s(x) \subset \Gamma\} \subset \Omega$$

the subfield of Ω of series whose support is contained in Γ . We also denote by $\hat{K} = \mathbb{C}((x^{\mathbb{Z}}))$ the field of fractions of the ring of formal power series.

We are going to deal with differential and q -difference equations simultaneously, hence it should be more convenient to use the Euler derivative in the case of differential equations, but some of our results are specific for the case of differential equations with the ordinary derivation, in particular, the case of autonomous equations. Hence we deal simultaneously with differential equations in terms of the ordinary differential operator, in terms of the Euler differential operator and with q -difference equations, with $|q| \neq 1$.

Throughout this paper P will denote a polynomial $P(y_0, \dots, y_n)$ in the indeterminates y_0, y_1, \dots, y_n with coefficients in Ω . Given $s(x) \in \Omega$, we set

$$P(s(x)) = P(s(x), s'(x), \dots, s^{(n)}(x)),$$

where $s'(x)$ is either the ordinary derivative $\frac{ds(x)}{dx}$, the Euler derivative $x \frac{ds(x)}{dx}$ or the q -difference operator $s(qx)$, and $s^{(\kappa)}(x)$ is the κ -th iteration. We do not consider mixed equations. We note that the field Ω is closed under the three operators above, as is $\mathbb{C}((x^\Gamma))$ for the Euler derivative and q -difference operators, and also for the ordinary differential operator provided $1 \in \Gamma$.

We introduce notations for our differential and q -difference operators. Define $\varepsilon = 1$ in the case of differential equations with the ordinary differential operator $\frac{d}{dx}$ or $\varepsilon = 0$ for differential equations with the Euler differential operator, and q -difference equations. For $\mu \in \mathbb{R}$ and $\kappa \in \mathbb{N}$, define $\delta_\mu^{(\kappa)} = \mu(\mu - 1) \dots (\mu - \kappa + 1)$ for differential equations with the differential operator $\frac{d}{dx}$, and $\delta_\mu^{(\kappa)} = \mu^\kappa$ for differential equations with the Euler operator $x \frac{d}{dx}$, and $\delta_\mu^{(\kappa)} = q^{\kappa\mu}$ for q -difference equations. In particular $\delta_\mu^{(0)} = 1$ in all cases, and we denote $\delta_\mu = \delta_\mu^{(1)}$. Then for any $s(x) = \sum_{i=0}^\infty c_i x^{\mu_i}$, and $\kappa = 0, \dots, n$, we write the κ -th iteration of the respective operator ' as:

$$s^{(\kappa)}(x) = \sum_{i=0}^\infty c_i \delta_{\mu_i}^{(\kappa)} x^{\mu_i - \varepsilon\kappa}.$$

The expression $\text{ord}_x s(x)$ denotes the *valuation* of $s(x)$: the minimum of the support of $s(x)$ or $+\infty$ if $s(x) = 0$.

The expression $P = 0$ denotes the corresponding equation of order n . A *solution* of $P = 0$ is a power series $s(x)$ such that $P(s(x), s'(x), \dots, s^{(n)}(x)) = 0$. The *support* of P is the union of the supports of its coefficients.

Our first two results rely on the following elementary fact about derivations with respect to *transcendent monomials*. Fix Γ an additive subgroup of \mathbb{R} and consider a family of exponents $\mu_1, \dots, \mu_m \in \mathbb{R}$ whose classes modulo $\langle \Gamma \rangle$, $\bar{\mu}_1, \dots, \bar{\mu}_m \in \mathbb{R}/\langle \Gamma \rangle$ are \mathbb{Q} -linearly independent. Let $\Gamma' = \langle \Gamma \cup \{\mu_1, \dots, \mu_m\} \rangle$. By definition, any $\alpha \in \Gamma'$ can be written uniquely as

$$\alpha = [\alpha]_\Gamma + \sum_{j=1}^m [\alpha]_{\mu_j} \mu_j,$$

where $[\alpha]_\Gamma \in \langle \Gamma \rangle$ and $[\alpha]_{\mu_j} \in \mathbb{Q}$.

Lemma 1. *With the above notation and conditions, for any $j = 1, \dots, m$, there exists derivation \mathcal{D}_j on $\mathbb{C}((x^{\Gamma'}))$ such that*

$$\mathcal{D}_j \left(\sum_{i=1}^{\infty} c_i x^{\alpha_i} \right) = \sum_{i=1}^{\infty} c_i [\alpha_i]_{\mu_j} x^{\alpha_i}.$$

Proof. The expression above is obviously a well-defined map in $\mathbb{C}((x^{\Gamma'}))$, due to the linear independence of the classes. A straightforward computation shows that it is linear and satisfies the Leibniz rule. \square

Let us remark that the above derivation \mathcal{D}_j depends on Γ and μ_1, \dots, μ_m although we do not stress this fact in the notation. Notice also that \mathcal{D}_j vanishes on $\mathbb{C}((x^{\Gamma}))$ and on $\mathbb{C}((x^{\Gamma_{j-1}}))$, for $\Gamma_{j-1} = \langle \Gamma \cup \{\mu_1, \dots, \mu_{j-1}\} \rangle$. Let us give some examples: Assume that $\Gamma = \{0\}$, $\mu_1 = 1$ and $\mu_2 = \pi$, hence $\mathcal{D}_1(x^{2+3\pi}) = 2x^{2+3\pi}$ and $\mathcal{D}_2(x^{2+3\pi}) = 3x^{2+3\pi}$; now assume that $\Gamma = \pi\mathbb{Z}$, $\mu_1 = 2$ and $\mu_2 = e$, hence $\mathcal{D}_1(x^{2+3\pi/2}) = x^{2+3\pi/2}$, $\mathcal{D}_2(x^{2+3\pi/2}) = 0$, and $\mathcal{D}_2(x^{\pi+5e}) = 5x^{\pi+5e}$.

Notice that **Lemma 1** is a particular case of the fact that given a group homomorphism $\varphi : \Gamma \rightarrow \mathbb{C}$, the map $\sum c_i x^{\mu_i} \mapsto \sum \varphi(\mu_i) c_i x^{\mu_i}$ is a derivation.

We will work with the following substitution operation on $P \in \Omega[y_0, \dots, y_n]$, for $s(x) \in \Omega$:

$$P[s(x)] = P(s(x) + y_0, s'(x) + y_1, \dots, s^{(n)}(x) + y_n) \in \Omega[y_0, y_1, \dots, y_n].$$

This substitution, which corresponds to a change of variable $y = s(x) + y$ in the equation $P = 0$, and which other authors denote P_{+s} in [1], or $P[s(x) + Y]$ in [3] is relevant because it allows us to iterate monomial substitutions and see how each of these affects the Newton Polygon.

By the chain rule:

$$\frac{\partial P}{\partial y_\kappa} [s(x)] = \frac{\partial}{\partial y_\kappa} (P[s(x)]), \quad \kappa = 0, 1, \dots, n.$$

For $P \in \mathbb{C}((x^{\Gamma'})) [y_0, \dots, y_n]$ and μ_1, \dots, μ_m and Γ' as in **Lemma 1**, take $s(x) \in \mathbb{C}((x^{\Gamma'}))$ (assuming without loss of generality that $1 \in \Gamma'$). Extending the derivation \mathcal{D}_j to the ring $\mathbb{C}((x^{\Gamma'})) [y_0, \dots, y_n]$ by acting trivially on the indeterminates y_0, y_1, \dots, y_n we get, as the \mathcal{D}_j are derivations vanishing on $\mathbb{C}((x^{\Gamma'}))$, the equality:

$$\mathcal{D}_j (P[s(x)]) = \sum_{\kappa=0}^n \frac{\partial}{\partial y_\kappa} (P[s(x)]) \mathcal{D}_j (s^{(\kappa)}(x)). \tag{2}$$

We introduce some notation that is useful for studying the coefficients of P (hence, of $P[s(x)]$, too). Given $\rho = (\rho_0, \dots, \rho_n) \in \mathbb{Z}_{\geq 0}^{n+1}$, the expression $P_\rho \in \mathbb{C}((x^{\Gamma'}))$ will denote the coefficient of $y_0^{\rho_0} y_1^{\rho_1} \dots y_n^{\rho_n}$ in P . We shall use the basic vectors $e_0 = (1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$, each having $n + 1$ components (we start at 0 because e_i will be related to y_i). Examining each term on both sides of (2), we obtain the key equality:

$$\mathcal{D}_j (P[s(x)]_\rho) = \sum_{\kappa=0}^n (\rho_\kappa + 1) (P[s(x)]_{\rho+e_\kappa}) \mathcal{D}_j (s^{(\kappa)}(x)). \tag{3}$$

3. Solutions and rational rank

Before proceeding, we state our results on the rational rank, and defer their proof to the end of the section, for the sake of clarity.

Theorem 1. *Let $P \in \Omega[y_0, \dots, y_n]$ be a non-zero polynomial, and $s(x) \in \Omega$ a solution of the equation $P = 0$. Then:*

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } s(x) \cup \text{supp } P \rangle}{\langle \text{supp } P \rangle} \leq n.$$

Thus we obtain a bound for the rational rank of solutions of equations with coefficients Puiseux series.

Corollary 1. *If $s(x) \in \Omega$ is a generalized power series with*

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } s(x) \cup \mathbb{Q} \rangle}{\mathbb{Q}} > n$$

then $s(x)$ is not the solution of a nontrivial equation of order n over the field of Puiseux series.

Corollary 2. *If $s(x) \in \Omega$ is a generalized power series with*

$$\dim_{\mathbb{Q}} \langle \text{supp } s(x) \rangle > n$$

then $s(x)$ is not the solution of a nontrivial equation $P = 0$, of order n with constant coefficients.

For instance $x + x^\pi$ is not a solution of a non-trivial equation $P = 0$ of order one and constant coefficients, that is, with $P(y_0, y_1) \in \mathbb{C}[y_0, y_1]$. We will improve this result in the case of autonomous first order differential equations (see [Theorem 4](#)).

Remark 1. From Grigoriev and Singer's [15] and Fortuny Ayuso and Cano's [6] finiteness results for solutions of differential and q -difference equations, our [Theorem 1](#) implies straightforwardly that if $\text{supp } P \subset \mathbb{Z}$ then: if $s(x) \in \Omega$ is a solution of $P = 0$, then $\text{supp } s(x)$ is contained in a \mathbb{Z} -module of rank at most $n + 1$. This gives an explicit bound for those existence results.

Our second main result deals with the case when P is a polynomial in x also. Notice that in this case $\langle \text{supp } P \rangle$ is either $\{0\}$ if P has constant coefficients or \mathbb{Q} otherwise.

Theorem 2. *Take $P \in \mathbb{C}[x, y_0, \dots, y_n]$ with $P \neq 0$ and assume $s(x) \in \Omega$ is a solution of $P = 0$ whose support has maximal rational rank, that is*

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } s(x) \cup \text{supp } P \rangle}{\langle \text{supp } P \rangle} = n.$$

Then $s(x)$ converges uniformly in any sector S of sufficiently small radius with vertex at the origin and of the opening less than 2π .

The following result covers [Theorem 11](#) in [7] and is a consequence of the previous [Theorem](#) for the case of first order equations with constant coefficients:

Corollary 3. *Let $P(y_0, y_1)$ a polynomial with constant coefficients. Let $s(x) \in \Omega$ be a solution of the equation $P = 0$. Then $s(x)$ is convergent in the sense of [Theorem 2](#). Moreover, if $s(x)$ is a Puiseux series then it is convergent in a neighbourhood of the origin.*

Proof. Either $s(x)$ is a constant or $\dim_{\mathbb{Q}} \langle \text{supp } s(x) \rangle \geq 1$. Since $\langle \text{supp } P \rangle = \{0\}$, by [Theorem 1](#), $\dim_{\mathbb{Q}} \langle \text{supp } s(x) \rangle = 1$ and the result follows from [Theorem 2](#). \square

We divide the proofs of [Theorems 1](#) and [2](#) into a shared initial part and their respective ends.

Common part of the proof of Theorems 1 and 2. Let $s(x) = \sum_{i=1}^{\infty} a_i x^{\alpha_i} \in \Omega$ be a solution of the non-trivial equation $P = 0$. We may assume without loss of generality that for some $0 \leq \kappa \leq n$, $\frac{\partial P}{\partial y_{\kappa}}(s(x)) \neq 0$: otherwise we proceed by induction on the total degree of P , as $\frac{\partial P}{\partial y_{\kappa}}$ and $s(x)$ would satisfy the hypotheses of both results. Thus, we can define:

$$\lambda = \min\{\text{ord}_x \frac{\partial P}{\partial y_{\kappa}}(s(x)) - \varepsilon \kappa \mid 0 \leq \kappa \leq n\}, \quad \Lambda = \min\{\text{ord}_x \frac{\partial P}{\partial y^{\rho}}(s(x)) \mid \rho \in \mathbb{Z}_{\geq 0}^{n+1}\}$$

(notice that $\Lambda < +\infty$ because for some $0 \leq \kappa \leq n$, $\frac{\partial P}{\partial y_{\kappa}}(s(x)) \neq 0$ as stated above). This λ corresponds to the one defined in [\[14\]](#) and Λ is just an auxiliary constant to be used in the forthcoming proof.

Consider

$$m_0 = \dim_{\mathbb{Q}} \frac{\langle \text{supp } s(x) \cup \text{supp } P \rangle}{\langle \text{supp } P \rangle} \in \mathbb{Z}_{\geq 0} \cup \{\infty\},$$

which in theory could be infinity in the case of q -difference equations. Part of our argument will consist in proving that it cannot be.

Let $\Gamma_0 = \langle \text{supp } P \rangle$. For $1 \leq j \leq m_0$ (if $m_0 = \infty$ the last inequality is strict), define μ_j and Γ_j inductively as follows: Assume that $\Gamma_0, \Gamma_1, \dots, \Gamma_{j-1}$ and $\mu_1, \mu_2, \dots, \mu_{j-1}$ have been defined (so that for $j = 1$ only Γ_0 exists). By the definition of m_0 , there exists the minimum k of the set of indices i such that $a_i \neq 0$ and $\alpha_i \notin \Gamma_{j-1}$. We set $\mu_j = \alpha_k$ and $\Gamma_j = \langle \Gamma_{j-1} \cup \{\mu_j\} \rangle$.

We can choose N large enough such that the truncation $\bar{s}(x) = \sum_{i=1}^N a_i x^{\alpha_i}$ of $s(x)$ satisfies the following property:

$$\text{ord}_x(s(x) - \bar{s}(x)) > |\lambda| + 2n + |\Lambda|, \tag{4}$$

which is possible because either $s(x)$ has a finite number of terms or its exponents tend to infinity. Notice that $\bar{s}(x)$ could be equal to $s(x)$ if the latter has a finite number of terms.

Once N is set, we can divide the exponents of $\bar{s}(x)$ into a finite number m of finite sequences $(\mu_{1,i})_{i=0}^{t_0}, \dots, (\mu_{m,i})_{i=0}^{t_m}$, such that $\mu_{j,i} \in \Gamma_j$ for any j, i . Formally, we rewrite:

$$\bar{s}(x) = \sum_{j=0}^m \sum_{i=0}^{t_j} c_{j,i} x^{\mu_{j,i}} \tag{5}$$

where the following properties hold:

- (i) The sequence of exponents $(\mu_{j,i})$ is increasing with respect to the lexicographical ordering of their indices (j, i) (with prevalence on the first component).
- (ii) The exponents $\mu_{j,i} \in \Gamma_j$, for $i = 0, \dots, t_j$ and $j = 0, \dots, m$.
- (iii) The coefficients $c_{j,i}$ are non-zero for $j = 1, \dots, m$ and $0 \leq i \leq t_j$.
- (iv) The exponent $\mu_{j,0} = \mu_j \notin \Gamma_{j-1}$ for $j = 1, \dots, m$, that is, it adds one to the rational rank.

By properties (i)–(iv), the set $\Gamma = \Gamma_0$, the exponents μ_1, \dots, μ_m and Γ_m satisfy the hypothesis of [Lemma 1](#). In order to make $\mathbb{C}((x^{\Gamma_m}))$ a differential field with the operator $\frac{d}{dx}$ we do the following: If $1 \notin \Gamma_m$, we set $\Gamma' = \langle \Gamma_m \cup \{1\} \rangle$ otherwise we set $\Gamma' = \Gamma_m$. Hence Γ and μ_1, \dots, μ_m , and eventually 1, satisfy the hypothesis of [Lemma 1](#) and $\mathbb{C}((x^{\Gamma'}))$ is closed under the three operators that we use. Any coefficient of $P[\bar{s}(x)]$ belongs also to $\mathbb{C}((x^{\Gamma'}))$, and we

may apply the derivations \mathcal{D}_j , $j = 1, \dots, m$, to these coefficients (if $1 \notin \Gamma_m$, we will not use the operator \mathcal{D}_{m+1} corresponding to 1).

Noticing that $P[\bar{s}(x)]_{e_k} = \frac{\partial P}{\partial y_k}(\bar{s}(x))$, Eq. (3) applied to $\rho = (0, 0, \dots, 0)$ gives:

$$\mathcal{D}_j(P(\bar{s}(x))) = \sum_{\kappa=0}^n \frac{\partial P}{\partial y_\kappa}(\bar{s}(x)) \mathcal{D}_j(\bar{s}^{(\kappa)}(x)) \tag{6}$$

From now on, the notation $t(x) = ax^\mu + \dots$ will mean that $a \in \mathbb{C}$ (a may be 0), and $\text{ord}_x(t(x) - ax^\mu) > \mu$. By definition of λ , we can write, for $\kappa \in 0, \dots, n$:

$$\frac{\partial P}{\partial y_\kappa}(s(x)) = d_\kappa x^{\lambda+\varepsilon\kappa} + \dots$$

and we know that there is at least one $\kappa \in \{0, \dots, n\}$ with $d_\kappa \neq 0$. Using the Taylor expansion of the left hand side of this equality and property (4), we also obtain

$$\frac{\partial P}{\partial y_\kappa}(\bar{s}(x)) = d_\kappa x^{\lambda+\varepsilon\kappa} + \dots$$

From the properties of the derivations \mathcal{D}_j , and because $[\mu_{j,0}]_{\mu_j} = 1$ and $[\mu_{j',i}]_{\mu_j} = 0$ for every $(j', i) < (j, 0)$, for $j = 1, \dots, m$, we have:

$$\mathcal{D}_j(\bar{s}^{(\kappa)}(x)) = c_{j,0} \delta_{\mu_j}^{(\kappa)} x^{\mu_j - \varepsilon\kappa} + \dots$$

Hence, Equation (6) becomes:

$$\begin{aligned} \mathcal{D}_j(P(\bar{s}(x))) &= \sum_{\kappa=0}^n (d_\kappa x^{\lambda+\varepsilon\kappa} + \dots)(c_{j,0} \delta_{\mu_j}^{(\kappa)} x^{\mu_j - \varepsilon\kappa} + \dots) \\ &= c_{j,0} \left(\sum_{\kappa=0}^n d_\kappa \delta_{\mu_j}^{(\kappa)} \right) x^{\lambda+\mu_j} + \dots, \end{aligned} \tag{7}$$

This finishes the common arguments.

End of proof of Theorem 1. In order to obtain a contradiction, we assume that $m_0 > n$ so that μ_{n+1} is defined. Recall that the exponents of $s(x)$ tend to $+\infty$. This allows us to choose N large enough such that $m \geq n + 1$ and $\text{ord}_x P(\bar{s}(x)) > \lambda + \mu_{n+1}$. Now $\mu_{n+1,0} = \mu_{n+1}$, and these inequalities imply that the left hand side of Eq. (6) has order greater than $\lambda + \mu_{n+1,0}$. Since $c_{j,0} \neq 0$, using (7) we obtain the $n + 1$ equalities:

$$\sum_{\kappa=0}^n d_\kappa \delta_{\mu_j}^{(\kappa)} = 0, \quad j = 1, 2, \dots, n + 1, \tag{8}$$

which is a square linear system with coefficient matrix $(\delta_{\mu_j}^{(\kappa)})$. Because $|q| \neq 1$ and μ_j is an increasing sequence of real numbers, then the μ_j are pairwise distinct, and we have $q^{\mu_j} \neq q^{\mu_i}$ for $i \neq j$. Thus the coefficient matrix is a Vandermonde matrix in the case of q -difference equations and in the case of differential equations with the Euler operator. In the case of differential equations with the ordinary differential operator, the matrix $(\delta_{\mu_j}^{(\kappa)})$ is reduced by elementary column operations to the Vandermonde matrix (μ_j^κ) . Hence, system (8) has the unique solution $d_0 = d_1 = \dots = d_n = 0$, contradicting the fact that at least one d_κ is non-zero. Thus, $m_0 \leq n$ which gives Theorem 1. \square

End of proof of Theorem 2. In this case, $m_0 = n$, and μ_n is defined, and we can choose N and m such that $\text{ord}_x P(\bar{s}(x)) > \lambda + \mu_n$, and $m = n$. Thus $\mu_{n,0} = \mu_n$ is also defined. The left hand side of Eq. (6) has order greater than $\lambda + \mu_{n,0}$ and we obtain the system of equations (8) for $j = 1, 2, \dots, n$. If $d_n = 0$, then this system of equations becomes another Vandermonde system for d_0, d_1, \dots, d_{n-1} , so that $d_0 = d_1 = \dots = d_{n-1} = d_n = 0$, again contradicting the existence of one non-zero d_k . As a consequence, $d_n \neq 0$, so $\text{ord}_x \frac{\partial P}{\partial y_n}(s(x)) = \lambda + \varepsilon n$, and we get:

$$\text{ord}_x \frac{\partial P}{\partial y_n}(s(x)) - \varepsilon n = \lambda \leq \text{ord}_x \frac{\partial P}{\partial y_\kappa}(s(x)) - \varepsilon \kappa, \quad \text{for } \kappa = 0, 1, \dots, n. \tag{9}$$

This means that the linearized differential operator $\sum_{\kappa=0}^n \frac{\partial P}{\partial y_\kappa}(s(x)) y_\kappa$ along $s(x)$ has a regular singularity at $x = 0$. If $s(x)$ is a formal Puiseux power series, then we apply the main result in [17] to guarantee that $s(x)$ is convergent in both differential cases. If $s(x) \in \Omega$, Condition (9) is the hypothesis of Theorem 1 of [13], which proves the Euler differential case. To prove the ordinary differential case we only need to rewrite P in terms of the Euler operator, that is, to replace y_κ with $\sum_{j=1}^{\kappa} a_{\kappa,j} x^{-j} y_j$, where the coefficients $a_{\kappa,j}$ are the Stirling number of the first kind. Condition (9) becomes then the hypothesis of Theorem 1 of [13] again.

For the case of q -difference equations we need to distinguish the cases $|q| > 1$ and $|q| < 1$, as in [14]. If $|q| > 1$, the fact that $d_n \neq 0$ and that μ_j is an increasing sequence of positive real numbers let us apply Theorem 1 of [14] straightforwardly. If $|q| < 1$, in order to apply the same Theorem we need to show that $d_0 \neq 0$. As in the previous argument, if $d_0 = 0$, then (8), for $j = 1, \dots, n$ becomes a Vandermonde system for d_1, \dots, d_n , whose only solution is $d_1 = \dots = d_n = 0$, getting the same contradiction. Thus $d_0 \neq 0$ and we can apply Theorem 1 of [14] again to obtain the convergence. \square

4. The Newton–Puiseux polygon

In this section we give a short description of the well known method of the Newton (or Newton–Puiseux) polygon applied to polynomial differential and q -difference equations (see for instance [4,6,11]), which we shall use to prove Theorem 3.

Let us introduce some notation: For $\rho = (\rho_0, \dots, \rho_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ we denote $|\rho| = \rho_0 + \rho_1 + \dots + \rho_n$ and $\omega(\rho) = \rho_1 + 2\rho_2 + \dots + n\rho_n$. As before, we set $\varepsilon = 0$ in the cases of differential equations with respect to the Euler derivative or q -difference equations, and $\varepsilon = 1$ in the case of differential equations with respect to the derivative $\frac{d}{dx}$.

Fix a non-zero polynomial $P \in \Omega[y_0, \dots, y_n]$, and write it uniquely as

$$P = \sum_{\rho \in \mathbb{Z}_{\geq 0}^{n+1}} \sum_{\alpha \in \text{supp}(P) - \varepsilon \omega(\rho)} P_{(\alpha, \rho)} x^{\alpha + \varepsilon \omega(\rho)} y_0^{\rho_0} \dots y_n^{\rho_n}$$

with $P_{(\alpha, \rho)} \in \mathbb{C}$. Given $V = (\alpha, r) \in \mathbb{R} \times \mathbb{Z}_{\geq 0}$, P_V will denote the sum of all the terms of P corresponding to the point V , that is

$$P_V = P_{(\alpha, r)} = \sum_{(\alpha, |\rho|) = V} P_{(\alpha, \rho)} x^{\alpha + \varepsilon \omega(\rho)} y_0^{\rho_0} \dots y_n^{\rho_n}.$$

There is no confusion possible between $P_{(\alpha, r)}$ and $P_{(\alpha, \rho)}$ except when $n = 0$, in which case we shall abuse the notation as the context will clarify what value we are using. The *cloud of points* $\mathcal{C}(P)$ of P is the set of points in the plane:

$$\mathcal{C}(P) = \{(\alpha, r) \in \mathbb{R} \times \mathbb{Z}_{\geq 0} : P_{(\alpha, r)} \neq 0\}.$$

We are interested in generalized power series with exponents in increasing order. In this setting, the *Newton–Puiseux polygon* (Newton polygon for short) of P , denoted $\mathcal{N}(P)$, is the convex hull in \mathbb{R}^2 of the set obtained by adjoining the half-line $\mathbb{R}_{\geq 0} \times \{0\}$ to each point in $\mathcal{C}(P)$:

$$\mathcal{N}(P) = \text{conv.hull.} \{(\alpha, r) + (\mathbb{R}_{\geq 0} \times \{0\}) : (\alpha, r) \in \mathcal{C}(P)\}.$$

Its border is composed of a sequence of points and segments. Given a positive number $\mu \in \mathbb{R}$, the *supporting line* of anti-slope μ , is the unique line with equation $L_\mu(P) \equiv \mu r + \alpha = \alpha_0$ with α_0 minimum and $L_\mu(P) \cap \mathcal{N}(P) \neq \emptyset$. The *element of anti-slope μ of $\mathcal{N}(P)$* , $E_\mu(P)$ is that intersection $L_\mu(P) \cap \mathcal{N}(P)$, which can be either a segment (called a *side* of $\mathcal{N}(P)$) or a point (a *vertex*). In both cases we denote by $\text{Top}(E_\mu(P))$ and $\text{Bot}(E_\mu(P))$ the highest and lowest points of $E_\mu(P)$, respectively (if E is a vertex, they coincide). Finally, instead of the *ordinate* r of a point $V = (\alpha, r)$, we shall call it its *height*, and denote $r = \text{ht}(V)$.

Given $V = (\alpha, r) \in \mathbb{R} \times \mathbb{Z}_{\geq 0}$, the *indicial polynomial* of P at V is

$$\Psi_{(P;V)}(T) = \sum_{(\alpha, |\rho|)=V} P_{(\alpha, \rho)} T^{(\rho)} \in \mathbb{C}[T],$$

where $T^{(\rho)}$ is equal to $T^{\omega(\rho)}$ for the Euler differential operator or the q -difference operator, and $T^{(\rho)} = \prod_{\kappa=1}^n (T(T-1) \cdots (T-\kappa+1))^{\rho_\kappa}$ for the differential operator $\frac{d}{dx}$.

The *characteristic polynomial* of P with respect to a anti-slope $\mu \in \mathbb{R}_{\geq 0}$ is

$$\Phi_{(P; \mu)}(C) = \sum_{(\alpha, |\rho|) \in E_\mu(P)} \delta_\mu^{(\rho)} P_{(\alpha, \rho)} C^{|\rho|} = \sum_{V \in E_\mu(P)} \Psi_{(P;V)}(\delta_\mu) C^{\text{ht}(V)} \in \mathbb{C}[C],$$

where $\text{ht}(V)$ is the ordinate of V . The key Lemma of the Newton polygon process gives the following necessary condition, which shows the importance of the characteristic polynomial (see [5, Lemma 1, pg. 63] and [6, Lemma 2, pg. 123] for a short proof):

Lemma 2. *Let $s(x) = c x^\mu + \sum_{\alpha > \mu} c_\alpha x^\alpha \in \Omega$ be a solution of $P = 0$. Then*

$$\Phi_{(P; \mu)}(c) = 0.$$

In fact, this lemma translates into a sequence of necessary conditions for a power series to be a solution of $P = 0$. Given $s(x) = \sum_{i=0}^\infty c_i x^{v_i} \in \Omega$, with $v_i < v_{i+1}$ for all $i \geq 0$, denote, for the sake of simplicity:

$$P_0 = P, \quad \text{and} \quad P_{i+1} = P_i[c_i x^{v_i}] = P[c_0 x^{v_0} + \cdots + c_i x^{v_i}], \quad i = 1, 2, \dots$$

If $s(x)$ is a solution of $P = 0$, then for each $i \in \mathbb{Z}_{\geq 0}$, the series $\sum_{i=i}^\infty c_i x^{v_i}$ must be a solution of $P_i = 0$. Thus, we obtain the sequence of necessary initial conditions (NIC) on the coefficients of $s(x)$ when it is a solution of $P = 0$:

$$\Phi_{(P_i; v_i)}(c_i) = 0, \quad i = 0, 1, \dots \tag{10}$$

We need, however, to check whether a finite power series is effectively the truncation of a solution. To this end, we introduce the following concepts: a finite sum $r(x) = c_0 x^{v_0} + \cdots + c_k x^{v_k}$ with $c_i \in \mathbb{C}^*$ (i.e. non-zero) and $v_i < v_{i+1}$, with $v_i \in \mathbb{R}$ will be called a *generalized polynomial*. Such a generalized polynomial $r(x)$ is *admissible for P* , or for $P = 0$ (or simply *admissible*) if the necessary initial conditions (10) are fulfilled for $i = 0, 1, \dots, k$. In particular, any truncation $r(x)$ of a solution $s(x) \in \Omega$ is an admissible generalized polynomial, but the converse is not true, even for linear equations: the generalized polynomial $r(x) = x$ is admissible for the differential equation $P = 2y_0 - y_1 - x + x^2$ (in terms of the Euler differential

operator) but there is no solution $s(x) \in \Omega$ of $P = 0$ having $r(x) = x$ as a truncation. Notice that the same holds considering $P = 0$ as a q -difference equation with $q = \sqrt{2}$ and $r(x) = (1 + \sqrt{2}/2)x$.

However, the converse statement holds when $\text{supp } r(x)$ has maximum rational rank. This is our third main result:

Theorem 3. *Assume that the equation $P = 0$ of order n admits a generalized admissible polynomial $r(x)$ with*

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } r(x) \cup \text{supp } P \rangle}{\langle \text{supp } P \rangle} \geq n. \tag{11}$$

Then $r(x)$ is the truncation of a solution $s(x)$ of $P = 0$, and the above inequality is actually an equality.

Before proving this result, we need to study the behaviour of the Newton polygon under changes of variables of the form $y = c x^v + y$. (See [5,6]).

Lemma 3. *Let $Q = P[c x^v]$. Then:*

- (1) $\mathcal{N}(Q)$ is contained in the closed right half-plane defined by the supporting line $L_v(P)$.
- (2) Let h the height of the point $\text{Top}(E_v(P))$. Then $\mathcal{N}(P)$ and $\mathcal{N}(Q)$ are equal above h . In particular $\text{Top}(E_v(P)) = \text{Top}(E_v(Q))$ and $L_v(P) = L_v(Q)$. Moreover, $P_{(\alpha, \rho)} = Q_{(\alpha, \rho)}$ for all (α, ρ) with $(\alpha, |\rho|)$ in the border of $\mathcal{N}(P)$ and $|\rho| \geq h$.
- (3) The height of the point $\text{Bot}(E_v(Q))$ is zero if and only if $\Phi_{(P, v)}(c) \neq 0$.
- (4) The following sequence of inequalities holds:

$$\text{ht}(\text{Top}(E_v(P))) \geq \text{ht}(\text{Bot}(E_v(Q))) \geq \text{ht}(\text{Top}(E_\mu(Q))),$$

where $\mu > v$ in the last expression.

We have the following characterization of generalized admissible polynomials in terms of the Newton polygon:

Lemma 4. *Let $r(x) = c_0 x^{v_0} + \dots + c_k x^{v_k}$ with $v_0 < \dots < v_k$, and, as above, $P_0 = P$ and $P_{i+1} = P_i[c_i x^{v_i}]$ for $i = 0, \dots, k$. Let us denote $Q = P[r(x)] = P_{k+1}$. Then $r(x)$ is an admissible generalized polynomial for P if and only the bottom vertex of $E_{v_k}(Q)$ has height greater than or equal to one.*

Proof. Assume that $r(x)$ is admissible. Applying Lemma 3 iteratively, we obtain the inequality $\text{ht}(\text{Bot}(E_{v_k}(P_{k+1}))) \geq 1$.

Conversely, assume that $r(x)$ is not admissible for P . Let i be the minimum index for which equation (10) does not hold. By part (3) of Lemma 3, $\text{ht}(\text{Bot}(E_{v_i}(P_{i+1}))) = 0$ and by part (4) of the same Lemma,

$$0 = \text{ht}(\text{Bot}(E_{v_i}(P_{i+1}))) \geq \text{ht}(\text{Top}(E_{v_{i+1}}(P_{i+1}))) \geq \text{ht}(\text{Bot}(E_{v_{i+1}}(P_{i+2}))) \geq 0.$$

Applying iteratively this argument we get $\text{ht}(\text{Bot}(E_{v_k}(P_{k+1}))) = 0$. \square

The following result is also well-known but we include its proof for the convenience of the reader.

Corollary 4. *Let $r(x)$ be an admissible generalized polynomial for $P = 0$, and let $Q = P[r(x)]$. Then either $r(x)$ is already a solution of the equation $P = 0$, or $\mathcal{N}(Q)$ has a side $E_\nu(Q)$ with $\nu > \max(\text{supp}(r(x)))$.*

Proof. By Lemma 4, we know that $\text{ht}(\text{Bot}(E_{\nu_k}(Q))) \geq 1$. If $\mathcal{N}(Q)$ does not intersect the horizontal axis of coordinates then the null power series is a solution of $Q = 0$ and therefore $r(x)$ is already a solution of the equation $P = 0$. Otherwise, there is a vertex of $\mathcal{N}(Q)$ of the form $(\beta, 0)$. Let $(\beta', 0)$ be the intersection of $L_{\nu_k}(Q)$ with the horizontal axis. Since $\text{ht}(\text{Bot}(E_{\nu_k}(Q))) \geq 1$, the point $(0, \beta') \notin \mathcal{C}(Q)$, and by (1) in Lemma 3 we get $\beta' < \beta$, which implies the existence of a side $E_\nu(Q)$ of $\mathcal{N}(Q)$ with anti-slope $\nu > \nu_k$. \square

The next Lemma provides several relations between $P = 0$ and some of the equations $\frac{\partial P}{\partial y_i} = 0$, which will be useful in the proof of the main Proposition of this section.

Lemma 5. *Let $r(x)$ be an admissible generalized polynomial for P , and set $Q = P[r(x)]$. Let $\nu > \max(\text{supp } r(x))$ and assume there exists a point $V = (\alpha, r) \in E_\nu(Q) \cap \mathcal{C}(Q)$, with $r \geq 2$. Take ρ with $|\rho| = r$ such that $Q_{(\alpha, \rho)} \neq 0$, which exists because $V \in \mathcal{C}(Q)$. Let i be an index $0 \leq i \leq n$ such that $\rho_i \geq 1$, and set $\bar{P} = \frac{\partial P}{\partial y_i}$ and $\bar{Q} = \bar{P}[r(x)]$.*

Then $r(x)$ is an admissible generalized polynomial for \bar{P} , and the point $\bar{V} = (\alpha, r - 1)$ belongs to $E_\nu(\bar{Q}) \cap \mathcal{C}(\bar{Q})$. Moreover, if V is a vertex of $\mathcal{N}(Q)$ then \bar{V} is a vertex of $\mathcal{N}(\bar{Q})$.

Proof. The cloud of points of \bar{P} is a subset of the image of the points of $\mathcal{C}(P)$ with height at least 1 under the map $\Delta(\alpha, r) = (\alpha, r - 1)$. By the chain rule $\bar{Q} = \frac{\partial Q}{\partial y_i}$, hence $\mathcal{C}(\bar{Q})$ is a subset of the image of the points of $\mathcal{C}(Q)$ with height at least one under Δ . Moreover $\bar{Q}_{(\alpha, \rho - e_i)} = \rho_i Q_{(\alpha, \rho)}$, so that the point $\bar{V} = (\alpha, r - 1)$ belongs to $\mathcal{C}(\bar{Q}) \cap \Delta(L_\nu(Q))$. As Δ is an affine translation, the existence of \bar{V} implies that $\Delta(L_\nu(Q))$ is the supporting line of anti-slope ν of $\mathcal{C}(\bar{Q})$. Therefore $\bar{V} \in E_\nu(\bar{Q})$.

As $\text{ht } \bar{V} = r - 1 \geq 1$, and $\bar{V} \in E_\nu(\bar{Q})$, we have $\text{ht}(\text{Top}(E_\nu(\bar{Q}))) \geq 1$. Let $\nu_k = \max \text{supp } r(x)$. Since $\nu > \nu_k$, by (4) of Lemma 3 we get $\text{ht Bot}(E_{\nu_k}(\bar{Q})) \geq \text{Top}(E_\nu(\bar{Q})) \geq 1$. Hence by Lemma 4, the generalized polynomial $r(x)$ is admissible for \bar{P} .

Assume now that V is a vertex of $\mathcal{N}(Q)$, so that there exists $\nu' > \max \text{supp } r(x)$ with $\nu \neq \nu'$ and $V \in E_\nu(Q) \cap E_{\nu'}(Q)$. We conclude that $\bar{V} \in E_\nu(\bar{Q}) \cap E_{\nu'}(\bar{Q})$ which proves that \bar{V} is a vertex of $\mathcal{N}(\bar{Q})$. \square

4.1. Admissible polynomials with maximal rational rank and characteristic polynomials

The goal of this subsection is to prove that if $r(x)$ is an admissible generalized polynomial for the equation $P = 0$ which has maximum rational rank then the characteristic polynomials of the relevant sides of the Newton polygon of $P[r(x)]$ always have non-zero roots. That is, after the last exponent of $r(x)$, the equation $P[r(x)] = 0$ behaves as an algebraic curve and $r(x)$ can always be completed to a solution of $P = 0$.

Throughout this subsection, $r(x)$ will be denote an admissible generalized polynomial for $P = 0$ such that

$$\dim_{\mathbb{Q}} \frac{\langle \text{supp } P \cup \text{supp } r(x) \rangle}{\langle \text{supp } P \rangle} = m \geq n, \tag{12}$$

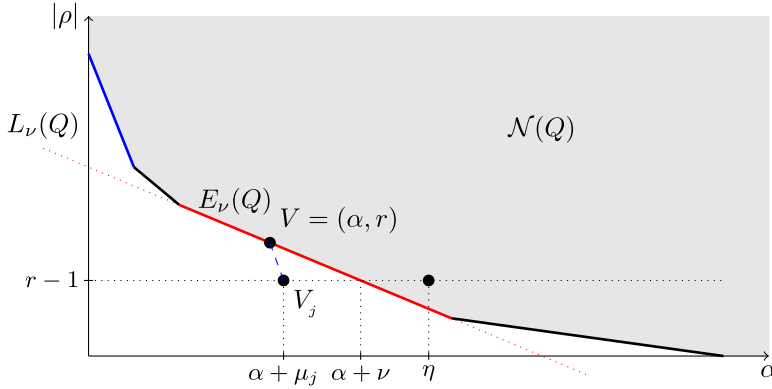


Fig. 1. Newton Polygon of $Q = P[r(x)]$ and some elements that appear in the proof of Lemma 6.

where n is the order of $P = 0$. Let $\Gamma = \langle \text{supp } P \rangle$. Recall that we can rewrite $r(x)$ using Eq. (5), such that the exponents satisfy properties (i)–(iv) after that Equation. Thus, we have:

$$r(x) = \sum_{j=0}^m (c_{j,0} x^{\mu_{j,0}} + c_{j,1} x^{\mu_{j,1}} + \dots + c_{j,t_j} x^{\mu_{j,t_j}}), \quad (13)$$

and we set, from now on, $Q = P[r(x)]$.

As in the previous section, we denote $\mu_j = \mu_{j,0}$ and either $\Gamma' = \langle \Gamma \cup \{\mu_1, \dots, \mu_m\} \rangle$ if $1 \in \langle \Gamma \cup \{\mu_1, \dots, \mu_m\} \rangle$ or $\Gamma' = \langle \Gamma \cup \{\mu_1, \dots, \mu_m, 1\} \rangle$, otherwise. The derivations \mathcal{D}_j , $j = 1, \dots, m$, on $\mathbb{C}((x^{\Gamma'}))$ are defined as in Lemma. Notice that $Q \in \mathbb{C}((x^{\Gamma'}))[[y_0, \dots, y_n]]$. Given $\rho = (\rho_0, \rho_1, \dots, \rho_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ and $\alpha \in \Gamma'$, we write $Q_\rho \in \mathbb{C}((x^{\Gamma'}))$ for the coefficient of $y_0^{\rho_0} y_1^{\rho_1} \dots y_n^{\rho_n}$ in Q , and $Q_{(\alpha, \rho)} \in \mathbb{C}$ for the coefficient of $x^{\alpha + \varepsilon \omega(\rho)}$ in Q_ρ . Recall that e_i is a vector with $n + 1$ components, all zero except the $i + 1$ -th, which is 1.

The existence of the admissible generalized polynomial $r(x)$ generates conditions on the coefficients of Q at elements of anti-slope greater than $\max \text{supp } r(x) = \mu_{m,t_m}$.

Lemma 6. *Let $v > \max \text{supp } r(x) = \mu_{m,t_m}$ and consider the element $E_v(Q)$ of anti-slope v of $\mathcal{N}(Q)$. Assume there is a point $V = (\alpha, r) \in E_v(Q)$, with $r = \text{ht}(V) \geq 1$. Then for all $\rho = (\rho_0, \rho_1, \dots, \rho_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\rho| = r - 1$, and for all $j = 1, 2, \dots, m$, the following equality holds:*

$$\sum_{\kappa=0}^n (\rho_\kappa + 1) \delta_{\mu_j}^{(\kappa)} Q_{(\alpha, \rho + e_\kappa)} = 0. \quad (14)$$

Remark 2. Notice that each $\rho = (\rho_0, \dots, \rho_n)$ with $|\rho| = r - 1$ gives rise to m conditions (14) on the coefficients of Q_V .

Proof. Fix for all this proof a $\rho \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\rho| = r - 1$. Let $V_j = (\alpha + \mu_j, r - 1)$ for $j = 1 \dots, m$. Notice that $V_j \notin \mathcal{N}(Q)$ because $V \in E_v(Q)$ and $v > \mu_{m,t_m} \geq \mu_j$ (see Fig. 1). Thus, $Q_{(\alpha + \mu_j, \rho)} = 0$.

Since μ_1, \dots, μ_m and Γ fulfil the hypothesis of [Lemma 1](#), we may apply the derivations \mathcal{D}_j to Q , and for each $j = 1, \dots, m$, [Eq. \(3\)](#) becomes

$$\mathcal{D}_j(Q_\rho) = \sum_{\kappa=0}^n (\rho_\kappa + 1) (Q_{\rho+e_\kappa}) \mathcal{D}_j(r^{(\kappa)}(x)). \tag{15}$$

Since $V = (\alpha, r)$ is on the border of $\mathcal{N}(Q)$ and $|\rho + e_\kappa| = r$, we have:

$$Q_{\rho+e_\kappa} = Q_{(\alpha, \rho+e_\kappa)} x^{\alpha+\varepsilon \omega(\rho+e_\kappa)} + \dots = Q_{(\alpha, \rho+e_\kappa)} x^{\alpha+\varepsilon \omega(\rho)+\varepsilon \kappa} + \dots$$

where the last equality holds because $\omega(\rho + e_\kappa) = \omega(\rho) + \kappa$. Because of Properties (i)–(iv) in the expression of $r(x)$ the following equality holds:

$$\mathcal{D}_j(r^{(\kappa)}(x)) = c_{j,0} \delta_{\mu_j}^{(\kappa)} x^{\mu_j - \varepsilon \kappa} + \dots,$$

where recall that $c_{j,0} \neq 0$ for each $j = 1, \dots, m$. Hence, the right hand side of [Eq. \(15\)](#) is equal to

$$c_{j,0} \left(\sum_{\kappa=0}^n (\rho_\kappa + 1) \delta_{\mu_j}^{(\kappa)} Q_{(\alpha, \rho+e_\kappa)} \right) x^{\alpha+\mu_j+\varepsilon \omega(\rho)} + \dots, \tag{16}$$

and the left hand side of [Eq. \(15\)](#) has, by definition of \mathcal{D}_j , order at least $\text{ord}_x Q_\rho$. Thus, if $Q_\rho = 0$ we are done. So we assume $Q_\rho \neq 0$ from now on. Set $\text{ord}_x Q_\rho = \eta + \varepsilon \omega(\rho)$.

It is enough to prove that $\eta > \alpha + \mu_j$ because this implies that the first term of [\(16\)](#) is zero which combined with [\(15\)](#) gives [Eq. \(14\)](#). By definition of $L_\nu(Q)$, any point in $\mathcal{C}(Q)$ belongs to the closed right halfplane defined by $L_\nu(Q)$. Since $V = (\alpha, r) \in E_\nu(Q) \subset L_\nu(Q)$, the point $(\alpha + \nu, r - 1)$ lies in $L_\nu(Q)$ because $L_\nu(Q)$ is a line with anti-slope ν . Therefore, any point with ordinate $r - 1$ in $\mathcal{N}(Q)$ has abscissa greater than or equal to $\alpha + \nu$. As $Q_\rho \neq 0$ then $Q_{(\eta+\varepsilon \omega(\rho), \rho)} \neq 0$ so that $(\eta, r - 1) \in \mathcal{C}(Q)$ and hence $\eta \geq \alpha + \nu > \alpha + \mu_j$. \square

The key consequence of [Lemma 6](#) is the following result.

Proposition 1. *Let $\nu > \max \text{supp } r(x)$ and $V = (\alpha, r) \in E_\nu(Q)$ with $Q_V \neq 0$. The indicial polynomial $\Psi_{(Q;V)}(T)$ is a power of the polynomial $(T - \delta_{\mu_1}) \cdots (T - \delta_{\mu_n})$ up to a non-zero constant c :*

$$\Psi_{(Q;V)}(T) = c \left(\prod_{j=1}^n (T - \delta_{\mu_j}) \right)^r.$$

Proof. We proceed by induction on r . The case $r = 0$ is trivial because $\Psi_{(Q;V)}(T) = Q_{(\alpha, (0, \dots, 0))}$ is a non-zero constant under our hypothesis $Q_V \neq 0$.

We need to prove the case $r = 1$ because our induction argument requires $r \geq 2$.

Assume $r = 1$. The indicial polynomial of Q at $V = (\alpha, 1)$ is, by definition,

$$\Psi_{(Q;V)}(T) = \sum_{k=0}^n Q_{(\alpha, e_k)} T^{(e_k)}.$$

Applying [Lemma 6](#) to $\rho = (0, \dots, 0)$ we get $\Psi_{(Q;V)}(\delta_{\mu_j}) = 0$ for $j = 1, \dots, n$ (i.e. each δ_{μ_j} is a root of $\Psi_{(Q;V)}(T)$). Since the polynomial $\Psi_{(Q;V)}(T)$ has degree at most n , we must have:

$$\Psi_{(Q;V)}(T) = \sum_{k=0}^n Q_{(\alpha, e_k)} T^{(e_k)} = Q_{(\alpha, e_n)} \prod_{j=1}^n (T - \delta_{\mu_j}),$$

and we need to show that $Q_{(\alpha, e_n)} \neq 0$. Assume for contradiction that $Q_{(\alpha, e_n)} = 0$, then $\Psi_{(Q; V)}(T) = 0$ and all the coefficients $Q_{(\alpha, e_\kappa)}$ should be zero for $\kappa = 0, \dots, n$, which contradicts the fact that $Q_V \neq 0$. This finishes case $r = 1$.

Assume the result holds for $r - 1$ and consider the general case. Let $V = (\alpha, r)$, with $r \geq 2$ and all other notations as before. For any $\rho \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\rho| = r - 1$, Eq. (14) for $j = 1, \dots, n$ implies the following equality of polynomials in the variable μ of degree at most n :

$$\sum_{\kappa=0}^n (\rho_\kappa + 1) T^{(e_\kappa)} Q_{(\alpha, \rho+e_\kappa)} = (\rho_n + 1) Q_{(\alpha, \rho+e_n)} \prod_{j=1}^n (T - \delta_{\mu_j}), \tag{17}$$

because δ_{μ_j} , for $j = 1, \dots, n$, are different roots of the left hand side.

We are going to prove that $Q_{(\alpha, (0, \dots, 0, r))} \neq 0$ and apply Lemma 5 to use the induction hypothesis. Arguing by contradiction, assume that $Q_{(\alpha, (0, \dots, 0, r))} = 0$. Take $\gamma \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\gamma| = r$ maximal with respect to the reverse lexicographical order such that $Q_{(\alpha, \gamma)} \neq 0$. This γ exists because $Q_V \neq 0$, and $\gamma \neq (0, 0, \dots, r)$ by the assumption. Let $\rho = \gamma - e_\kappa$ for some $\kappa < n$ with $\gamma_\kappa > 0$, which exists too because $\gamma_n < r$. Eq. (17) applied to this ρ implies that if $Q_{(\alpha, \rho+e_n)} = 0$, then the left hand side of (17) is the null polynomial so that $Q_{(\alpha, \gamma)} = 0$ because $\gamma = \rho + e_\kappa$. Hence $Q_{(\alpha, \rho+e_n)} \neq 0$. This contradicts the definition of γ . Thus our assumption is false and $Q_{(\alpha, (0, \dots, 0, r))} \neq 0$.

Multiplying both sides of Eq. (17) by $T^{(\rho)}$, as $T^{(\rho)} T^{(e_\kappa)} = T^{(\rho+e_\kappa)}$, we get

$$\sum_{\kappa=0}^n (\rho_\kappa + 1) T^{(\rho+e_\kappa)} Q_{(\alpha, \rho+e_\kappa)} = (\rho_n + 1) Q_{(\alpha, \rho+e_n)} T^{(\rho)} \prod_{j=1}^n (T - \delta_{\mu_j}). \tag{18}$$

The sum of all the left hand sides of (18) for ρ with $|\rho| = r - 1$ gives:

$$\sum_{|\rho|=r-1} \sum_{\kappa=0}^n (\rho_\kappa + 1) T^{(\rho+e_\kappa)} Q_{(\alpha, \rho+e_\kappa)} = r \sum_{|\gamma|=r} T^{(\gamma)} Q_{(\alpha, \gamma)} = r \Psi_{(Q; V)}(T),$$

where the second equality is the definition of the indicial polynomial, and the first one follows from collapsing the preimages of the map $(\rho, \kappa) \mapsto \gamma = \rho + e_\kappa$. As a consequence, summing all the corresponding right hand sides of (18) we get

$$r \Psi_{(Q; V)}(T) = \prod_{j=1}^n (T - \delta_{\mu_j}) \sum_{|\rho|=r-1} (\rho_n + 1) Q_{(\alpha, \rho+e_n)} T^{(\rho)}. \tag{19}$$

We are now going to apply the induction hypothesis in order to prove that the summation on the right hand side is, up to a non-constant factor, an $r - 1$ power of $\prod_{j=1}^n (T - \delta_{\mu_j})$. To this end, recall that $Q_{(\alpha, (0, 0, \dots, r))} \neq 0$ and consider $\bar{P} = \frac{\partial P}{\partial y_n}$. By the chain rule, $\bar{Q} = \bar{P}[r(x)] = \frac{\partial Q}{\partial y_n}$, and in particular, $\bar{Q}_{(\alpha, (0, 0, \dots, r-1))} = r Q_{(\alpha, (0, 0, \dots, r))} \neq 0$. Lemma 5 guarantees that $r(x)$ is an admissible generalized polynomial for \bar{P} . Let $\bar{V} = (\alpha, r - 1)$. We know that then $\bar{Q}_{\bar{V}} \neq 0$, and by Lemma 5 again, \bar{V} belongs to an element $E_\nu(\bar{Q})$ of anti-slope ν . These properties allow us to apply the induction hypothesis to \bar{P} , $r(x)$ and \bar{V} which means that the indicial polynomial $\Psi_{(\bar{Q}; \bar{V})}(T)$ is, up to a non-zero constant, an $r - 1$ -th power of $\prod_{j=1}^n (T - \delta_{\mu_j})$. Thus, on one hand, by definition and because $\bar{Q} = \frac{\partial Q}{\partial y_n}$ we have

$$\Psi_{(\bar{Q}; \bar{V})}(T) = \sum_{|\rho|=r-1} \bar{Q}_{(\alpha, \rho)} T^{(\rho)} = \sum_{|\rho|=r-1} (\rho_n + 1) Q_{(\alpha, \rho+e_n)} T^{(\rho)}, \tag{20}$$

and, on the other, by the induction hypothesis, there is a non-zero constant c with

$$\Psi_{(\overline{Q}; \overline{v})}(T) = c \left(\prod_{j=1}^n (T - \delta_{\mu_j}) \right)^{r-1}. \quad (21)$$

The proof of the first statement in the Proposition finishes by connecting (20) and (21), and inserting the result into (19). \square

Corollary 5. *In the conditions of Proposition 1, if $E_v(Q)$ is a side of $\mathcal{N}(Q)$, then the characteristic polynomial $\Phi_{(Q;v)}(C)$ has at least one non-zero root.*

Proof. By definition:

$$\Phi_{(Q;v)}(C) = \sum_{W \in E_v(Q)} \Psi_{(Q;W)}(\delta_v) C^{\text{ht}(W)}.$$

We have proved above that for all $W \in E_v(Q) \cap \mathcal{C}(Q)$, $\Psi_{(Q;W)}(\delta_v) \neq 0$, because $v > \mu_j$, for all $j = 1, \dots, n$. Since $E_v(Q)$ is a side, it has at least two vertices, whence $\Psi_{(Q;v)}(C)$ has at least two non-zero monomials, and thus it has at least one non-zero root. \square

Proof of Theorem 3. Let $Q = P[r(x)]$. Consider the following procedure, analogue to the Newton–Puiseux construction for algebraic curves:

Procedure 1. COMPLETION OF $r(x)$ TO A SOLUTION OF $P = 0$.

Setup. SET $r_0(x) = r(x)$, $Q_1 = Q$, $\eta_0 = \max(\text{supp}(r(x)))$, and $i = 1$.

1. IF $y = 0$ is a solution of $Q_i = 0$ THEN RETURN $s(x) = r_{i-1}(x)$ ELSE
2. CHOOSE a side of $\mathcal{N}(Q_i)$ with anti-slope $\eta_i > \eta_{i-1}$.
3. CHOOSE a non-zero root d_i of the characteristic polynomial $\Phi_{(Q_i; \eta_i)}(C)$.
4. SET $r_i(x) = r_{i-1}(x) + d_i x^{\eta_i}$ and $Q_{i+1} = Q_i[d_i x^{\eta_i}]$.
5. SET $i = i + 1$ and GOTO 1.
6. RETURN $s(x) = r(x) + \sum_{i=1}^{\infty} d_i x^{\eta_i}$.

We shall show that in each iteration, the main loop of the above procedure can be performed, that the output $s(x)$ belongs to Ω , and it is a solution of the equation $P = 0$. From this follows, by Theorem 1, that $m \leq n$, hence $m = n$.

Let us prove that each step can be performed. Assume $r_{i-1}(x)$, Q_i , η_{i-1} are defined for some $i \geq 1$ and $r_{i-1}(x)$ is admissible for $P = 0$. If the condition of line 1 holds the procedure finished. Otherwise, as $r_{i-1}(x)$ is admissible for $P = 0$ there exists $\eta_i > \eta_{i-1}$ by Corollary 4 such that $\mathcal{N}(Q_i)$ has a side of anti-slope η_i . This allows line 2 of Procedure 1 to be performed. Proposition 1 guarantees the existence of a non-zero root d_i of the characteristic polynomial $\Phi_{(Q_i; \eta_i)}(C)$ which gives line 3.

We have proved that the above method either returns a generalized polynomial $r_{i-1}(x)$ which is a solution of $P = 0$ or, after infinitely many steps, it returns a formal power series $s(x) = r(x) + \sum_{i=1}^{\infty} d_i x^{\eta_i}$. From the definition of η_0 and the fact that $\eta_0 < \eta_j$, it is obvious that $r(x)$ is a truncation of $s(x)$ but it remains to prove that $s(x) \in \Omega$ and that $s(x)$ is a solution of $P = 0$. These facts follow from the following Proposition, which can be found in the literature under slightly different hypotheses: In [15] the equation $P = 0$ is differential and has coefficients in the field $\mathbb{C}((x^{\mathbb{Z}}))$, in [4] the series $s(x)$ has rational exponents, and in [6] $P = 0$ is a q -difference equation whose coefficients are grid-based series. The arguments presented in

those proofs are equally valid in our setting with minimal modification. For the convenience of the reader we sketch the proof of the specific statement we need, thus ending the proof of [Theorem 3](#). \square

Proposition 2. *Let $P \in \Omega[y_0, y_1, \dots, y_n]$ and $s(x) = \sum_{i=0}^{\infty} c_i x^{v_i}$ be a series with $v_0 < v_1 < \dots$ such that $s(x)$ satisfies the necessary initial conditions [\(10\)](#) for any $i \geq 0$. Then $s(x) \in \Omega$ and it is solution of the equation $P = 0$.*

Proof. First, we are going to define a special point which we call the *pivot point* of P along the series $\sum_{i=0}^{\infty} c_i x^{v_i}$ and will be central to our arguments.

As previously, we denote $P_0 = P$ and $P_{i+1} = P_i[c_i x^{v_i}]$. By [Lemma 3](#) we know that $\text{ht Top}(E_{v_i}(P_i)) \geq \text{ht Top}(E_{v_{i+1}}(P_{i+1})) \geq 1$, for any $i \geq 0$. Hence there exists an index i_0 such that this sequence of heights stabilizes, and therefore the points $\text{Top}(E_{v_i}(P_i))$ stabilize too: there is a point $V = (\alpha, r)$ such that $\text{Top}(E_{v_{i_0}}(P_{i_0})) = \text{Top}(E_{v_i}(P_i)) = V$ for all $i \geq i_0$. This V is the *pivot point* of P along $s(x)$.

We know that $r \geq 1$ by [Lemma 3](#). Notice that for any $k \geq i_0$, setting $\alpha_k = \text{ord}_x P(\sum_{i=0}^{k-1} c_i x^{v_i})$ one has $\alpha_k \geq \alpha + r v_k$ because the point $(\alpha_k, 0)$ is in $\mathcal{C}(P_k)$ and the supporting line $L_{v_k}(P_k)$ passes through V . Hence, in order to prove the Proposition it is enough to show that $s(x) \in \Omega$, that is $\lim v_i = \infty$, because then automatically $\lim \alpha_k = \infty$ which makes $s(x)$ a solution of $P = 0$.

We may assume that $r = 1$, because if $r \geq 2$ we can apply $r - 1$ times [Lemma 5](#) to find some derivative $P' = \frac{\partial^{r-1} P}{\partial y_0^{j_0} \dots \partial y_n^{j_n}}$ of P , such that, the height of the pivot point of P' along $s(x)$ is one. Thus from now on we assume that $r = 1$.

We may also assume that $\alpha = 0$ so that $V = (0, 1)$. Let us sketch why (a detailed argument can be found in [\[6, pages 15 and 16\]](#)). Performing the change of variable $\bar{y} = x^{v_{i_0}} y$ on $P = 0$, and multiplying the resulting equation by $x^{-\beta}$, with $\beta = \alpha + v_{i_0}$, we obtain a new equation $\bar{P} = 0$ whose pivot point along $\bar{s}(x) = x^{-v_{i_0}} s(x)$ is $(0, 1)$ and such that the coefficients of \bar{P}_{i_0} are elements of Ω with non-negative order in x . Using \bar{P}_{i_0} instead of P , we may assume that $P = 0$ is in *quasi-solved form*: $\mathcal{N}(P)$ is contained in the first quadrant, and in particular the coefficients of P have non-negative order in x , and $V = (0, 1)$ is a vertex of $\mathcal{N}(P)$, and $v_0 \geq 0$.

We first prove the Proposition assuming that P is in quasi-solved form and with the additional hypothesis that its coefficients are grid-based, that is: there exists a finitely generated semigroup G of $\mathbb{R}_{\geq 0}$ such that $\text{supp } P \subset G$. In this case we can prove that $s(x)$ is also grid-based. The general case will follow from this one.

By [Lemma 3](#), the point $V = (0, 1)$ is a vertex of $\mathcal{N}(P_i)$ for all $i \geq 0$. Also by [Lemma 3](#), the equality $P_{(0, e_\kappa)} = (P_i)_{(0, e_\kappa)}$ holds for all $i \geq 0$ and $\kappa = 0, 1, \dots, n$. Hence the indicial polynomials $\Psi_{(P; V)}(T)$ and $\Psi_{(P_i; V)}(T)$ are equal for all $i \geq 0$. Let $\Sigma = \{\mu \in \mathbb{R}_{\geq 0} \mid \Psi_{(P; V)}(\delta_\mu) = 0\}$. This set is finite because the set of roots of the polynomial $\Psi_{(P; V)}(T) = \sum_{\kappa=0}^n P_{(0, e_\kappa)} T^\kappa$ is finite (it is not identically null since $(0, 1)$ is a vertex), and $|q| \neq 1$ in the case of q -difference equations. Let G' be the semigroup generated by G and Σ , which is finitely generated.

Let us prove inductively that $\text{supp } P_i \subset G'$ and that $v_i \in G'$. The case $i = 0$ holds by hypothesis. Let $i \geq 0$ and assume that $\text{supp } P_i \subset G'$; let us first prove that $v_i \in G'$. By hypothesis, $\Phi_{(P_i; v_i)}(c_i) = 0$. If the element $E_{v_i}(P_i)$ is a side then its vertices are $(0, 1)$ and $(\alpha_i, 0)$, so that $v_i = \alpha_i \in \text{supp } P_i \subset G'$. Otherwise $E_{v_i}(P_i)$ is the vertex $\{V\}$, which implies

that $\Phi_{(P_i;v_i)}(c_i) = \Psi_{(P_i;V)}(\delta_{v_i})c_i$, so that $v_i \in \Sigma$. In both cases, $v_i \in G'$. Since $P_{i+1} = P_i[c_i x^{v_i}]$ then $\text{supp } P_{i+1} \subset G'$. This shows that $\text{supp } s(x) \subset G'$ which implies that $\lim v_i = \infty$ as desired.

It only remains to prove the Proposition when P has coefficients in Ω , not necessarily grid-based. Take $N > 0$ and denote by R the polynomial obtained by truncating in the variable x the coefficients of P up to order N . Define as usually $R_0 = R$ and $R_{i+1} = R_i[c_i x^{v_i}]$ for $i \geq 0$. Since the coefficients of P have non-negative order in x and $v_i \geq 0$, then the truncation up to order N of R_i and that of P_i coincide for all $i \geq 0$. In particular, if $v_k \leq N$ then, $\Phi_{(R_i;v_i)}(c_i) = 0$ for $0 \leq i \leq k$. Since $\text{supp}(R)$ is finite, it generates a finitely generated semigroup G of $\mathbb{R}_{\geq 0}$. As before, we may prove by induction that $v_i \in G'$ for $0 \leq i \leq k$. Since the set $G' \cap \mathbb{R}_{\leq N}$ finite, then the set $\text{supp } s(x) \cap \mathbb{R}_{\leq N}$ is finite for any N and therefore either $\lim v_i = \infty$ or $\text{supp } s(x)$ is finite; in both cases $s(x) \in \Omega$ and we are done.

4.2. On the completion of admissible polynomials to solutions

Grigoriev and Singer in [15] provide, in the case of differential equations, a criterion for an admissible polynomial to be the truncation of an actual solution of $P = 0$, based on the following definition. As before, $r(x) = \sum_{i=0}^{k-1} c_i x^{v_i}$ is an admissible generalized polynomial for the equation $P = 0$, and we denote $P_0 = P$ and $P_{i+1} = P_i[c_i x^{v_i}]$, for $0 \leq i \leq k - 1$. Let $V = \text{Bot}(E_{v_{k-1}}(P_k))$ be the bottom vertex of $E_{v_{k-1}}(P_k)$.

Definition 1. We say that $r(x)$ stabilizes P if the height $\text{ht}(V)$ is 1, and v_{k-1} is greater than the maximum of the finite set $\Sigma = \{\mu \in \mathbb{R}_{\geq 0} \mid \Psi_{(P;V)}(\delta_\mu) = 0\}$.

In Section 3 of [15] the authors prove the following *stabilization criterion* (for differential equations): if $r(x)$ stabilizes P then it is the truncation of a unique solution $s(x)$ of $P = 0$. For the benefit of the reader we include a proof which covers also the q -difference case.

Proof of the Stabilization Criterion of [15]. We need to show the existence and uniqueness of a solution $s(x) = r(x) + \sum_{i=k}^{\infty} c_i x^{v_i} \in \Omega$, with $v_i < v_{i+1}$ for all i . By Proposition 2, it is enough to prove that $r(x)$ can be completed uniquely to an $s(x)$ such that $s(x)$ satisfies the necessary initial conditions (10).

Notice that whenever $v_k > v_{k-1}$ and c_k satisfy the necessary initial condition $\Phi_{(P_k;v_k)}(c_k) = 0$ then the generalized polynomial $r(x) + c_k x^{v_k}$ stabilizes P . Thus, by recurrence, we only need to prove that: either (I) $P(r(x)) = 0$ and the condition $\Phi_{(P_k;v_k)}(c_k) = 0$ does not hold for any $v_k > v_{k-1}$ and $c_k \neq 0$; or (II) $P(r(x)) \neq 0$ and there only exist one $v_k > v_{k-1}$ and one $c_k \neq 0$ with $\Phi_{(P_k;v_k)}(c_k) = 0$.

Let $v_k > v_{k-1}$. Because $\text{ht } V = 1$, there are only three cases for the element $E_{v_k}(P_k)$: (a) $E_{v_k}(P_k)$ is a vertex of $\mathcal{N}(P_k)$ with ordinate zero; (b) $E_{v_k}(P_k)$ is the vertex V ; (c) $E_{v_k}(P_k)$ is the side joining V and the vertex of $\mathcal{N}(P_k)$ on the horizontal axis, say $(\beta, 0)$.

In case (a), $\Phi_{(P_k;v_k)}(C)$ is a non-zero constant polynomial, so it has no roots. In case (b), $\Phi_{(P_k;v_k)}(C) = \Psi_{(P_k;V)}(\delta_{v_k})C$ whose only root is $C = 0$ as $\Psi_{(P_k;V)}(\delta_{v_k}) = \Psi_{(P_{k-1};V)}(\delta_{v_k}) \neq 0$ because $r(x)$ stabilizes P . In case (c) we have

$$\Phi_{(P_k;v_k)}(C) = \Psi_{(P_k;V)}(\delta_{v_k})C + P_{k(\beta,(0,\dots,0))},$$

both of whose coefficients are non-zero by assumption so that it has a single root.

If $P(r(x)) = 0$ then we are necessarily in case (b) because $\mathcal{C}(P_k)$ has no points with ordinate zero, whence (I) holds.

Assume that $P(r(x)) = dx^\gamma + \dots$ with $d \neq 0$, hence $(\gamma, 0)$ is a vertex of $\mathcal{N}(P_k)$. As $V = (\alpha, 1)$ for some α and $V \in L_{\nu_{k-1}}(P_k)$, then $\gamma > \alpha + \nu_{k-1}$. If $\nu_k \neq \gamma - \alpha$ then we are either in case (a) or case (b), so that $\Phi_{(P_k; \nu_k)}(C)$ has no non-zero roots. As $\gamma - \alpha > \nu_{k-1}$, there only remains the possibility $\nu_k = \gamma - \alpha$, giving case (c) with $\beta = \gamma$, and $\Phi_{(P_k; \nu_k)}(C)$ has a single non-zero root, which is (II). \square

The criterion presented in [Theorem 3](#) is of a different nature than this one: it only depends on the dimension of the \mathbb{Q} -vector space spanned by the support of $r(x)$ and the order of the equation, while the above criterion required knowing the whole equation P , $r(x)$, and the substitution $P_k = P[r(x)]$.

In [Section 5](#) we present an example of an admissible generalized polynomial that satisfies the criterion of [Theorem 3](#) and that can be completed in two different ways to a solution of the differential equation so that the stabilization criterion of Grigoriev and Singer does not apply.

4.3. Autonomous ordinary differential equations of first order

In this subsection we will apply our previous results to the case of first order autonomous ordinary differential equations. In the next Theorem we prove that any solution $s(x) \in \Omega$ of that kind of equation is a formal Puiseux series (statement (1)). We also provide alternative proofs of two known results: Theorems 11 and 12 in [\[7\]](#) (statements (2) and (3)).

Let $P = 0$ be a non-trivial autonomous ordinary differential equation of first order, that is, a first order differential equation invariant by the translations $x \mapsto x + c$, $c \in \mathbb{C}$. Hence P can be written as a non-zero polynomial with constant coefficients in y_0 and y_1 , where y_1 refers to the derivative $\frac{d}{dx}$. Let us denote by $\hat{K}^* = \cup_{d \geq 1} \mathbb{C}((x^{\frac{1}{d}\mathbb{Z}}))$ the field of formal Puiseux power series, i. e. the field of grid-based elements of $\mathbb{C}((x^{\mathbb{Q}}))$.

Theorem 4. *Let $P(y_0, y_1) = 0$ be non-trivial an autonomous first order ordinary differential equation. Then*

- (1) *If $s(x) \in \Omega$ is a solution of the equation $P = 0$, then $s(x) \in \hat{K}^*$.*
- (2) *If $s(x) \in \hat{K}^*$ is a solution of $P = 0$, then $s(x)$ is a convergent Puiseux power series.*
- (3) *For any point $(x_0, c_0) \in \mathbb{C}^2$, there exists an convergent solution $s(x)$ of $P = 0$ passing through the point (x_0, c_0) .*

Proof. Part (2) is consequence of our [Theorem 2](#), see [Corollary 3](#). In order to prove parts (1) and (3), let us note some specific properties of the Newton polygon of P in this case.

Since P is a polynomial with constant coefficients, we can write

$$P = \sum_{i=0}^{d_0} \sum_{j=0}^{d_1} P_{i,j} y_0^i y_1^j, \quad P_{i,j} \in \mathbb{C}. \tag{22}$$

Then any term $P_{i,j} y_0^i y_1^j$ corresponds to the point $(-j, i+j)$ in $\mathcal{C}(P)$, which provides a bijection between the non-zero terms of P and $\mathcal{C}(P)$. Let $V \in \mathcal{C}(P)$ be a point lying in some side or vertex of $\mathcal{N}(P)$. There exist unique integers $0 \leq i \leq d_1$ and $0 \leq j \leq d_2$ such that $V = (-j, i+j)$. Then, the indicial polynomial $\Psi_{(P;V)}(T) = P_{i,j} T^j$, with $P_{i,j} \neq 0$ because $V \in \mathcal{C}(P)$. We are assuming that P is effectively a differential equation, that is, y_1 appears in P , so that there is a point with strictly negative abscissa in $\mathcal{C}(P)$. This implies that for any $\mu \in \mathbb{R}$, the characteristic polynomial $\Phi_{(P;\mu)}(C)$ has a non-zero root if and only if $\mu = 0$ or μ is the anti-slope of a side of $\mathcal{N}(P)$.

Let us prove part (1). Let $s(x) = c_0 x^{\mu_0} + \dots \in \Omega$, with $c_0 \neq 0$, be a non-constant solution of $P = 0$. We may assume that $\mu_0 \neq 0$, otherwise, we perform the change of variable $P[c_0] = P(y_0 + c_0, y_1)$ to obtain a polynomial with constant coefficients which has $s(x) - c_0 = c_1 x^{\mu_1} + \dots$, with $\mu_1 > 0$, as solution. By Lemma 2, c_0 is a non-zero root of the characteristic polynomial $\Phi_{(P; \mu_0)}(C)$. By the above properties of $\mathcal{N}(P)$, either $\mu_0 = 0$ or μ_0 is the anti-slope of a side of $\mathcal{N}(P)$. Since the sides of $\mathcal{N}(P)$ are rational, necessarily μ_0 is a non-zero rational number. Hence $r(x) = c_0 x^{\mu_0}$ is an admissible polynomial for $P = 0$. Since $\text{supp } P = \{0\}$ and $\langle \text{supp } r(x) \rangle = \mathbb{Q}$, then $r(x)$ satisfies the hypothesis of Theorem 3. Hence, $r(x)$ is the truncation of a solution $s(x)$ of $P = 0$. By Theorem 1, $\langle \text{supp } s(x) \rangle = \mathbb{Q}$, hence $s(x) \in \mathbb{C}(\langle x^{\mathbb{Q}} \rangle)$. Now, the arguments given in Proposition 2 for the grid-based case, shows that $s(x)$ is grid-based, hence $s(x) \in \hat{K}^*$.

Let us prove part (3). Since P is autonomous, we may assume that $x_0 = 0$. Performing the transformation $P(y_0 + c_0, y_1)$, we may assume that $c_0 = 0$. Write P as in (22). If $P_{0,0} = 0$, then $s(x) = 0$ is a solution and we are done. Assume that $P_{0,0} \neq 0$. Since P is of order one, the Newton polygon $\mathcal{N}(P)$ has a vertex of the form $(-j, i + j)$, with $j \geq 1$. Hence $\mathcal{N}(P)$ has at least one side with anti-slope $0 < \mu_0 \in \mathbb{Q}$. By the properties of $\mathcal{N}(P)$, the characteristic polynomial $\Phi_{(P; \mu_0)}(C)$ has a non-zero root c_0 , and as in part (2), there exists a power series $s(x) = c_0 x^{\mu_0} + \dots \in \hat{K}^*$ solution of $P = 0$. By Corollary 3, $s(x)$ is convergent. \square

5. Example

Consider the irrational number $\tau = \pi/2$ and the linear differential equations

$$\begin{aligned} L_1 &= \tau y_0 - y_1 - ((\tau - 1)x + (\tau - 2)x^2 + (\tau - 3)x^3), \\ L_2 &= \tau y_0 - y_1 - ((\tau - 1)x + (\tau - 2)x^2 + (\tau - 5)x^5), \end{aligned}$$

where y_1 refers to the Euler derivative $x \frac{d y(x)}{d x}$ and $y_0 = y(x)$, for $y(x) \in \Omega$. Set $P = L_1 L_2 + x^6 y_0 y_1$ and $r(x) = x + x^\tau$. Here $L_1 L_2$ is considered as a product of polynomials, not as a composition of differential operators. More precisely

$$\begin{aligned} P &= y_1^2 + \tau^2 y_0^2 + (-2\tau + x^6) y_0 y_1 \\ &+ ((2\tau - 2)x + (2\tau - 4)x^2 + (\tau - 3)x^3 + (\tau - 5)x^5) y_1 \\ &+ (2\tau(1 - \tau)x + 2\tau(2 - \tau)x^2 + \tau(3 - \tau)x^3 + \tau(5 - \tau)x^5) y_0 \\ &+ (1 - 2\tau + \tau^2)x^2 + (4 - 6\tau + 2\tau^2)x^3 + \dots + (15 - 8\tau + \tau^2)x^8. \end{aligned}$$

Let us verify that $r(x)$ satisfies the criterion of Theorem 3 for $P = 0$ but it can be continued to two different solutions, so that $r(x)$ does not satisfy the stabilization criterion. Recall that Theorem 3 guarantees also the convergence of those continuations.

As the order of the equation $P = 0$ is one and τ is irrational we only need to show that $r(x)$ is admissible. To this end, define $P_0 = P$, $P_1 = P_0[x]$, $P_2 = P_1[x^\tau]$. Writing only the terms corresponding to the border of each Newton polygon, we have

$$\begin{aligned} P_1 &= (\tau y_0 - y_1)^2 + 2(\tau - 2)x^2(y_1 - \tau y_0) + (\tau - 2)^2 x^4 + \dots, \\ P_2 &= (\tau y_0 - y_1)^2 + 2(\tau - 2)x^2(y_1 - \tau y_0) + (\tau - 2)^2 x^4 + \dots. \end{aligned}$$

From now on we refer the reader to Fig. 2 for the structure and elements of the Newton polygon of each the equations we shall compute.

The element $E_1(P_0)$ of anti-slope $\nu_1 = 1$ is the side of $\mathcal{N}(P_0)$ with vertices $(0, 2)$ and $(2, 2)$. Its characteristic polynomial is $\Phi_{(P; 1)}(C) = \tau^2(\tau^{-1} - 1)(C - 1)^2$, whose only root is

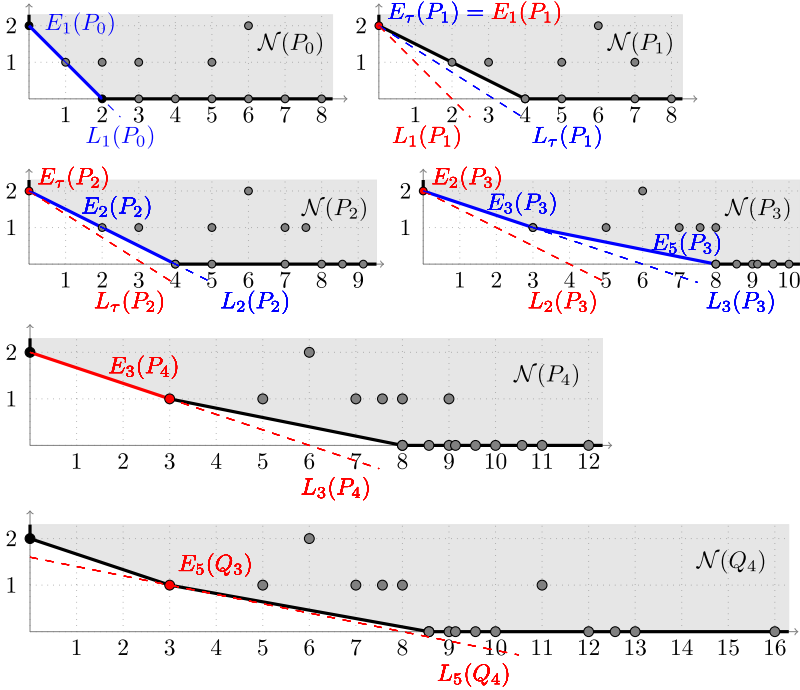


Fig. 2. Newton polygons of P_0, P_1, P_2 and P_3 following the admissible generalized polynomial $x + x^\tau + x^{2\tau} + x^{3\tau}$ until the stabilization step is reached.

$c_1 = 1$. Hence the polynomial $r_1(x) = x$ is admissible for P but does not satisfy either the criterion of Theorem 3 (it has no irrational exponents), or the stabilization criterion (because $\text{ht Bot}(E_1(P_1)) = 2$).

Now for P_1 the indicial polynomial for the vertex $V = (0, 2)$ of $\mathcal{N}(P_1)$ is $\Psi_{(P_1;V)}(\mu) = \tau - \mu$, whose root is $\mu = \tau$. Set $\nu_2 = \tau$. Since $1 < \tau < 2$, the element $E_\tau(P_1)$ is V , and its characteristic polynomial is $\Phi_{(P_1;\tau)}(C) = 0$ so that any $c_2 \in \mathbb{C}$ is a root. We choose $c_2 = 1$. Thus $r_2(x) = x + x^\tau$ is an admissible polynomial for $P = 0$. We have not reached the stabilization step because $E_\tau(P_2) = \{V\}$ and the bottom vertex is V which has height 2 (see $\mathcal{N}(P_2)$ in Fig. 2). However, we can already apply Theorem 3 so that $x + x^\tau$ can be continued to at least one solution of $P = 0$. Moreover, by Theorem 2, any solution which is a continuation of $x + x^\tau$ is convergent.

Let us try and continue the process to see when we reach the stabilization step.

As $\mathcal{N}(P_2)$ has a unique side, of anti-slope 2, $E_2(P_2)$, we need to compute the corresponding characteristic polynomial

$$\bar{\Phi}_{(P_2;2)}(C) = (\tau - 2)^2 + 2(\tau - 2)(2 - \tau)C + (\tau - 2)^2 = (\tau - 2)^2(C - 1)^2,$$

whose only root is $c_3 = 1$ which makes $r_3(x) = x + x^\tau + x^{2\tau}$ an admissible polynomial. Let $P_3 = P_2[x^{2\tau}]$:

$$P_3 = (\tau y_0 - y_1)^2 + (\tau - 3) x^3 (y_1 - \tau y_0) + (\tau - 4)^2 x^8 + \dots,$$

whose Newton polygon $\mathcal{N}(P_3)$ has $E_2(P_3) = \{V\}$. Again, we still have not reached the stabilization step.

The Newton polygon $\mathcal{N}(P_3)$ has two sides: setting $W = (3, 1)$, they are $E_3(P_3) = [V, W]$ and $E_5(P_3) = [W, (8, 0)]$ of anti-slopes 3 and 5 respectively. Notice that [Proposition 1](#) applies to the points V, W and $(8, 0)$ and as a consequence their corresponding indicial polynomials are powers of $(\mu - \tau)$ up to a constant factor:

$$\Psi_{(P_3;V)}(\mu) = (\tau - \mu)^2, \quad \Psi_{(P_3;W)}(\mu) = (\tau - 3)(\mu - \tau), \quad \Psi_{(P_3;(8,0))}(\mu) = (\tau - 4)^2(\mu - \tau)^0.$$

This guarantees that the characteristic polynomials $\Phi_{(P_3;3)}(C)$ and $\Phi_{(P_3;5)}(C)$ have non-zero roots. For anti-slope 3 we get

$$\Phi_{(P_3;3)}(C) = (\tau - 3)^2 C^2 + (\tau - 3)(3 - \tau)C = (\tau - 3)^2 C(C - 1),$$

and for anti-slope 5,

$$\Phi_{(P_3;5)}(C) = (\tau - 3)(5 - \tau)C + (\tau - 4)^2,$$

whose single root is $d_5 = \frac{(\tau-4)^2}{(\tau-3)(\tau-5)}$. Since $\Phi_{(P_3;3)}(1) = 0$ and $\Phi_{(P_3;5)}(d_5) = 0$, we can continue $r_3(x)$ with either x^3 or $d_5 x^5$ to get admissible polynomials. Let $P_4 = P_3[x^3]$ and $Q_4 = P_3[d_5 x^5]$:

$$P_4 = (\tau y_0 - y_1)^2 - (\tau - 3) x^3 (y_1 - \tau y_0) + x^8 + \dots, \\ Q_4 = (\tau y_0 - y_1)^2 + (\tau - 3) x^3 (y_1 - \tau y_0) + (1 + \tau) x^{7+\tau} + \dots.$$

The structure of $\mathcal{N}(P_4)$ guarantees that $z(x) = x + x^\tau + x^2 + x^3$ stabilizes P because $\text{Bot}(E_3(P_4)) = W$ has height 1 and its indicial polynomial $\Psi_{(P_4;W)}(\mu) = (\tau - 3)(\mu - \tau)$ has no roots greater than 3. The stabilization criterion now implies that $z(x)$ can be completed to an actual solution $\bar{z}(x)$, necessarily convergent. Similar arguments apply to the other choice $Q_4 = P_3[d_5 x^5]$ and $w(x) = x + x^\tau + x^2 + d_5 x^5$, which can be continued to a convergent solution $\bar{w}(x)$. For the sake of the reader we include some of the first terms of $\bar{z}(x)$ and $\bar{w}(x)$:

$$\bar{z}(x) = x + x^\tau + x^2 + x^3 + \frac{1}{(\tau - 5)(3 - \tau)} x^5 + \frac{\tau + 1}{4(\tau - 3)} x^{4+\tau} + \dots, \\ \bar{w}(x) = x + x^\tau + x^2 + d_5 x^5 - \frac{\tau + 1}{4(\tau - 3)} x^{4+\tau} + \frac{3}{(\tau - 6)(\tau - 3)} x^6 + \dots$$

Declaration of competing interest

The authors declare that they have no competing interests.

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