A cardinal dissensus measure based on the Mahalanobis distance

T. González-Arteaga[a,*], J. C. R. Alcantud[b], R. de Andrés Calle[c]

[a]*Presad Research Group, Facultad de Ciencias*
Universidad de Valladolid, E47011 Valladolid, Spain
[b]*Borda Research Unit and Multidisciplinary Institute of Enterprise (IME)*
Universidad de Salamanca, E37007 Salamanca, Spain
[c]*Borda Research Unit, Presad Research Group and IME*
Universidad de Salamanca, E37007 Salamanca, Spain

Abstract

In this paper we address the problem of measuring the degree of consensus/dissensus in a context where experts or agents express their opinions on alternatives or issues by means of cardinal evaluations. To this end we propose a new class of distance-based consensus model, the family of the Mahalanobis dissensus measures for profiles of cardinal values. We set forth some meaningful properties of the Mahalanobis dissensus measures. Finally, an application over a real empirical example is presented and discussed.

Keywords: Decision analysis, consensus/dissensus, cardinal profile, Mahalanobis distance, correlation

1. Introduction

In Decision Making Theory and its applications, consensus measurement and its reaching in a society (i.e., a group of agents or experts) are relevant research issues. Many studies investigating the aforementioned subjects have been carried out under several frameworks (see Herrera-Viedma et al. (2002), Fedrizzi et al. (2007), Dong et al. (2008), Cabrerizo et al. (2010), Dong et al. (2010), Fu and Yang (2012), Dong and Zhang (2014), Palomares et al. (2014),

*Corresponding author
Email addresses: teresag@eio.uva.es (T. González-Arteaga), jcr@usal.es (J. C. R. Alcantud), rocioac@usal.es (R. de Andrés Calle)

Wu and Chiclana (2014b,a), Liu et al. (2015) and Wu et al. (2015) among others) and based on different methodologies (González-Pachón and Romero (1999), Cook (2006), Eklund et al. (2007), Fedrizzi et al. (2007), Eklund et al. (2008), Chiclana et al. (2013), Fu and Yang (2010, 2011), Palomares and Martínez (2014), Gong et al. (2015) and Liu et al. (2015) among others).

Since the seminal contribution by Bosch (2005) several authors have addressed the consensus measurement topic from an axiomatic perspective. Earlier analyses can be mentioned, e.g., Hays (1960) or Day and McMorris (1985). This issue is also seen as the problem of combining a set of ordinal rankings to obtain an indicator of their 'consensus', a term with multiple possible meanings (Martínez-Panero (2011)).

Generally speaking, the usual axiomatic approaches assume that each individual expresses his or her opinions through ordinal preferences over the alternatives. A group of agents is characterized by the set of their preferences –their preference profile. Then a consensus measure is a mapping which assigns to each preference profile a number between 0 and 1. The assumption is made that the higher the values, the more consensus in the profile.

Technical restrictions on the preferences provide various approaches in the literature. In most cases the agents are presumed to linearly order the alternatives (see Bosch (2005) or Alcalde-Unzu and Vorsatz (2013)). Since this assumption seems rather demanding (especially as the number of alternatives grows), an obvious extension is to allow for ties. This is the case where the agents have complete preorders on the alternatives (e.g., García-Lapresta and Pérez-Román (2011)). Alcantud et al. (2013a, 2015) take a different position. They study the case where agents have dichotomous opinions on the alternatives, a model that does not necessarily require pairwise comparisons.

Notwithstanding the use of different ordinal preference frameworks, the problem of how to measure consensus is an open-ended question in several research areas. This fact is due to that methodology used in each case is a relevant element in the problem addressed. To date various methods have been developed to measure consensus under ordinal preference structures based on distances and association measures like Kemeny's distance, Kendall's coefficient, Goodman-Kruskal's index and Spearman's coefficient among others (see e.g., Spearman (1904), Kemeny (1959), Goodman and Kruskal (1979), Cook and Seiford (1982) and Kendall and Gibbons (1990)).

In this paper we first tackle the analysis of coherence that derives from profiles of cardinal rather than ordinal evaluations. Modern convention applies the term cardinal to measurements that assign significance to differences (cf., Basu (1982), High and Bloch (1989), Chiclana et al. (2009)). By contrast ordinal preferences only permit to order the alternatives from best to worst without any additional information. To see how this affects the analysis of our problem, let us consider a naive example of a society with two agents. They evaluate two public goods with monetary amounts. One agent gives a value of 1€ for the first good and 2€ for the second good. The other agent values these goods at 10€ and 90€ respectively. If we only use the ordinal information in this case, we should conclude that there is unanimity in the society: all members agree that 'good 2 is more valuable than good 1'. However the agents disagree largely. Therefore, the subtleties of cardinality clearly have an impact when we aim at measuring the cohesiveness of cardinal evaluations.

Unlike previous references, we adopt the notion of dissensus measure as the fundamental concept. This seems only natural because it resembles more the notion of a "measure of statistical dispersion", in the sense that 0 captures the natural notion of unanimity as total lack of variability among agents, and then increasingly higher numbers mean more disparity among evaluations in the profile.[1]

In order to build a particular dissensus measure we adopt a distance-based approach. Firstly, one computes the distances between each pair of individuals. Then all these distances are aggregated. In our present proposal the distances (or similarities) are computed through the Mahalanobis distance (Mahalanobis (1936)). We thus define the class of Mahalanobis dissensus measures.

The Mahalanobis distance plays an important role in Statistics and Data Analysis. It arises as a natural generalization of the Euclidean distance. A Mahalanobis distance accounts for the effects of differences in scales and associations among magnitudes. Consequently, building on the well-known performance of the Mahalanobis distance, our novel proposal seems especially fit for the cases when the measurement units of the issues are different, e.g.,

[1]As a remote antecedent of this position, we note that statistically variance-based methods are commonly employed to measure consensus of verbal opinions (cf., Hoffman (1994) and Mejias et al. (1996).

performance appraisal processes when employees are evaluated attending to their productivity and their leadership capacity; or where the issues are correlated. For example, evaluation of related public projects. An antecedent for the weaker case of profiles of preferences has been provided elsewhere, cf. Alcantud et al. (2013b), and an application to comparisons of real rankings on universities worldwide is developed. Here we apply our new indicator to a real situation, namely, economic forecasts made by several agencies. Since the forecasts concern economic quantities, they have an intrinsic value which is naturally cardinal and also there are relations among them.

The paper is structured as follows. In Section 2, we introduce basic notation and definitions. In Section 3, we set forth the class of the Mahalanobis dissensus measures and their main properties. Section 4 provides a comparison of several Mahalanobis dissensus measures. Next, a practical application with discussion is given in Section 5. Finally, we present some concluding remarks. Appendices contain proofs of some properties and a short review in matrix algebra.

2. Notation and definitions

This section is devoted to introduce some notation and a new concept in order to compare group cohesiveness: namely, dissensus measures. Then, a comparison with the standard approach is made. We partially borrow notation and definitions from Alcantud et al. (2013b). In addition, we use some elements of matrix analysis that we recall in the AppendixB to make the paper self-contained.

Let $X = \{x_1, ..., x_k\}$ be the finite set of k issues, options, alternatives, or candidates. It is assumed that X contains al least two options, i.e., the cardinality of X is at least 2. Abusing notation, on occasions we refer to issue x_s as issue s for convenience. A population of agents or experts is a finite subset $\mathbf{N} = \{1, 2, ..., N\}$ of natural numbers. To avoid trivialities we assume $N > 1$.

We consider that each expert evaluates each alternative by means of a quantitative value. The quantitative information gathered from the set of N experts on the set of k alternatives is summarized by an $N \times k$ numerical matrix M:

$$M = \left(M_{ij}\right)_{N\times k}$$

We write M_i to denote the evaluation vector of agent i over the issues (i.e., row i of M) and M^j to denote the vector with all the evaluations for issue j (i.e., column j of M). For convenience, $(1)_{N\times k}$ denotes the $N \times k$ matrix whose cells are all equal to 1 and $\mathbf{1}_N$ denotes the column vector whose N elements are equal to 1. We write $\mathbb{M}_{N\times k}$ for the set of all $N \times k$ real-valued matrices. Any $M \in \mathbb{M}_{N\times k}$ is called a *profile*.

Any permutation σ of the experts $\{1, 2, ..., N\}$ determines a profile M^σ by permutation of the rows of M: row i of the profile M^σ is row $\sigma(i)$ of the profile M. Similarly, any permutation π of the alternatives $\{1, 2, ..., k\}$ determines a profile ${}^{\pi}M$ by permutation of the columns of M: column i of the profile ${}^{\pi}M$ is column $\pi(i)$ of the profile M.

For each profile $M \in \mathbb{M}_{N\times k}$, its restriction to *subprofile* on the issues in $I \subseteq X$, denoted M^I, arises from exactly selecting the columns of M that are associated with the respective issues in I (in the same order). And for simplicity, if $I = \{j\}$ then $M^I = M^{\{j\}} = M^j$ is column j of M. Any partition $\{I_1, \ldots, I_s\}$ of $\{1, 2, \ldots, k\}$, that we identify with a partition of X, generates a *decomposition* of M into subprofiles $M^{I_1}, \ldots, M^{I_s}$. [2]

A profile $M \in \mathbb{M}_{N\times k}$ is *unanimous* if the evaluations for all the alternatives are the same across experts. In matrix terms, the columns of $M \in \mathbb{M}_{N\times k}$ are constant, or equivalently, all rows of the profile are coincident.

An *expansion* of a profile $M \in \mathbb{M}_{N\times k}$ of $\mathbf{N}$ on $X = \{x_1, ..., x_k\}$ is a profile $\bar{M} \in \mathbb{M}_{\bar{N}\times k}$ of $\bar{\mathbf{N}} = \{1, ..., N, N+1, \ldots, \bar{N}\}$ on $X = \{x_1, ..., x_k\}$, such that the restriction of $\bar{M}$ to the first N experts of $\mathbf{N}$ coincides with M.

Finally, a *replication* of a profile $M \in \mathbb{M}_{N\times k}$ of the society $\mathbf{N}$ on $X = \{x_1, ..., x_k\}$ is the profile $M \uplus M \in \mathbb{M}_{2N\times k}$ obtained by duplicating each row of M, in the sense that rows t and $N + t$ of $M \uplus M$ are coincident and equal to row t of M, for each $t = 1, \ldots, N$.

We now define a dissensus measure as follows:

Definition 1. A *dissensus measure* on $\mathbb{M}_{N\times k}$ is a mapping defined by $\delta : \mathbb{M}_{N\times k} \to [0, \infty)$ with the property:

i) *Unanimity*: for each $M \in \mathbb{M}_{N\times k}$, $\delta(M) = 0$ if and only if the profile $M \in \mathbb{M}_{N\times k}$ is unanimous.

[2] A partition of a set S is a collection of pairwise disjoints non-empty subsets of S whose union is S.

We also define a *normal* dissensus measure as a dissensus measure that additionally verifies:

ii) *Anonymity*: $\delta(M^\sigma) = \delta(M)$ for each permutation σ of the agents and $M \in \mathbb{M}_{N\times k}$.
iii) *Neutrality*: $\delta({}^\pi M) = \delta(M)$ for each permutation π of the alternatives and $M \in \mathbb{M}_{N\times k}$.

This definition does not attempt to state dissensus by opposition to consensus. The literature usually deals with a formulation of consensus where the higher the index, the more coherence in the society's opinions. The terms consensus and dissensus should not be taken as formal antonyms, especially because a universally accepted definition of consensus is not available and we do not intend to give an absolute concept of dissensus. However, consensus measures in the sense of Bosch (see Bosch (2005), Definition 3.1) verify anonymity and neutrality (see also Alcantud et al. (2013b), Definition 1), and from a *purely technical* viewpoint, they relate to dissensus measures as follows.

Lemma 1. *If μ is a consensus measure then $1-\mu$ is a normal dissensus measure. Conversely, if δ is a normal dissensus measure then $\frac{1}{\delta+1}$ is a consensus measure.*

Proof 1. *We just need to recall that the mapping $i:[0,\infty)\longrightarrow(0,1]$ given by $i(x)=\frac{1}{x+1}$ is strictly decreasing.* □

3. The class of Mahalanobis dissensus measures and its properties

In this section we introduce a broad class of dissensus measures that depends on a reference matrix, namely the Mahalanobis dissensus measures. We also give its more prominent properties.

Our interest is to cover the specific characteristics in cardinal profiles, like possible differences in scales, and correlations among the issues. Before providing our main definition, we recover the definition of the Mahalanobis distance on which our measure is based.

Definition 2. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix and let us assume that x and y vectors from $\mathbb{R}^k$ are row vectors. The Mahalanobis (squared) distance on $\mathbb{R}^k$ associated with Σ is defined by* [3]

$$d_\Sigma(x,y) = (x-y)\Sigma^{-1}(x-y)^t$$

The off-diagonal elements of Σ permit to account for cross relations among the issues or alternatives. Through the diagonal elements different measurement scales can be incorporated. The Σ matrix contains variances and covariances among random variables when the Mahalanobis distance is used in Statistical Data Analysis.

Definition 3. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. The* Mahalanobis dissensus measure *on $\mathbb{M}_{N\times k}$ associated with Σ is the mapping $\delta_\Sigma : \mathbb{M}_{N\times k} \to \mathbb{R}$ given by*

$$\delta_\Sigma(M) = \frac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) = \frac{1}{C_N^2} \cdot \sum_{i<j} (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t \quad (1)$$

for each profile $M \in \mathbb{M}_{N\times k}$ on k alternatives, where $C_N^2 = \frac{N(N-1)}{2}$ is the number of non ordered pairs of the N agents.

Note that the above expression is the average of all distances between the evaluation vectors provided by all pairs of agents according to the Mahalanobis distance associated with Σ (Definition 2).

It is immediate to check that δ_Σ verifies conditions i) and ii) for each positive definite Σ matrix. But δ_Σ fails to satisfy neutrality like the following example proves.

Example 1. Let $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $k=2$ and $N=2$. Then $\Sigma^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. For $M = \begin{pmatrix} 1 & -1 \\ 3 & 0 \end{pmatrix}$ one has $M_1 = (1,-1)$ and $M_2 = (3,0)$. Then

[3] Our choice of $d_\Sigma(x,y)$ coincides with the original Mahalanobis' definition (see Mahalanobis (1936)). In order to exploit the inclusion of the Euclidean distance, some authors work with $\sqrt{d_\Sigma(x,y)}$ instead. In both cases we have distances on $\mathbb{R}^k$.

$$\delta_\Sigma(M) = \frac{1}{C_2^2} \cdot \left((1-3,\ -1-0)\ \Sigma^{-1}\ (1-3,\ -1-0)^t\right) = \frac{9}{2}.$$

If the columns of M are permuted in order to obtain ${}^\pi M = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$, then

$$\delta_\Sigma({}^\pi M) = \frac{1}{C_2^2} \cdot \left((-1-0,\ 1-3)\ \Sigma^{-1}\ (-1-0,\ 1-3)^t\right) = 3.$$

Therefore

$$\delta_\Sigma({}^\pi M) = 3 \neq \frac{9}{2} = \delta_\Sigma(M),$$

which proves that δ_Σ does not verify neutrality.

Nevertheless, if the Σ matrix is adapted according to a specific permutation of the alternatives then the Mahalanobis disensus measure verifies a kind of "soft" neutrality like the following result proves.

Proposition 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix. For each profile $M \in \mathbb{M}_{N\times k}$ and each permutation π of the alternatives, i.e., a permutation of $\{1, \dots, k\}$,*

$$\delta_\Sigma(M) = \delta_{\Sigma^\pi}({}^\pi M)$$

where $\Sigma^\pi = P_\pi^t\, \Sigma\, P_\pi$ and P_π is the permutation matrix corresponding to π.

Proof 2. *Using the definition of Mahalanobis dissensus measure (Definition 3), it is sufficient to prove that $d_{\Sigma^\pi}({}^\pi M_i, {}^\pi M_j) = d_\Sigma(M_i, M_j)$*

$$\begin{aligned} d_{\Sigma^\pi}({}^\pi M_i, {}^\pi M_j) &= ({}^\pi M_i - {}^\pi M_j)\ (\Sigma^\pi)^{-1}\ ({}^\pi M_i - {}^\pi M_j)^t = \\ &= (M_i P_\pi - M_j P_\pi)\ (P_\pi^t \Sigma P_\pi)^{-1}\ (M_i P_\pi - M_j P_\pi)^t = \\ &= (M_i - M_j)\ P_\pi\, P_\pi^t \Sigma^{-1} P_\pi\, P_\pi^t\ (M_i - M_j)^t = \\ &= (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t = \\ &= d_\Sigma(M_i, M_j). \end{aligned}$$

We have only used the fact that the permutation matrix P_π is orthogonal. □

3.1. Some particular specifications

Some special instances of Mahalanobis dissensus measures have specific interpretations.

- If we have a single issue or alternative, then $M \in \mathbb{M}_{N\times 1}$ is a vector and Σ can be identified as a number $c > 0$. Then

$$\delta_c(M) = \frac{1}{C_N^2} \cdot \sum_{i<j} \frac{1}{c}(M_i - M_j)^2 = \frac{1}{c} \cdot \frac{2N}{N-1} \cdot S_M^2$$

  where S_M^2 is the sample variance of M. [4] Therefore the dissensus for a single issue is the result of correcting its sample variance by a factor of $\frac{1}{c} \cdot \frac{2N}{N-1}$.

- If Σ is the identity, then $\delta_I(M) = \dfrac{1}{C_N^2} \cdot \displaystyle\sum_{i<j} \sum_{r=1}^{k} (M_{ir} - M_{jr})^2$. This expression uses the square of the Euclidean distance between real-valued vectors, thus it recovers a version of the consensus measure for ordinal preferences based on this distance (Cook and Seiford (1982)). Henceforth δ_I is called the *Euclidean dissensus measure.*

- If $\Sigma = diag(c_{11}, \ldots, c_{kk})$ is a diagonal matrix then $d_\Sigma(M_i, M_j)$ gives the weighted average of the square of the differences in assessments for each alternative between agents i and j, where the weight attached to alternative r is $\frac{1}{c_{rr}}$:

$$\begin{aligned} \delta_\Sigma(M) &= \tfrac{1}{C_N^2} \cdot \textstyle\sum_{i<j} d_\Sigma(M_i, M_j) = \\ &= \frac{1}{C_N^2} \cdot \sum_{i<j} \left(\sum_{r=1}^{k} \frac{1}{c_{rr}} \cdot (M_{ir} - M_{jr})^2 \right) = \\ &= \sum_{r=1}^{k} \frac{1}{c_{rr}} \cdot \delta_I(M^r). \end{aligned}$$

[4] In order to check this, we use a well-known property of the variance: given a vector $x = (x_1, x_2, \ldots, x_n)$, whose mean is $\overline{x}$, $S_x^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2 = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - x_j)^2$.

This particular specification of the dissensus measure allows to incorporate different weights to the alternatives. This fact increases the richness of the analysis in comparison with the (square of the) Euclidean distance. Furthermore, if $\Sigma = \lambda I$ for some $\lambda > 0$, then Proposition 2 below gives additional relationships.

The Proposition 2 gives the relation between the Euclidean dissensus measure and the Mahalanobis dissensus measure associated to a matrix which is a multiple of the identity matrix, $\Sigma = \lambda I$.

Proposition 2. *For each profile $M \in \mathbb{M}_{N\times k}$ and $\lambda > 0$,*

$$\delta_{\lambda I}(M) = \delta_I(\frac{1}{\sqrt{\lambda}} \cdot M) = \frac{1}{\lambda} \cdot \delta_I(M).$$

Proof 3. *Using Definition 3, the assertion is direct if we check $d_{\lambda I}(M_i, M_j) = d_I(\frac{1}{\sqrt{\lambda}} \cdot M_i, \frac{1}{\sqrt{\lambda}} \cdot M_j)$ and $d_{\lambda I}(M_i, M_j) = \frac{1}{\lambda} \cdot d_I(M_i, M_j)$.*

$$\begin{aligned}
d_{\lambda I}(M_i, M_j) &= (M_i - M_j)(\lambda I)^{-1}(M_i - M_j)^t = \\
&= (M_i - M_j)\begin{pmatrix} \frac{1}{\lambda} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda} \end{pmatrix}(M_i - M_j)^t = \\
&= \sum_{r=1}^{k} \frac{1}{\lambda} \cdot (M_{ir} - M_{jr})^2 = \sum_{r=1}^{k} \left(\frac{1}{\sqrt{\lambda}} \cdot M_{ir} - \frac{1}{\sqrt{\lambda}} \cdot M_{jr} \right)^2 = \\
&= \left(\tfrac{1}{\sqrt{\lambda}} \cdot M_i - \tfrac{1}{\sqrt{\lambda}} \cdot M_j \right) \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \left(\tfrac{1}{\sqrt{\lambda}} \cdot M_i - \tfrac{1}{\sqrt{\lambda}} \cdot M_j \right)^t = \\
&= d_I(\tfrac{1}{\sqrt{\lambda}} \cdot M_i \, , \ \tfrac{1}{\sqrt{\lambda}} \cdot M_j).
\end{aligned}$$

$$\begin{aligned}
d_{\lambda I}(M_i, M_j) &= \tfrac{1}{\lambda} \cdot \sum_{r=1}^{k} (M_{ir} - M_{jr})^2 = \\
&= \tfrac{1}{\lambda} \cdot (M_i - M_j) \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} (M_i - M_j)^t = \tfrac{1}{\lambda} \cdot d_I(M_i, \ M_j).
\end{aligned}$$

□

3.2. Some properties of the class of Mahalanobis dissensus measures

Measuring cohesiveness by means of the Mahalanobis dissensus measure ensures some interesting operational features. We proceed to examine them. The proofs of these properties are given in AppendixA.

Let $M \in \mathbb{M}_{N\times k}$ denote a profile and let Σ, $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ be positive definite matrices. The following properties hold true:

1. *Neutrality*. A dissensus measure δ_Σ verifies neutrality if and only if the associated Σ matrix is a diagonal matrix whose diagonal elements are the same. Formally:

   $\delta_\Sigma(M) = \delta_\Sigma({}^{\pi}M)$ any profile $M \in \mathbb{M}_{N\times k}$ and any permutation π of $\{1, \ldots, k\}$, if and only if $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for some $\lambda > 0$.

2. *Oneness*. If for a particular size N of a society the Mahalanobis dissensus measures associated with two matrices coincide for all possible profiles, then the corresponding dissensus measures are equal. Formally:

   If for a fixed N it is the case that $\delta_{\Sigma_1}(M) = \delta_{\Sigma_2}(M)$ for each profile $M \in \mathbb{M}_{N\times k}$, then $\Sigma_1 = \Sigma_2$, i.e., for each N' and $M' \in \mathbb{M}_{N'\times k}$, it is also the case that

   $$\delta_{\Sigma_1}(M') = \delta_{\Sigma_2}(M').$$

3. *Cardinal transformations*. In contrast to ordinal assessments, cardinal evaluations are dependent on scales. So an important question arises about if the scale choice disturbs the cohesiveness measures. In this regard, once we update the reference matrix accordingly, the Mahalanobis dissensus measures associated to Σ do not vary. This fact happens even if we modify the scales of all issues in different way. In addition, a simple translation of each issue by adding a number does not change the cohesiveness measure. Formally:

   Let $a = (a_1, \ldots, a_k)^t$ be a column vector and $B = diag(b_1, \ldots, b_k)$ be a diagonal matrix. The affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ is $M^* = \mathbf{1}_N\, a^t + M\, B$, $M^* \in \mathbb{M}_{N\times k}$. Its columns are defined by $M^{*j} = a_j \cdot \mathbf{1}_N + b_j \cdot M^j$ and its rows are defined by $M_i^* = (a_1 + b_1 M_{i1}, \ldots, a_k + b_k M_{ik}) = a + M_i\, B$.

If $M^* = \mathbf{1}_N\, a^t \,+\, M\,B$ is a positive affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ and $\Sigma^* = B\Sigma B^t$ is the corresponding adjusted Σ, then

$$\delta_{\Sigma^*}(M^*) = \delta_\Sigma(M).$$

4. *Replication monotonicity.* When a non-unanimous society is replicated, its dissensus measure increases. That is, if $M \in \mathbb{M}_{N\times k}$ is a non-unanimous profile then

$$\delta_\Sigma(M \uplus M) = \left(\frac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M)$$

therefore

$$\delta_\Sigma(M \uplus M\ ) > \delta_\Sigma(M).$$

We can note that the difference between such measures is negligible for large societies. In addition, if we have an unanimous profile $M \in \mathbb{M}_{N\times k}$ then by Definition 1 i), δ_Σ verifies

$$\delta_\Sigma(M \uplus M) = \delta_\Sigma(M) = 0.$$

5. *Splitting the set of alternatives.* Suppose that the set of alternatives is divided in two (or more) subgroups, in such way that we do not consider any possible cross-effect among subgroups (perhaps because we know that there is not interdependence). Then the computation can be simplified by referring to measures of the dissensus in sub-profiles as follows.

Given $\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$, where $\Sigma_{11} \in \mathbb{M}_{r\times r}$, $\Sigma_{22} \in \mathbb{M}_{(k-r)\times(k-r)}$, for each profile $M = (M^{I_1}, M^{I_2})$ where $M^{I_1} \in \mathbb{M}_{N\times r}$, $M^{I_2} \in \mathbb{M}_{N\times(k-r)}$

$$\delta_\Sigma(M) = \delta_{\Sigma_{11}}(M^{I_1}) + \delta_{\Sigma_{22}}(M^{I_2}).$$

Remark 1. *Note that if the Σ matrix was originally a block diagonal matrix in the form $\Sigma = diag(\Sigma_{11}, \ldots, \Sigma_{ss})$, then it is possible to take the corresponding partition of the set of alternatives, $X = I_1 \cup I_2 \cup \ldots \cup I_s$. Consequently, the original profile $M \in \mathbb{M}_{N\times K}$ can be rewritten like $M = (M^{I_1}, M^{I_2}, \ldots, M^{I_s})$. Then*

$$\delta_{\Sigma}(M) = \sum_{i=1}^{s} \delta_{\Sigma_{ii}}(M^{I_i}).$$

6. *Adding alternatives.* An *extension* of a profile $M \in \mathbb{M}_{N\times k}$ is a new profile, $M^* \in \mathbb{M}_{N\times(k+r)}$, such that M^* includes r new alternatives. Under this assumption, M^* can be seen as a profile with two subgroups, the initial and the new alternatives, $M^* = (M,\ M^{new}) \in \mathbb{M}_{N\times(k+r)}$. If the aforementioned subgroups of alternatives are not related then Property 5 applies. Consequently,

$$\delta_{\Sigma^*}(M^*) = \delta_{\Sigma}(M) + \delta_{\Sigma^{new}}(M^{new})$$

where $\Sigma^* = \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma^{new} \end{pmatrix}$ and $\Sigma^{new} \in \mathbb{M}_{r\times r}$ is the associated matrix to the dissensus measure for the r new alternatives.

In the particular case where all the new alternatives added to the profile M are evaluated equally by all agents,

$$\delta_{\Sigma^*}(M^*) = \delta_{\Sigma}(M),$$

irrespective of Σ^{new} because unanimous profiles produce dissensus measures equal to zero. This particular case is defined like a property called "independence of irrelevant alternatives" in Alcantud et al. (2013a).

7. *Adding agents to the society.* Suppose that a new agent is added to the society, then the Mahalonabis dissensus measure of the enlarged society does not decrease. In addition, the increment is minimal when the"average agent" is added up. Formally:

Let $M \in \mathbb{M}_{N\times k}$ be a profile and $\bar{M} \in \mathbb{M}_{(N+1)\times k}$ be its expansion after incorporating the evaluations of a new agent. The Mahalanobis dissensus measure for $\bar{M}$ is

$$\delta_{\Sigma}(\bar{M}) = \frac{N-1}{N+1} \cdot \delta_{\Sigma}(M) + \frac{1}{C^2_{N+1}} \cdot \sum_{i=1}^{N} d_{\Sigma}(M_i, \bar{M}_{N+1})$$

where $\bar{M}_{N+1}$ is the row of $\bar{M}$ which incorporates the new agent's assessments for the alternatives.

If the assessments of the new agent coincide with the average of the original agents' evaluations for each alternative, then the minimal increment of the dissensus measure is obtained.

Remark 2. *A particular case is when the Mahalanobis dissensus measure is zero, or equivalently, there exits unanimity. If we include a new agent whose evaluations coincide with the assessments of the original agents, the Mahalanobis dissensus measure continues being zero.*

4. Comparison of Mahalanobis dissensus measures

In practical situations we could potentially use various Mahalanobis dissensus measures for profiles of cardinal information.[5] Hence it is worth studying the relations among evaluations achieved when we vary the reference matrices. This section addresses this point.

Theorems 1 and 2 below identify conditions on matrices that ensure consistent comparisons between Mahalanobis dissensus measures, whatever the number of agents. Based on these theorems, a final result gives bounds for the Mahalanobis dissensus measure.

Along this section $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ denote two positive definite matrices and d_{Σ_1}, d_{Σ_2} denote the corresponding Mahalanobis (squared) distances on $\mathbb{R}^k$ associated to Σ_1 and Σ_2. Let $\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \ldots \geq \lambda_k^{(i)} > 0$ be the eigenvalues of Σ_i, $i = 1, 2$.

Theorem 1. *If there exists N for which each profile $M \in \mathbb{M}_{N\times k}$ verifies $\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$ then*

$$\lambda_i^{(1)} \leq \lambda_i^{(2)} \quad \text{for } \; i = 1, \ldots, k \tag{2}$$

Proof 4. *We take a profile $M \in \mathbb{M}_{k\times k}$ with $M_i = 0$ for $i = 2, 3, \ldots, N$ and $M_1 = x \in \mathbb{R}^k$. By assumption*

$$\delta_{\Sigma_1}(M) = \frac{1}{C_N^2} \cdot d_{\Sigma_1}(x, 0) \;\geq\; \delta_{\Sigma_2}(M) = \frac{1}{C_N^2} \cdot d_{\Sigma_2}(x, 0).$$

Consequently, the hypothesis is reduced to $d_{\Sigma_1}(x, 0) \geq d_{\Sigma_2}(x, 0)$ for $x \in \mathbb{R}^k$. It means

[5] This is the case of our real example in Section 5 below.

$$x\Sigma_1^{-1}x^t \geq x\Sigma_2^{-1}x^t \;\Rightarrow\; x\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)x^t \geq 0 \quad \textit{for } x \in \mathbb{R}^k.$$

Then $(\Sigma_1^{-1} - \Sigma_2^{-1})$ *is a non-negative definite matrix. Now we use the result included in the AppendixB (see Point 11) to finish the proof:*

$$\Sigma_1^{-1} \geq \Sigma_2^{-1} \Longrightarrow \frac{1}{\lambda_i^{(1)}} \geq \frac{1}{\lambda_i^{(2)}} \Longrightarrow \lambda_i^{(1)} \leq \lambda_i^{(2)} \textit{ for } i = 1, 2, \ldots, k.$$

□

The converse of Theorem 1 is not always true like Example 2 below shows. Nevertheless, Theorem 2 below proves that a partial converse of Theorem 1 holds true under a technical restriction on the definite matrices.

Example 2. Let us consider a particular case of two matrices

$$\Sigma_1 = \begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix} \quad \Sigma_2 = \begin{pmatrix} 0.60 & 0.20 \\ 0.20 & 0.30 \end{pmatrix}$$

whose eigenvalues verify $\lambda_i^{(1)} \leq \lambda_i^{(2)}$ for $i = 1, 2$ because $\lambda_1^{(1)} = 0.5$, $\lambda_2^{(1)} = 0.1$ and $\lambda_1^{(2)} = 0.7$, $\lambda_2^{(2)} = 0.2$.

Let $M \in \mathbb{M}_{2\times 2}$ be the profile $M = \begin{pmatrix} 4 & 60 \\ 0 & 0 \end{pmatrix}$. The Mahalanobis dissensus measures for M associated with Σ_1 and Σ_2 produce

$$\delta_{\Sigma_1}(M) = 14630.4 \leq 14777.14 = \delta_{\Sigma_2}(M).$$

Therefore it is not true that $\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$ holds throughout.

Theorem 2. *If* $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ *are commutable matrices and their eigenvalues verify* $\lambda_1^{(1)} \leq \lambda_k^{(2)}$ *then*

$$\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$$

for each size N *and each profile* $M \in \mathbb{M}_{N\times k}$.

Proof 5. *Assuming* $\Sigma_1, \Sigma_2 \in \mathbb{M}_{k\times k}$ *are commutable, we can apply Point 12 in AppendixB to* Σ_1^{-1} *and* Σ_2^{-1}*. Consequently, there exists an orthonormal matrix* $Q \in \mathbb{M}_{k\times k}$ *such that*

$$Q^t\Sigma_1^{-1}Q = D_1 \quad and \quad Q^t\Sigma_2^{-1}Q = D_2$$

being $D_1, D_2 \in \mathbb{M}_{k\times k}$ *diagonal matrices. It is possible to select* Q *in such a way that the diagonal elements of* D_1 *verify* $\frac{1}{\lambda_1^{(1)}} \leq \ldots, \leq \frac{1}{\lambda_k^{(1)}}$*. Thus*

$$D_1 = diag\left(\frac{1}{\lambda_1^{(1)}}, \ldots, \frac{1}{\lambda_k^{(1)}}\right) \quad and \;\; D_2 = diag\left(\frac{1}{\lambda_{\pi(1)}^{(2)}}, \ldots, \frac{1}{\lambda_{\pi(k)}^{(2)}}\right),$$

where π *is a permutation of* $\{1, 2, \ldots, k\}$.[6]

Let $x, y \in \mathbb{R}^k$ *be two row vectors. Since* Q *is an orthonormal matrix, there exists a vector* $z \in \mathbb{R}^k$ *such that* $(x-y)^t = Qz$

$$d_{\Sigma_1}(x,y) = (x-y)\Sigma_1^{-1}(x-y)^t = z^tQ^t\Sigma_1^{-1}Qz = z^tD_1z = \sum_{j=1}^{k}\frac{1}{\lambda_j^{(1)}}z_j^2$$

$$d_{\Sigma_2}(x,y) = (x-y)\Sigma_2^{-1}(x-y)^t = z^tQ^t\Sigma_2^{-1}Qz = z^tD_2z = \sum_{j=1}^{k}\frac{1}{\lambda_{\pi(j)}^{(2)}}z_j^2$$

From premise that $\lambda_1^{(1)} \leq \lambda_k^{(2)}$ *we have*

$$\frac{1}{\lambda_k^{(1)}} \geq \ldots \geq \frac{1}{\lambda_1^{(1)}} \geq \frac{1}{\lambda_k^{(2)}} \geq \ldots \geq \frac{1}{\lambda_1^{(2)}}.$$

Thus $\frac{1}{\lambda_j^{(1)}} \geq \frac{1}{\lambda_{\pi(j)}^{(2)}}$ *for* $j = 1, 2, \ldots, k$ *and as a result it is obtained*

$$\sum_{j=1}^{k}\frac{1}{\lambda_j^{(1)}}z_j^2 \geq \sum_{j=1}^{k}\frac{1}{\lambda_{\pi(j)}^{(2)}}z_j^2\,.$$

[6]When Q does not lead to a diagonal matrix with properly ordered eigenvalues, we change Q for $Q' = QP^t$, P being a permutation matrix. Q' is also an orthogonal matrix (see AppendixB, Point 10) which simultaneously diagonalizes Σ_1^{-1} and Σ_2^{-1} . In addition, we get a diagonal matrix D_1^* with the same eigenvalues that D_1 but in the proper order.

In consequence, $d_{\Sigma_1}(x,y) \geq d_{\Sigma_2}(x,y)$.

Now, using Definition 3 the theorem is proven. □

Example 3 below shows the relevance of hypothesis on the eigenvalues (Theorem 2).

Example 3. Considering Σ_1 and Σ_2 from Example 2, we observe that they are commutable matrices:

$$\Sigma_1\,\Sigma_2 = \begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix}\begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix} = \begin{pmatrix} 0.076 & -0.012 \\ -0.012 & 0.094 \end{pmatrix}$$

$$\Sigma_2\,\Sigma_1 = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}\begin{pmatrix} 0.18 & -0.16 \\ -0.16 & 0.42 \end{pmatrix} = \begin{pmatrix} 0.076 & -0.012 \\ -0.012 & 0.094 \end{pmatrix}$$

We can see that the assumption $\lambda_1^{(1)} \leq \lambda_k^{(2)}$ even if $k = 2$ does not imply $\lambda_i^{(1)} \leq \lambda_i^{(2)}$ for $i = 1, 2$ (see Equation 2, Theorem 1):

$$\begin{aligned} \lambda_1^{(1)} = 0.5 &\;<\; 0.7 = \lambda_1^{(2)}, \\ \lambda_2^{(1)} = 0.1 &\;<\; 0.2 = \lambda_2^{(2)}, \\[1em] \lambda_1^{(1)} = 0.5 &\;>\; 0.2 = \lambda_2^{(2)}. \end{aligned}$$

Example 4 bellow reveals that the commutativity of Σ_1 and Σ_2 is not superfluous in the statement of Theorem 2.

Example 4. Let us consider $\Sigma_1 = \begin{pmatrix} 0.05 & 0 \\ 0 & 0.1 \end{pmatrix}$ and $\Sigma_2 = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.3 \end{pmatrix}$, with $\lambda_2^{(1)} = 0.05$, $\lambda_1^{(1)} = 0.1$ and $\lambda_2^{(2)} = 0.2$, $\lambda_1^{(2)} = 0.7$. These eigenvalues satisfy $\lambda_1^{(1)} \leq \lambda_2^{(2)}$ and Σ_1 and Σ_2 matrices are not commutable:

$$\Sigma_1\,\Sigma_2 = \begin{pmatrix} 0.03 & 0.01 \\ 0.02 & 0.03 \end{pmatrix} \neq \begin{pmatrix} 0.03 & 0.02 \\ 0.01 & 0.03 \end{pmatrix} = \Sigma_2\,\Sigma_1$$

Let $M \in \mathbb{M}_{2\times 2}$ be a specific profile, $M = \begin{pmatrix} 4 & 60 \\ 0 & 0 \end{pmatrix}$. The Mahalanobis dissensus measures for M associated with Σ_1 and Σ_2 produce

$\delta_{\Sigma_1}(M) = 360.8 \leq 14,777.14 = \delta_{\Sigma_2}(M)$. Therefore it is not true that $\delta_{\Sigma_1}(M) \geq \delta_{\Sigma_2}(M)$ holds throughout.

Theorems 1 and 2 can be extended to r positive definite matrices $\Sigma_1, \ldots, \Sigma_r$ as a matter of course.

Apart from Theorems 1 and 2, the following corollary reveals that the Mahalanobis dissensus measure associated to Σ is confined within bounds depending only on the extreme eigenvalues of Σ.

Corollary 1. *Let $\Sigma \in \mathbb{M}_{k\times k}$ be a positive definite matrix with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_k$, it is verified*

$$\delta_{\lambda_1 I}(M) \leq \delta_{\Sigma}(M) \leq \delta_{\lambda_k I}(M)$$

or equivalently

$$\frac{1}{\lambda_1} \cdot \delta_I(M) \leq \delta_{\Sigma}(M) \leq \frac{1}{\lambda_k} \cdot \delta_I(M)$$

for each N and for each $M \in \mathbb{M}_{N\times k}$.

Proof 6. *This result is straightforward from Theorem 2. Observe that such Theorem can be applied because $\lambda_k I$ (resp., $\lambda_1 I$) and M are commutable matrices and the eigenvalues of the diagonal matrix $\lambda_k I$ (resp., $\lambda_1 I$) are all equal to λ_k (resp. λ_1). Proposition 2 is used.* □

Figure 1 illustrates the previous corollary regarding the distances used for $\delta_{\lambda_1 I}$, δ_{Σ} and $\delta_{\lambda_k I}$. We can observe that all points on the ellipse have the same Mahalanobis distance to point A, namely d_{Σ}. Moreover, distance d_{Σ} is always between the values of the corresponding distances $d_{\lambda_1 I}$ and $d_{\lambda_k I}$.

5. Discussion on practical application using a real example

In this Section we fully develop a real example. It aims at giving an explicit application of our proposal and discussing some of its features.

We are interested in assessing the cohesiveness of the forecasts of various magnitudes for the Spanish Economy in 2014: GDP (Gross Domestic Product), Unemployment Rate, Public Deficit, Public Debt and Inflation. These forecasts have been published by different institutions and organizations, and

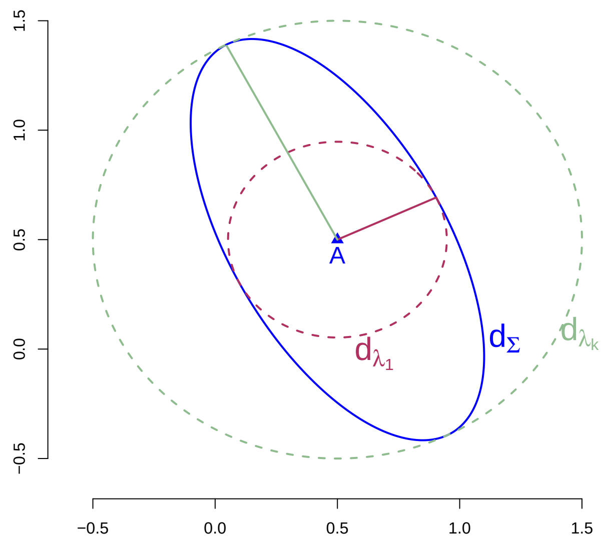

Figure 1: Curves of equidistance to point A with d_Σ (ellipse), $d_{\lambda_1 I}$ and $d_{\lambda_k I}$ (circumferences).

each one was made at around the same time. Specifically, three waves of forecasts were published in the Spring of 2013 (Table 1), Autumn of 2013 (Table 2) and Spring of 2014 (Table 3).

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.70	26.40	-6.90	97.60	1.50
OECD	0.40	28.00	-6.40	97.00	0.40
European Commission	0.90	26.40	-7.00	91.30	0.80
BBVA Research	0.90	26.40	-5.70	96.30	1.20
FUNCAS	0.50	26.00	-4.60	99.20	1.60

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

Table 1: Forecasts for several magnitudes for the Spanish Economy for the year 2014 published in Spring of 2013.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.20	26.70	-5.80	99.10	1.50
OECD	0.50	26.30	-6.10	98.00	0.50
European Commission	0.50	26.40	-5.90	99.90	0.90
BBVA Research	0.90	25.60	-5.80	98.50	1.10
FUNCAS	1.00	25.90	-5.90	100.50	1.30

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

Table 2: Forecasts for sevaral magnitudes for the Spanish Economy for the year 2014 published in Autumn of 2013.

We intend to measure the cohesiveness of the aforementioned predictions. Since they are expressed by cardinal valuations, we need to go beyond the traditional analyses refered to in this paper. To this purpose, we first gather the data corresponding to Tables 1, 2 and 3 in the profiles $M^{(S)}, M^{(A)}, M^{(lS)} \in \mathbb{M}_{5\times 5}$, respectively. Next, we select a suitable reference matrix and finally we make the computations of the Mahalanobis dissensus measures.

5.1. Reference matrix

Once the profiles have been fixed, the following step to compute their Mahalanobis dissensus measures is to avail oneself of a suitable reference matrix Σ. The choice of such a matrix can easily raise controversy. Nevertheless, we can learn from the role of the Σ matrix in the Mahalanobis distance from a statistical point of view. This matrix contains the variances and covariances

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
IMF	0.90	25.50	-5.89	98.80	0.50
OECD	1.00	25.40	-5.50	98.30	0.10
European Commission	1.10	25.50	-5.60	103.80	0.10
BBVA Research	1.10	25.10	-5.80	98.40	1.10
FUNCAS	1.20	25.10	-6.00	100.00	0.10

Abbreviations: Unemployment Rate (U. Rate), Public Deficit/Debt (P. Deficit/Debt), Banco Bilbao Vizcaya Argentaria Reseach (BBVA Research), Fundación de las Cajas de Ahorros (FUNCAS).

Table 3: Forecasts for several magnitudes for the Spanish Economy for the year 2014 published in Spring of 2014.

among the statistical variables, therefore, those characteristics are brought into play in this distance. We recall that covariances (or corresponding correlations) among variables reveal their interdependence. In Statistics, this Σ matrix is usually unknown and it is estimated from a sample. One exception is the unlikely case when the data are generated by a known multivariate probability distribution. This is not the case of our example.

Year	GDP	U. Rate	P. Deficit	P. Debt	Inflation
2001	3.70	10.55	0.50	55.60	2.70
2002	2.70	11.47	0.20	52.60	3.50
2003	3.10	11.48	0.30	48.80	3.00
2004	3.30	10.97	0.10	46.30	3.00
2005	3.60	9.16	-1.30	43.20	3.40
2006	4.10	8.51	-2.40	39.70	3.50
2007	3.50	8.26	-1.90	36.30	2.80
2008	0.90	11.33	4.50	40.20	4.10
2009	-3.70	18.01	11.20	53.90	-0.30
2010	-0.30	20.06	9.70	61.50	1.80
2011	0.40	21.64	9.40	69.30	3.20
2012	-1.40	25.03	10.60	84.20	2.40

Table 4: Past data for the Spanish Economy (2001 – 2012). Source: Spanish National Statistics Institute (INE) and Bank of Spain.

Therefore we employ a reference matrix Σ that captures the variances and covariances among the macroeconomic magnitudes of the Spanish Economy. It seems natural to produce such a matrix from historical macroeconomic data corresponding to the issues under inspection. Table 4 contains such recorded data, and Table 5 gives the corresponding correlation coefficients. [7] These values are depicted in Figure 2. Each ellipse represents the correlation between a pair of variables. The ellipses slant upward (resp., downward) show a positive (resp., negative) correlation. Moreover, the narrower the ellipse the stronger correlation represented. For example, the pair formed by GDP and Public Deficit holds the strongest negative correlation.

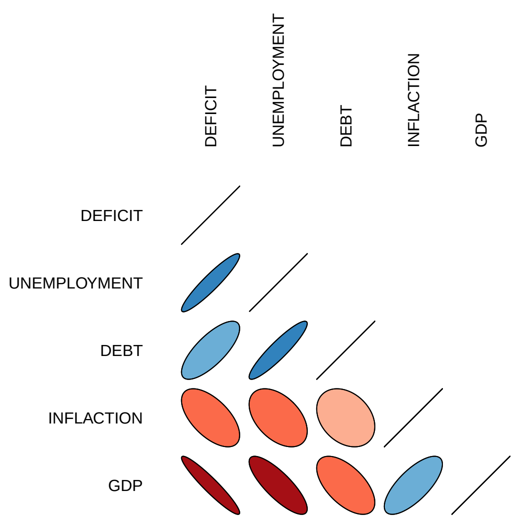

Figure 2: A depiction of the correlation matrix of the Spanish macroeconomic data from 2001 to 2012.

[7]Given two vectors $X = (x_1, \dots, x_n)'$ and $Y = (y_1, \dots, y_n)'$ with $\overline{x}$ and $\overline{y}$ their respective means, the correlation coefficient between X and Y is computed by $cor(X,Y) = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \overline{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \overline{y})^2}}$.

	GDP	U. Rate	P. Deficit	P. Debt	Inflation
GDP	1.00	-0.81	-0.94	-0.59	0.73
U. Rate	-0.81	1.00	0.93	0.92	-0.46
P. Deficit	-0.94	0.93	1.00	0.75	-0.60
P. Debt	-0.59	0.92	0.75	1.00	-0.30
Inflation	0.73	-0.46	-0.60	-0.30	1.00

Table 5: Correlations between macroeconomic magnitudes for historial data.

On the basis of Table 4, we compute the corresponding variance-covariance matrix $\mathbf{\Sigma}$ [8].

$$\mathbf{\Sigma} = \begin{pmatrix} 6.11 & -11.49 & -12.43 & -20.19 & 2.03 \\ -11.49 & 32.74 & 28.43 & 72.42 & -2.97 \\ -12.43 & 28.43 & 28.41 & 55.41 & -3.60 \\ -20.19 & 72.42 & 55.41 & 190.52 & -4.73 \\ 2.03 & -2.97 & -3.60 & -4.73 & 1.28 \end{pmatrix}$$

5.2. Computation of the dissensus

Now we calculate the Mahalanobis dissensus measures associated with $\mathbf{\Sigma}$ for the profiles of the forecasts for the Spanish Economy, namely, $M^{(S)}$, $M^{(A)}$ and $M^{(lS)}$.

We obtain the following Mahalanobis dissensus measures associated with the aforementioned $\mathbf{\Sigma}$:

$$\delta_{\mathbf{\Sigma}}(M^{(S)}) = 8.45, \quad \delta_{\mathbf{\Sigma}}(M^{(A)}) = 2.05, \quad \delta_{\mathbf{\Sigma}}(M^{(lS)}) = 1.51.$$

Note that the measure of the dissensus decreases along the time. This is what we intuitively expect, since the latter forecasts rest on more accurate and factual information.

Apart from the measure of the cohesiveness of the profiles, our proposal also produces a measure of divergence among the evaluations of different agents on a set of issues. We can answer questions like "*Are the predictions of*

[8] Let X be a $n \times k$ matrix whose columns have means $\overline{X}_i$, $i = 1, \ldots, k$. The cells of the variance-covariance matrix are $\Sigma_{ij} = \dfrac{1}{n-1}\sum_{r=1}^{n}(x_{ri} - \overline{X}_i)(x_{rj} - \overline{X}_j)$.

the European Commission for the Spanish Economy similar to the predictions of the BBVA Reseach?" or "*Is the previous comparison more or less similar than the comparison between the predictions of the BBVA Research vs. the predictions of the IMF?*". Table 6 provides these items for comparison.

		Spring 2013	Autumn 2013	Spring 2014
OECD	FUNCAS	23.18	2.85	0.27
European Comm.	FUNCAS	19.65	1.21	0.61
IMF	FUNCAS	9.31	3.63	1.15
OECD	BBVA Research	8.05	2.50	2.04
European Comm.	BBVA Research	5.62	1.25	2.86
IMF	OECD	5.24	2.76	0.93
IMF	European Comm.	4.87	1.47	2.62
BBVA Research	FUNCAS	4.52	0.12	2.10
IMF	BBVA Research	3.31	4.11	0.86
OECD	European Comm.	0.79	0.60	1.61

Table 6: Dissensus between pairs of agents for the profiles of forecasts published in Spring of 2013 (in descending order), Autumn of 2013 and Spring of 2014.

5.3. Other simpler approaches: Drawbacks or limitations

The choice of the reference matrix is a key point in the application of the Mahalanobis dissensus measure. As an explanatory exercise in this subsection we discuss on the more simplistic approaches where naive reference matrices are employed. If we use the identity matrix as the reference matrix (for example, because we lack data to make a better inference), then we get a Mahalanobis dissensus measure which gives the same importance to the differences in all the issues (see Subsection 3.1). However the choice of the identity matrix as the reference matrix discards much relevant information. We note the variance of the Public Debt is 190.52, while Inflation has a variance of 1.28 (see $\mathbf{\Sigma}$). So, a difference of one unit in the forecasts from two agents does not signify the same if such a difference corresponds to Inflation or to Public Debt.

We could alternatively employ as the reference matrix, the diagonal matrix with the variances of the issues, that is,

$$\mathbf{\Sigma}_\sigma = diag(6.11, 32.74, 28.41, 190.52, 1.28).$$

In this case, we remove the effects of the interdependence among the economic magnitudes on the dissensus measure.

In order to check that an inconvenient choice of the reference matrix easily produces misleading conclusions. Table 7 shows the dissensus measures derived from the three matrices mentioned above, $\mathbf{\Sigma}$, I and $\mathbf{\Sigma}_\sigma$. The dissensus $\delta_{\mathbf{\Sigma}}$ is decreasing along time as previously reported. This intuitively appealing feature is not captured when we utilize simpler matrices. Consequently, introducing corrections due to variances or to cross-effects is crucial for a reliable final analysis.

		Profiles		
		$M^{(S)}$	$M^{(A)}$	$M^{(lS)}$
Reference matrix		Spring 2013	Autumn 2013	Spring 2014
Σ	$\delta_{\mathbf{\Sigma}}$	8.45	2.05	1.51
Diagonal	$\delta_{\mathbf{\Sigma}_\sigma}$	0.61	0.29	0.37
Identity	δ_I	21.59	2.97	11.20

Table 7: Dissensus for several profiles of economic forecasts for the Spanish Economy for the year 2014. Data published in Spring of 2013, Autumn of 2013 and in Spring of 2014.

6. Concluding remarks

We explore the problem of measuring the degree of cohesiveness in a setting where experts express their opinions on alternatives or issues by means of cardinal evaluations. We use the general concept of dissensus measure and introduce one particular formulation based on the Mahalanobis distance for numerical vectors, namely the Mahalanobis dissensus measure.

We provide some properties which make our proposal appealing. We emphasize that the Mahalanobis dissensus measure on the profiles with k issues or alternatives is scale-independent for each issue and it accounts for cross-relations of issues. In addition, the comparison between different Mahalanobis dissensus measures can be made through the eigenvalues of their associated matrices.

We illustrate our proposal with a real numerical application about forecasts for several magnitudes for the Spanish Economy. We discuss the relevance of the choice of the reference matrix in this context.

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AppendixA. Proofs of properties in Section 3.2

Proof of property 1. Neutrality.

Let us first prove sufficiency. If $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for a value $\lambda > 0$, the thesis is straightforward from the Definition 3.

Let us now prove necessity. Due to the fact that δ_Σ verifies neutrality for any profile $M \in \mathbb{M}_{N\times k}$ and for any permutation π of $\{1, \ldots, k\}$

$$\delta_\Sigma(M) = \delta_\Sigma({}^\pi M),$$

it must be deduced $\Sigma = diag\{\lambda, \ldots, \lambda\}$ for a value $\lambda > 0$.

Let $M \in \mathbb{M}_{2\times k}$ be a particular profile such that $M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$. The dissensus measure for $M \in \mathbb{M}_{2\times k}$ is given by $\delta_\Sigma(M) = (M_1 - M_2)\Sigma^{-1}(M_1 - M_2)^t$ according to Definition 3. If M is permuted by means of π, we obtain the matrix ${}^\pi M \in \mathbb{M}_{2\times k}$ and consequently its dissensus measure is $\delta_\Sigma({}^\pi M) = ({}^\pi M_1 -{}^\pi M_2)\Sigma^{-1}({}^\pi M_1 -{}^\pi M_2)^t$.

According to Point 10 in AppendixB, we can write ${}^\pi M = MP_\pi$, being $P_\pi \in \mathbb{M}_{k\times k}$ the corresponding permutation matrix. Consequently,

$$\begin{aligned} \delta_\Sigma({}^\pi M) &= (M_1P_\pi - M_2P_\pi)\Sigma^{-1}(M_1P_\pi - M_2P_\pi)^t = \\ &= (M_1 - M_2)P_\pi\Sigma^{-1}P_\pi^t(M_1 - M_2)^t. \end{aligned}$$

Since δ_Σ verifies neutrality, $\delta_\Sigma(M) = \delta_\Sigma({}^\pi M)$ for any $M \in \mathbb{M}_{2\times k}$,

$$\Sigma^{-1} = P_\pi\Sigma^{-1}P_\pi^t.$$

Using the spectral decomposition (see AppendixB, Points 15 and 16) Σ^{-1} can be written as $\Sigma^{-1} = \Gamma D_\lambda^{-1}\Gamma^t$ for a unique orthogonal matrix Γ. Therefore

$$\Sigma^{-1} = P_\pi\Sigma^{-1}P_\pi^t = P_\pi\Gamma D_\lambda^{-1}\Gamma^t P_\pi^t.$$

Observe that the matrix $P_\pi\Gamma$ is orthogonal because it is the product of two orthogonal matrices. Since the spectral decomposition assures that Γ is unique, it must be

$$\Gamma = P_\pi\Gamma$$

for every $P_\pi \in \mathbb{M}_{k\times k}$ permutation matrix. Note that this equation implies that performing any permutation of the rows of Γ produces Γ.

Therefore Γ must be the identity matrix, i.e., $\Gamma = I$.

We can now deduce

$$\begin{aligned} \Sigma^{-1} &= \Gamma D_\lambda^{-1}\Gamma^t = D_\lambda^{-1}, \\ \Sigma^{-1} &= P_\pi\Gamma D_\lambda^{-1}\Gamma^t P_\pi^t = \quad P_\pi D_\lambda^{-1} P_\pi^t. \end{aligned}$$

Thus we conclude that Σ is a diagonal matrix.

Let us now prove that the diagonal elements of $\Sigma = D_\lambda$ are all equal.

From the above equalities of Σ^{-1}, it is verified $D_\lambda^{-1} = P_\pi D_\lambda^{-1} P_\pi^t$, for any permutation π of $\{1, \dots, k\}$.

For the particular permutation matrix

$$P_\pi = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

we obtain $P_\pi D_\lambda^{-1} P_\pi^t = diag\{\lambda_2^{-1}, \lambda_1^{-1}, \dots \lambda_k^{-1}\}$ and given that $D_\lambda^{-1} = P_\pi D_\lambda^{-1} P_\pi^t$, it must be $\lambda_1 = \lambda_2$. A routine modification of the argument proves $\lambda_1 = \lambda_j$, $j = 3, \dots, k$.

□

Proof of property 2. Oneness.

Let N be a fixed value. We take a profile $M \in \mathbb{M}_{N\times k}$ with $M_i = 0$ for $i = 2, \dots, N$ and $M_1 = x \in \mathbb{R}^k$ any row vector. For this particular profile the hypothesis $\delta_{\Sigma_1}(M) = \delta_{\Sigma_2}(M)$ reduces to $d_{\Sigma_1}(x, 0) = d_{\Sigma_2}(x, 0)$. It means that, $x\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)x^t = 0$ and $\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)$ is a non-negative definite matrix.

Let c_{ij} be the elements of the matrix $\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)$. Considering the i-th row vector of the canonical base $e_i = (0, \dots, 1, \dots, 0)$ then, $e_i\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right)e_i^t = c_{ii} = 0$. Therefore $c_{11} = \dots = c_{kk} = 0$ and $trace\left(\Sigma_1^{-1} - \Sigma_2^{-1}\right) = 0$. As a consequence, using AppendixB (Point 13), $\Sigma_1 = \Sigma_2$.

□

Proof of property 3. Cardinal transformations.

Let $a = (a_1, \dots, a_k)^t$ be a column vector and $B = diag(b_1, \dots, b_k)$ be a diagonal matrix. The affine transformation of the profile $M \in \mathbb{M}_{N\times k}$ is $M^* = \mathbf{1}_N\, a^t + M\, B$, $M^* \in \mathbb{M}_{N\times k}$. Its columns are defined by $M^{*j} = a_j \cdot \mathbf{1}_N + b_j \cdot M^j$ and its rows are defined by $M_i^* = (a_1 + b_1 M_{i1}, \dots, a_k + b_k M_{ik}) = a + M_i\, B$.

Let $\Sigma^* = B \Sigma B^t$ be the Σ matrix updated according to the affine transformation. Then, all elements σ_{ij}^* of Σ^* and all elements σ_{ij} of Σ are related by $\sigma^*_{ij} = b_i b_j \sigma_{ij}$. Due to the fact that B is a diagonal matrix, $B = B^t$ and $(\Sigma^*)^{-1} = B^{-1}\Sigma^{-1}B^{-1}$. We now proceed to compute the Mahalanobis distance under the previous remarks:

$$\begin{aligned} d_{\Sigma^*}(M^*_i, M^*_j) &= (M^*_i - M^*_j)(\Sigma^*)^{-1}(M^*_i - M^*_j)^t = \\ &= (a + M_iB - a - M_jB)(B\Sigma B^t)^{-1}\,(a + M_iB - a - M_jB)^t = \\ &= (M_i - M_j)\,BB^{-1}\Sigma^{-1}B^{-1}B\,(M_i - M_j)^t = \\ &= (M_i - M_j)\Sigma^{-1}(M_i - M_j)^t = \\ &= d_\Sigma(M_i, M_j). \end{aligned}$$

Based on the previous distance, we obtain:

$$\delta_{\Sigma^*}(M^*) = \frac{1}{C^2_N} \cdot \sum_{i<j} d_{\Sigma^*}(M^*_i, M^*_j) = \frac{1}{C^2_N} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) = \delta_\Sigma(M)$$

□

Proof of property 4. Replication monotonicity.

Let us compute the Mahalanobis dissensus measure for $M \uplus M$.

$$\begin{aligned} \delta_\Sigma(M \uplus M) &= \tfrac{1}{C^2_{2N}} \cdot \sum_{i=1}^{2N} \sum_{\substack{j=1 \\ i<j}}^{2N} d_\Sigma((M \uplus M)_i, (M \uplus M)_j) = \\ &= \tfrac{1}{C^2_{2N}} \cdot \left(\sum_{i=1}^{N} \sum_{\substack{j=1 \\ i<j}}^{N} d_\Sigma(M_i, M_j) + \sum_{i=1}^{N} \sum_{j=N+1}^{2N} d_\Sigma(M_i, M_j) \right) + \\ &+ \tfrac{1}{C^2_{2N}} \cdot \left(\sum_{i=N}^{2N} \sum_{\substack{j=1 \\ i<j}}^{2N} d_\Sigma(M_i, M_j) \right) = \tfrac{1}{C^2_{2N}} \cdot C^2_N \cdot \delta_\Sigma(M) + \\ &+ \tfrac{1}{C^2_{2N}} \cdot \sum_{i=1}^{N} \sum_{r=1}^{N} d_\Sigma(M_i, M_{N+r}) + \frac{1}{C^2_{2N}} \cdot \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i<j}}^{N} d_\Sigma(M_i, M_j) = \\ &= \tfrac{1}{C^2_{2N}} \cdot (\, 4\, C^2_N \cdot \delta_\Sigma(M)\,) = \left(\tfrac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M) \end{aligned}$$

Therefore

$$\delta_\Sigma(M \uplus M) = \left(\frac{2N-2}{2N-1}\right) \cdot \delta_\Sigma(M)$$

and in particular

$$\delta_\Sigma(M \uplus M\ ) > \delta_\Sigma(M).$$

□

Proof of property 5. Splitting the set of alternatives.

We set $X = I_1 \cup I_2 = \{x_1, ..., x_r\} \cup \{x_{r+1}, ..., x_k\}$ as a partition of the alternatives. Given $\Sigma = \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$, where $\Sigma_{11} \in \mathbb{M}_{r\times r}$, $\Sigma_{22} \in \mathbb{M}_{(k-r)\times(k-r)}$, for each profile $M = (M^{I_1}, M^{I_2})$ where $M^{I_1} \in \mathbb{M}_{N\times r}$, $M^{I_2} \in \mathbb{M}_{N\times(k-r)}$. Recalling Point 5 in AppendixB

$$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}.$$

We are now in a position to calculate $d_\Sigma(M_i, M_j)$, the Mahalanobis distance between a pair of agents i and j:

$$\begin{aligned} d_\Sigma(M_i, M_j) &= (M_i^{I_1} - M_j^{I_1})\Sigma_{11}^{-1}(M_i^{I_1} - M_j^{I_1})^t + \\ &+ (M_i^{I_2} - M_j^{I_2})\Sigma_{22}^{-1}(M_i^{I_2} - M_j^{I_2})^t = \\ &= d_{\Sigma_{11}}(M_i^{I_1}, M_j^{I_1}) + d_{\Sigma_{22}}(M_i^{I_2}, M_j^{I_2}). \end{aligned}$$

Using Definition 3, the Mahalanobis dissensus measure on M associated with Σ is given by

$$\begin{aligned} \delta_\Sigma(M) &= \tfrac{1}{C_N^2} \cdot \sum_{i<j} d_\Sigma(M_i, M_j) = \\ &= \tfrac{1}{C_N^2} \cdot \sum_{i<j} \big(d_{\Sigma_{11}}(M_i^{I_1}, M_j^{I_1}) + d_{\Sigma_{22}}(M_i^{I_2}, M_j^{I_2}) \big) = \\ &= \delta_{\Sigma_{11}}(M^{I_1}) + \delta_{\Sigma_{22}}(M^{I_2}). \end{aligned} \tag{A.1}$$

It is easy to check that this property holds true for any size of the partition. We set $X = I_1 \cup I_2 \cup \ldots \cup I_s$ as a partition of the alternatives. Considering not cross-effects among the subsets of the alternatives, the $\Sigma \in \mathbb{M}_{k\times k}$ matrix has a block diagonal form, $\Sigma = diag(\Sigma_{11}, \ldots, \Sigma_{ss})$. Analogously, a profile $M \in \mathbb{M}_{N\times k}$ can be written as $M = (M^{I_1}, M^{I_2}, \ldots, M^{I_s})$. Then

$$\delta_\Sigma(M) = \sum_{i=1}^{s} \delta_{\Sigma_{ii}}(M^{I_i}).$$

□

Proof of property 6. Adding alternatives.

The proof is straightforward from Equation (A.1). □

Proof of property 7. Adding agents to the society.

Let $M \in \mathbb{M}_{N\times k}$ be a profile on X of the society $\mathbf{N}$, $\bar{M} \in \mathbb{M}_{(N+1)\times k}$ a expansion of M by adding the evaluations of a new agent, $\bar{M}_{N+1}$. Then

$$\begin{aligned}
\delta_\Sigma(\bar{M}) &= \tfrac{1}{C^2_{N+1}} \cdot \sum_{i<j} d_\Sigma(\bar{M}_i, \bar{M}_j) = \frac{1}{C^2_{N+1}} \cdot \sum_{i=1}^{N+1} \sum_{\substack{j=1\\ i<j}}^{N+1} d_\Sigma(\bar{M}_i, \bar{M}_j) = \\
&= \tfrac{1}{C^2_{N+1}} \cdot \left(\sum_{i=1}^{N} \sum_{\substack{j=1\\ i<j}}^{N} d_\Sigma(M_i, M_j) + \sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1}) \right) = \\
&= \tfrac{1}{C^2_{N+1}} \cdot \left(C^2_N \cdot \delta_\Sigma(M) + \sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1}) \right) = \\
&= \tfrac{N-1}{N+1} \cdot \delta_\Sigma(M) + \tfrac{1}{C^2_{N+1}} \cdot \sum_{i=1}^{N} d_\Sigma(M_i, \bar{M}_{N+1}).
\end{aligned}$$

Now we have to minimize $\delta_\Sigma(\bar{M})$. Obviously, the vector which minimizes $\delta_\Sigma(\bar{M})$ is the vector that gathers the opinion of the agent $N+1$ in the profile $\bar{M}$. For simplicity we recall $\bar{M}_{N+1}$ like $x \in \mathbb{R}^k$. From $\delta_\Sigma(\bar{M})$ expression, it is enough to resolve

$$\min_x \sum_{i=1}^{N} d_\Sigma(M_i, x) = \min_x \sum_{i=1}^{N} \left(M_i \Sigma^{-1} M_i^t - 2 M_i \Sigma^{-1} x^t + x \Sigma^{-1} x^t \right),$$

or equivalently,

$$\min_x \sum_{i=1}^{N} \left(-2M_i\Sigma^{-1}x^t + x\Sigma^{-1}x^t\right).$$

We solve it by the standard method using Point 14 in the AppendixB.

$$\begin{aligned}
\frac{\partial}{\partial x}\sum_{i=1}^{N} \left(-2M_i\Sigma^{-1}x^t + x\Sigma^{-1}x^t\right) &= -2\sum_{i=1}^{N} \left(M_i\Sigma^{-1}\right)^t + \sum_{i=1}^{N} 2\Sigma^{-1}x^t = \\
&= -2\left(\textstyle\sum_{i=1}^{N} \Sigma^{-1}M_i^t\right) + 2N\Sigma^{-1}x^t = \\
&= -2\Sigma^{-1}\left(\textstyle\sum_{i=1}^{N} M_i^t - Nx^t\right) = 0.
\end{aligned}$$

$$\sum_{i=1}^{N} M_i^t - Nx^t = 0 \Longrightarrow x = \frac{1}{N}\sum_{i=1}^{N} M_i.$$

Due to the fact that the second derivative is $2N\Sigma^{-1}$, a positive definite matrix, we have a minimum in $x = \frac{1}{N}\sum_{i=1}^{N} M_i$.

□

AppendixB. Review in matrix algebra

This appendix contains some technical results and background material of matrix analysis which are particularly useful in this paper. Let A be a real matrix of order $n \times n$.

1. A diagonal matrix A with diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ is represented as $A = diag(a_{11}, a_{22}, \ldots, a_{nn})$.
2. The trace of a matrix A of dimension $n \times n$ is the sum of its diagonal elements, i.e., $trace(A) = \sum_{i=1}^{n} = a_{ii}$.
3. Two matrices A and B of dimensions $n \times n$ are commutable if $AB = BA$. It is also said that they commute. We say that a family of $n \times n$ matrices $A_1, A_2, \ldots, A_k$ is a commutable family if for any $i, j \in \{1, \ldots, k\}$, A_i and A_j commute.
4. A matrix A is orthogonal if $A^T A = AA^T = I$, i.e. $A^{-1} = A^T$.

5. The inverse matrix of a partitioned matrix $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} and A_{22} are non singular, is

$$\begin{pmatrix} (A_{11} - A_{21}A_{22}^{-1}A_{12})^{-1} & -A_{11}A_{12}\,(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

6. Let v be a vector $n \times 1$. A symmetric matrix A is a positive semi-definite matrix (or non-negative definite matrix) if $v^tAv \geq 0$ and A is a positive definite matrix if $v^tAv > 0$ for all non-zero vector v.
7. If there exist a scalar λ and a non-zero vector γ such that $A\gamma = \lambda\gamma$, we call them an eigenvalue of A and an associated eigenvector, respectively.
8. There are up to n eigenvalues $\lambda_1, \ldots, \lambda_n$ of A. If A is a positive semi-definite matrix, its eigenvalues are all non-negative.
9. If A is a positive definite matrix, its eigenvalues $\lambda_1, \ldots, \lambda_n$ are positive values and A^{-1} has eigenvalues $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$.
10. A permutation matrix of order $n \times n$ is a square matrix obtained from the same size identity matrix by a permutation of rows. Let π be a permutation of $\{1, 2, ..., k\}$ and let e_i be the i-th vector of the canonical base of $\mathbb{R}^n$, that is, $e_{ij} = 1$ if $i = j$, $e_{ij} = 0$ otherwise. We define the permutation matrix P_π whose rows are $e_{\pi(i)}$. We rearrange the corresponding rows (resp. columns) of A using the permutation π by left (resp., right) multiplication, $P_\pi A$ (resp., AP_π). Every row and every column of a permutation matrix contain exactly one nonzero entry, which is 1. A product of permutation matrices is again a permutation matrix. The inverse of a permutation matrix is again a permutation matrix. In fact, $P^{-1} = P^t$.
11. Let A and B be $p \times p$ symmetric matrices. If $A - B$ is a non-negative definite matrix, then it is expressed as $A \geq B$. In this case $ch_i(A) \geq ch_i(B)$ for $i = 1, \ldots, p$, where $ch_i(A)$ denotes the i-th characteristic root of a symmetric matrix A, arranged in increasing order (Fujikoshi et al., 2010, pp. 497 (A.1.9)).
12. *A theorem on a simultaneous diagonizable family of matrices.* A set consisting of symmetric $n \times n$ matrices, $A_1, \ldots, A_r$, is simultaneously diagonalizable by an orthogonal matrix if and only if they commute in pairs, that is to say, for each $i \neq j$, $A_iA_j = A_jA_i$. Simultaneously diagonalizable means that there exists an orthogonal matrix U such that $U^tA_iU = D_i$ where D_i is a diagonal matrix for every A_i in the set

(Horn and Johnson, 2010, pp. 52, theorem 1.3.19) and (Harville, 1997, pp. 561).

13. If a non negative definite matrix has trace equal to zero, then this matrix is zero (Harville, 1997, pp. 238).
14. If A is a symmetric matrix $n \times n$, x and b are vectors of length n, then

$$\frac{\partial\, Ax}{\partial x} = \frac{\partial\, x^t A}{\partial x} = A\,; \quad \frac{\partial\, b^t x}{\partial x} = \frac{\partial\, x b^t}{\partial x} = b\,; \qquad \frac{\partial\, x^t A x}{\partial x} = 2 \cdot Aa.$$

See Harville (1997).

15. Spectral decomposition. Let Σ be a $k \times k$ real symmetric matrix. There exists an orthogonal matrix $\Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_k)$, whose column vectors γ_i are the normalized eigenvectors of Σ, $\gamma_i^t \gamma_i = 1$. Its eigenvalues are $\lambda_1, \ldots, \lambda_k$. It is verified that $\Gamma^t \Sigma \Gamma = D_\lambda$ where $D_\lambda = diag(\lambda_1, \ldots, \lambda_k)$ is a diagonal matrix with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$. In this way Γ is unique. [9] We note that $\Sigma = \Gamma D_\lambda \Gamma^t$, that is, $\Sigma = \sum_{i=1}^{k} \lambda_i \gamma_i \gamma_i^t$.
16. When Σ is a positive semi-definite matrix, all its characteristic roots or eigenvalues are real and greater than or equal to zero. Accordingly, the inverse of Σ is $\Sigma^{-1} = \Gamma D_\lambda^{-1} \Gamma^t$.

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[9] With this ordering D_λ is unique. If all the eigenvalues are different, Γ is unique. In other case Γ is unique except for a postfactor (a matrix which allows a different base for eigenvalues).

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