PULLBACK, FORWARDS AND CHAOTIC DYNAMICS IN 1-D NON-AUTONOMOUS LINEAR-DISSIPATIVE EQUATIONS

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Abstract. The global attractor of a skew product semiflow for a non-autonomous differential equation describes the asymptotic behaviour of the model. This attractor is usually characterized as the union, for all the parameters in the base space, of the associated pullback cocycle attractors in the product space. The continuity of the cocycle attractor in the parameter is usually a difficult question. In this paper we develop in detail a 1D non-autonomous linear differential equation and show the richness of non-autonomous dynamics by focusing on the continuity, characterization and chaotic dynamics of the cocycle attractors. In particular, we analyze the sets of continuity and discontinuity for the parameter of the attractors, and relate them with the eventually forwards behaviour of the processes. We will also find chaotic behaviour on the attractors in the Li-Yorke and Auslander-Yorke senses. Note that they hold for linear 1D equations, which shows a crucial difference with respect to the presence of chaotic dynamics in autonomous systems.

1. Introduction

We are interested in the asymptotic dynamics of initial value problems of the form

$$\begin{cases} \dot{x} = f(t,x), \ t > s \\ x(s) = x_0 \in X, \end{cases} \tag{1.1}$$

$f : \mathbb{R} \times D \subset \mathbb{R} \times X \to X$ is a map belonging to some metric space $\mathcal{C}$, and X a Banach space. Assume that, for each $f \in \mathcal{C}$, $x_0 \in X$, the solution of (1.1) is defined for all $t \geq s$; that is, for

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each $x_0 \in X$, there is a unique continuous function $[s, \infty) \ni t \mapsto x(t, s, f, x_0) \in X$ satisfying (1.1). For each t, $f(t, \cdot)$ is the vector field that drives the solution at time t. Hence, the path described by the solution in X between s and $s + \tau$ will depend on both the initial time s and elapsed time τ.

In this framework, two asymptotics give rise to completely different scenarios. We may study the asymptotics with respect to the elapsed time $t - s$ (uniformly or not in s) or with respect to s (when $s \to -\infty$ and t is arbitrary but fixed). These are called, respectively, forwards and pullback dynamics and are in general unrelated.

During the last twenty years two main approaches have been developed in order to study attractors for (1.1): on the one hand the pullback attractor ([12, 29]), an invariant set for the evolution process which is pullback (but, in general, not forwards) attracting; on the other hand the global attractor for the associated skew-product flow, an invariant compact set attracting forwards in time ([45, 29]).

There is a general method to consider the family of non-linearities as a base flow driven by the time shift applied to the non-linearity $f(t, \cdot)$ of the original equation. We consider $f \in C_b(\mathbb{R}, X)$, the set of bounded uniformly continuous functions from $\mathbb{R}$ into X with the metric ρ of the uniform convergence. Denote by P_0 the set of all translates of f,

$$P_0(f) = \{f(s + \cdot) : \ s \in \mathbb{R}\},$$

and define the shift operator $\theta_t : C_b(\mathbb{R}, X) \to C_b(\mathbb{R}, X)$ by

$$\theta_t f(\cdot) = f(\cdot + t).$$

For autonomous and periodic time dependence this construction yields a closed base space P_0. However, for more general almost-periodic terms it is convenient to consider the closure of P_0 with respect to ρ:

$$P := P_\rho(f) = \text{closure of } P_0(f) \text{ in } C_b(\mathbb{R}, X) \text{ with respect to } \rho,$$

known as the hull of the function f in the space $(C_b(\mathbb{R}, X); \rho)$, see [15, 44]. Continuity of θ_t on P_0 then extends to continuity of θ_t on P.

In this paper we consider the 1D linear and dissipative differential equation

$$x' = h(\theta_t p)x + g(x),\ p \in P, x \in \mathbb{R}^n, \tag{1.2}$$

with h a real almost-periodic function with null mean value and unbounded primitive and

$$P = \overline{\{\theta_t h,\ t \in \mathbb{R}\}}$$

the hull of h. Nete that (P, θ) is a continuous flow in a compact metric space. $g : \mathbb{R} \to \mathbb{R}$ is a smooth negative function with $\lim_{t\to\infty} g(x) = -\infty$. We denote by $C(P)$ the set of real continuous functions on P and thus $C_0(P)$ will denote the subset of $C(P)$ with null mean value. $B(P)$ will represent the subset of $C_0(P)$ with continuous primitive, and by $U(P)$ its complemetary, i.e., the subset of $C_0(P)$ of functions with unbounded primitive (see Section 3).

The pullback cocycle attractor $A(p)$ (see Definition 2.3) for (1.2) is described by an interval $[a(p), b(p)]$, for all $p \in P$. The aim of this paper is to study in detail the structure and internal dynamics on this family of attractors.

An important result in Cheban et al. [13] proves that, if the function $p \to A(p)$ is upper and lower semicontinuous, then, uniformly, pullback and forwards attraction are equivalent. The results this paper will confirm that the property of continuity of this set-valued map cannot be weakened. Indeed, in Section 3 we study, for a particular $h \in U(P)$, the set $P_s \subset P$ of continuity and not continuity $P_f \subset P$ of function $p \to A(p)$, showing that our attractor is a pinched set (see Definition 2.1), described as $A(p) = 0$ for all $p \in P_s$ and $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ for all $p \in P_f$.

For a residual set in P_s, we prove (see Proposition 4.8 and Corollary 4.10) that the is no forwards attraction to $A(p)$, i.e., we lose forwards attraction specifically in the continuity points of the cocycle attractor. In some cases this residual set is all P_s. In Section 5 we find that, generically, this is the situation we find, i.e., if we define

$$R_s(h) = \{h \in C_0(P) :\ \nu(P_s(h)) = 1\}$$

and

$$R_f(h) = \{h \in C_0(P) :\ \nu(P_f(h)) = 1\}$$

we get (see Theorem 5.2) that $R_s(P)$ is a residual set in $C_0(P)$. Although topologically more unusual, in section 5.2 we concentrate in the case when $R_f \neq \emptyset$, so that, we can deal with $h \in U(P)$, with $\nu(P_f(h)) = 1$. Theorem 5.4 proves that we get forwards attraction in P_f,, i.e., we get forwards attraction in full measure precisely in the set of not continuity of the map $p \to A(p)$.

In Section 6 we find chaos inside the pullback cocycle attractor. We think this is the first time in the literature where chaos is studied related to this kind of attractors. Indeed, Theorem 6.4 shows that, in the previous case with $h \in U(P)$ and $\nu(P_f(h)) = 1$ the sets $[-b(p), b(p)]$ are scrambled (see Definition 6.1), leading to Li-Yorke chaotic dynamics (see [8]). Finally, in Section 6.2 we can also find sensitive dependence on the set $\mathbb{A}_0 = \cup_{p \in P}\{p\} \times 0$, so that we also find chaotic dynamics in the Auslander-Yorke sense (see [5]).

2. Basic notions

We start with some preliminary concepts and results on topological dynamics and ergodic theory that can be found in Ellis [17], Nemytskii and Stepanov [37] and Shen and Yi [46].

We introduce two types of almost-periodic functions that will play a relevant role in all what follows. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ a vector with rational independent components. The Kronecker flow of vector α is defined on the $m-$dimensional torus $\mathbb{T}^m$ by the map $\theta_\alpha : \mathbb{R} \times \mathbb{T}^m \to \mathbb{T}^m$, $(t, x_1, \dots, x_m) \to (x_1 + t\alpha_1, \dots, x_m + t\alpha_m)$, which is almost periodic and minimal.

We say that a function $f \in C(\mathbb{R}, \mathbb{R})$ is quasi-periodic if there exists a Kronecker flow $(\mathbb{T}^m, \theta_\alpha)$ and a function $h \in C(\mathbb{T}^m)$ with $f(t) = h(\alpha_1 t, \dots, \alpha_m t)$ for every $t \in \mathbb{R}$. Under this condition the hull of h is isomorphic to $(\mathbb{T}^m, \theta_\alpha)$.

We say that a function $h \in C(\mathbb{R}, \mathbb{R})$ is limit-periodic if it is the uniform limit of a sequence of continuous and periodic functions. In this case the hull of h has frequently a amore complicated structure: in some cases it provides a solenoid. Many relevant examples in the literature considered in this paper have been developed by quasi-periodic or limit-periodic functions.

Let (P, d_P) a compact metric space and $\theta = \{\theta_t\}_{t \in \mathbb{R}}$ a real continuous flow on P. Given $p \in P$ the set $\{\theta_t p\}_{t \in \mathbb{R}}$ is called the orbit of p. We say that a subset $P_1 \subset P$ is θ-invariant if $\theta_t(P_1) = P_1$ for all $t \in \mathbb{R}$. A subset P_1 is minimal if it is compact invariant and it does not contain properly any other compact invariant set. We say that the continuous flow (P, θ) is recurrent or mininal if P is minimal.

Definition 2.1. *(i) A minimal set $K \subset P \times X$ is said an automorphic extension of the base P if, for some $p \in P$, $K \cap \Pi^{-1}(p)$ is singleton.*

(ii) A compact invariant set $K \subset P \times X$ is called a pinched set if there exists a residual set $P_0 \subset P$ such that $K \cap \Pi^{-1}(p)$ is a singleton for all $p \in P_0$ and $K \cap \Pi^{-1}(p)$ is not a singleton for all $p \notin P_0$. An invariant compact set $K \subset P \times X$ is almost automorphic if it is pinched and minimal.

A normalized regular measure ν defined on the Borel sets of P is invariant if $\nu(\theta_t(P_1)) = \nu(P_1)$ for every Borel subset $P_1 \subset P$ and every $t \in \mathbb{R}$. It is ergodic if, in addition, $\nu(P_1) = 1$ or $\nu(P_1) = 0$ for every invariant subset P_1. The set of normalized invariant measures is not void. We say that (P, d_P) is uniquely ergodic if it has a unique normalized invariant measure which is necessarily ergodic.

We say that the flow (P, d_P) is almost-periodic if the family $\{\theta_t\}_{t\in\mathbb{R}}$ of section maps is equicontinuous, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $p_1, p_2 \in P$ and $\mathrm{d}_P(p_1, p_2) < \delta$ then $\mathrm{d}_P(\theta_t p_1, \theta_t p_2) < \varepsilon$ for every $t \in \mathbb{R}$.

A subset $L \subset \mathbb{R}$ is said to be relatively dense if there exists a number $l > 0$ such that every interval $[r, r + l]$ contains at least a point of L. We say that $f \in C_b(\mathbb{R}, \mathbb{R})$ is almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset $L_\varepsilon(f)$ such that $\sup_{t\in\mathbb{R}} |f(t+r) - f(t)| \leq \varepsilon$ for every $r \in L_\varepsilon(f)$. If $f \in C_b(\mathbb{R}, \mathbb{R})$ is almost-periodic then the hull $P = P(f)$ of f is a compact metric space and if $\{\theta_t\}_{t\in\mathbb{R}}$ denotes the shift operator, then the flow (P, θ) is almost-periodic, minimal and ergodic. In fact P is an abelian topological group and the Haar measure is its only invariant measure.

We can try to analyse non-autonomous differential equations as the combination of a base flow $\{\theta_t\}_{t\in\mathbb{R}}$ on P and, for each $p \in P$, the semiflow $\mathbb{R}^+ \times X \ni (t, x_0) \mapsto \varphi(t, p)x_0 \in X$ where, for each $x_0 \in X$, $\mathbb{R}^+ \ni t \mapsto \varphi(t, p)x_0 \in X$ is the solution of the initial value problem

$$\begin{cases} \dot{x} = p(t, x), \ t > 0, \\ x(0) = x_0 \in X. \end{cases} \tag{2.1}$$

Then, the family of mappings

$$(t, p) \in \mathbb{R}^+ \times p \mapsto \varphi(t, p) \in \mathcal{C}(X),$$

satisfies

- $\varphi(0, p) = \mathrm{Id}_X$ for all $p \in P$,

- $x \mapsto \varphi(t,p)x \in X$ is continuous, and
- for all $t \geq s$, $s \in \mathbb{R}$, and $p \in P$,

$$\varphi(t+s,p) = \varphi(t,\theta_s p)\varphi(s,p),$$

  the 'cocycle property'.

One interprets $\varphi(t,p)x$ as the solution at time t that has started in the state x at time zero subjected to the non-autonomous driving term $p \in P$.

The pair $(\varphi,\theta)_{(X,P)}$ will be called a *non-autonomous dynamical system* on (X,P). Now, given a non-autonomous dynamical system (φ,θ) on (X,P). One can also define an associated autonomous dynamical system (see [44, 45]) $\Pi(\cdot)$ on $\mathbb{X} = P \times X$ (with the metric $\mathrm{d}_{\mathbb{X}}((x,p),(\bar{x},\bar{p})) = \mathrm{d}(x,\bar{x}) + \rho(p,\bar{p})$) by setting

$$\Pi(t)(p,x) = (\theta_t p, \varphi(t,p)x)),\ t \geq 0.$$

The semigroup property of θ_t and the cocycle property of φ ensure that $\Pi(\cdot)$ satisfies the semigroup property.

Thus, given a non-autonomous differential equation such as (1.1), we need to deal with four different dynamical systems:

(a) The *driving semigroup* $\{\theta_t\ :\ t \geq 0\}$ on p associated to the dynamics of the time-dependent nonlinearities appearing in the equation.
(b) the *skew-product semiflow* $\{\Pi(t) :\ t \geq 0\}$ defined on the product space $P \times X$,
(c) the associated *non-autonomous dynamical system* $(\varphi,\theta)_{(X,P)}$ with $\varphi(t,\theta_s f)x_0 = x(t+s,f,x_0)$,
(d) and the evolution process $S(t,s)x_0 = u(t-s,\theta_s f)x_0$.

Observe that these dynamical systems can possess an associated attractor:

(i) A global attractor $\mathbb{A}$ for the skew-product semiflow $\Pi(t)$,
(ii) a cocycle attractor $\{A(p)\}_{p\in p}$ for the cocycle semiflow φ,
(iii) a pullback attractor $\{A(t)\}_{t\in\mathbb{R}}$ for the evolution process $S(t,s)$.

We next introduce and compare some concepts of the topological and random theory of dynamical systems. Given a NDS $(\varphi,\theta)_{(X,P)}$, suppose that the associated skew product semiflow semigroup $\{\Pi(t) : t \geqslant 0\}$ possesses a global attractor $\mathbb{A}$ on $P \times X$. We know that $\{\Pi(t) : t \geqslant 0\}$ has a global attractor if and only if there exists a compact set $\mathbb{K} \subset P \times X$ such that

$$\lim_{t\to\infty} \mathrm{dist}(\Pi(t)\mathbb{B}, \mathbb{K}) = 0, \tag{2.2}$$

for any bounded subset $\mathbb{B}$ of $P \times X$, where dist denotes the Hausdorff semidistance between sets defined as

$$\mathrm{dist}(A, B) = \sup_{a\in A} \inf_{b\in b} \mathrm{d}(a, b).$$

Definition 2.2. *(i) A* non-autonomous set *is a family $\{D(p)\}_{p\in P}$ of subsets of X indexed in p. We say that $\{D(p)\}_{p\in P}$ is an open (closed, compact) non-autonomous set if each fiber $D(p)$ is an open (closed, compact) subset of X.*

(ii) A non-autonomous set $\{D(p)\}_{p\in P}$ is invariant *under the NDS $(\varphi, \theta)_{(X,P)}$ if*

$$\varphi(t, p)D(p) = D(\theta_t p),$$

for all $t \geqslant 0$ and each $p \in P$.

It is immediate that a non-autonomous set $\{D(p)\}_{p\in P}$ is invariant for $(\varphi, \theta)_{(X,P)}$ if and only if the corresponding subset $\mathbb{D}$ of $P \times X$, given by

$$\mathbb{D} = \bigcup_{p\in P} \{p\} \times D(p),$$

is invariant for the semigroup $\{\Pi(t) : t \geqslant 0\}$.

Given a subset $\mathbb{E}$ of $P \times X$ we denote by $E(p) = \{x \in X : (x, p) \in \mathbb{E}\}$ the $p-$section of $\mathbb{E}$; hence

$$\mathbb{E} = \bigcup_{p\in p} \{p\} \times E(p) \tag{2.3}$$

Given a non-autonomous set $\{E(p)\}_{p\in P}$ we denote by $\mathbb{E}$ the set defined by (2.3).

Note that

$$\bigcup_{p\in p} E(p) = \Pi_X \mathbb{E}.$$

We can now relate the concept of cocycle attractors $(\varphi, \theta)_{(X,P)}$ with the global attractor for the associated skew product semiflow $\{\Pi(t) : t \geqslant 0\}$.

Definition 2.3. *Suppose P is compact and invariant and that $\{\theta_t : t \geqslant 0\}$ is a group over P and $\theta_t^{-1} = \theta_{-t}$, for all $t > 0$. A compact non-autonomous set $\{A(p)\}_{p \in P}$ is called a* cocycle attractor *of $(\varphi, \theta)_{(X,P)}$ if*

- **(i)** *$\{A(p)\}_{p \in P}$ is invariant under the NDS $(\varphi, \theta)_{(X,P)}$; i.e., $\varphi(t, p)A(p) = A(\theta_t p)$, for all $t \geqslant 0$.*
- **(ii)** *$\{A(p)\}_{p \in P}$ pullback attracts all bounded subsets $B \subset X$, i.e.*

$$\lim_{t \to +\infty} \operatorname{dist}(\varphi(t, \theta_{-t} p)B, A(p)) = 0.$$

The following result can be found, for instance, in Propositions 3.30 and 3.31 in Kloeden and Rasmussen [29], or Theorem 3.4 in Caraballo et al. [11].

Theorem 2.4. *Let $(\varphi, \theta)_{(X,P)}$ be a non-autonomous dynamical system, where P is compact, and let $\{\Pi(t) : t \geqslant 0\}$ be the associated skew product semiflow on $P \times X$ with a global attractor $\mathbb{A}$. Then $\{A(p)\}_{p \in \Xi}$ with $A(p) = \{x \in X : (x, p) \in \mathbb{A}\}$ is the cocycle attractor of $(\varphi, \theta)_{(X,P)}$.*

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The following result offers a converse (see Proposition 3.32 in [29], or Theorem 3.4 in [11])

Theorem 2.5. *Suppose that $\{A(p)\}_{p \in P}$ is the cocycle attractor of $(\varphi, \theta)_{(X,P)}$, $\{\Pi(t) : t \geqslant 0\}$ is the associated skew product semiflow. Assume that $\{A(p)\}_{p \in P}$ is uniformly attracting, i.e.,*

$$\lim_{t \to +\infty} \sup_{p \in P} \operatorname{dist}(\varphi(t, \theta_{-t} p)D, A(p)) = 0,$$

and that $\bigcup_{p \in P} A(p)$ is precompact in X. Then the set $\mathbb{A}$ associated with $\{A(p)\}_{p \in P}$, given by

$$\mathbb{A} = \bigcup_{p \in P} \{p\} \times A(p),$$

is the global attractor of the semigroup $\{\Pi(t) : t \geqslant 0\}$.

3. Non-uniform pullback cocycle attractors

Let $(\mathrm{P}, \nu, \mathbb{R})$ a minimal flow on a compact metric space P. For X a Banach space we consider a skew-product semiflow $\{\Pi(t)\}_{t \in \mathbb{R}^+}$ on $\mathrm{P} \times X$. Suppose $\Pi(t)$ admits a global attractor $\mathbb{A}$ described by

$$\mathbb{A} = \bigcup_{p \in P} \{p\} \times A(p).$$

In Cheban et al. [13] it is proved that the continuity of the set-function $p \to A(p)$ implies the uniform pullback and forwards attraction to the cocycle attractor.

The aim of the following sections is to develop some non-trivial models in which the above function is not continuous in all P, and, by a careful study of its sets of continuity, to give a detailed description on the dynamics and the structure and of the attractors.

3.1. **Attractors for order preserving non-autonomous systems.** In what follows we supposse X is a partially ordered Banach space, i.e. there exists a closed convex positive cone $X^+ \subset X$, vectorial subspace of X such that $X^+ \cap (-X^+) = \{0\}$.

This define a partial order relation on X in the way $x \leq y$ if $y - x \in X^+$; we write $x < y$ if $x \leq y$ and $x \neq y$. If in addition $\mathrm{int}(X^+) \neq \emptyset$ we say that X is strongly ordered.

We introduce the concepts of sub, super and equilibrium given by Arnold and Chueshov [3] in the stochastic (see also Chueshov [16]) and by Novo et al. [36] in the topological setting.

Definition 3.1. *A Borel map $a : P \to X$ such that $\varphi(t,p)a(p)$ is defined for any $t \geq 0$ is*

a) *an equilibrium if $a(\theta_t p) = \varphi(t,p)a(p)$, for any $p \in P$ and $t \geq 0$,*
b) *a super-equilibrium if $a(\theta_t p) \geq \varphi(t,p)a(p)$, for any $p \in P$ and $t \geq 0$,*
c) *a sub-equilibrium if $a(\theta_t p) \leq \varphi(t,p)a(p)$, for any $p \in P$ and $t \geq 0$.*

Definition 3.2. *A super-equilibrium (resp. sub-equilibrium) $a : P \to X$ is semi-continuous is the following holds*

i $\Gamma_a = \mathrm{closure}_X\{a(p) :\ p \in P\}$ *is a compact subset in X*
ii) $C_a = \{(p,x) :\ x \leq a(p)\}$ *(resp. $C_a = \{(p,x) :\ x \geq a(p)\}$) is a closed subset of $P \times X$.*

Definition 3.3. *Let $(\varphi,\theta)_{(X,P)}$ be a non-autonomous dynamical system. We say that φ is order-preserving if there exists an order relation '$\leq$' in X such that, if $u_0 \leq v_0$, then $\varphi(t,p)u_0 \leq \varphi(t,p)v_0$, for all $p \in$ P.*

An equilibrium is semicontinuous in any os these cases. We name a semi-equilibrium to a sub-equilibrium or a super-equilibrium.

The following result, that will be relevant in the topological version of the semi-equilibria, was proved in Proposition 3.4 of Novo et al. [36], following classical arguments from Aubin and Frankowska [4].

Proposition 3.4. *Let assume that $a : P \to X$ is a semi-continuous semi-equilibrium. Then it has a residual invariant set P_c of continuity points.*

We assume that φ admits a cocycle attractor. The following result gives sufficient conditions for the existence of upper and lower asymptotically stable semi-equilibria, giving some useful information on the structure of this invariant set. The proof was given by Arnold and Chueshov [3] in the random context and generalized to the topological formulation in Novo et al. [36].

Theorem 3.5. *Let φ be an order-preserving process and $A(p)$ its associated (pullback) cocycle attractor. Suppose there exist Borel $\alpha, \beta : P \to X$ such that the cocycle attractor is in the "interval" $[\alpha(p), \beta(p)]$, i.e.*

$$A(p) \subset I_\alpha^\beta(p) = [\alpha(p), \beta(p)] = \{x \in X : \ \alpha(p) \le x \le \beta(p)\}.$$

Then, there exist two equilibria $a, b : P \to X$ with $a(p), b(p) \in A(p)$ such that

i)
$$\alpha(p) \le a(p) \le b(p) \le \beta(p), \ \textit{and}\ A(p) \subset I_a^b(p), \ \textit{for all}\ p \in \mathrm{P}.$$

ii) *a is minimal (b is maximal) in the sense that it does not exist any complete trajectory in the interval $I_\alpha^a(I_b^\beta)$.*

iii) *$a(p)$ is pullback asymptotically stable from below, that is, for all $v(\cdot)$ with $\alpha(p) \le v(p) \le a(p)$, for all $p \in \mathrm{P}$, we have that*
$$\lim_{t\to+\infty} d(\varphi(p, \theta_{-t}p)v(\theta_{-t}p), a(p)) = 0.$$
$b(p)$ is pullback asymptotically stable from above, that is, for all $v(\cdot)$ with $\beta(p) \ge v(p) \ge b(p)$, for all $p \in \mathrm{P}$, we have that
$$\lim_{t\to+\infty} d(\varphi(p, \theta_{-t}p)v(\theta_{-t}p), b(p)) = 0.$$

iv) *If $\mathbb{A} = \bigcup_{p\in P}\{p\} \times A(p)$ is compact and the maps α, β are continuous, then the functions $p \to a(p)$, $p \to b(p)$ are upper semicontinuous and admits a residual set $\mathrm{P}_c \subset \mathrm{P}$ of continuity.*

v) *Assume condition in iv), and take $p_0 \in \mathrm{P}_c$. Then the sets*
$$\mathbb{K}_a = \overline{\{(\theta_t p_0, a(\theta_t p_0)), \ t \in \mathbb{R}\}}$$

and

$$\mathbb{K}_b = \overline{\{(\theta_t p_0, b(\theta_t p_0),\ t \in \mathbb{R}\}}$$

define minimal semiflows in $\mathrm{P} \times X$*, with* $\mathbb{K}_a, \mathbb{K}_b \subset \mathbb{A}$*. Moreover*

$$\mathrm{card}(\mathbb{K}_a \cap \Pi^{-1}(p)) = \mathrm{card}(\mathbb{K}_b \cap \Pi^{-1}(p)) = 1,$$

for all $p \in \mathrm{P}_c$*, i.e.,* $\mathbb{K}_a, \mathbb{K}_b$ *are almost automorphic extensions of* (P, θ).

Proof. Items i),ii) and iii) can be found in Arnold and Chueshov [3]

Items iv) and v) are proved in Theorem 3.6 at Novo et al. [36]. We repeat the argument here, for completeness. Note that $\Gamma_a = \mathrm{closure}_X\{a(p) :\ p \in P\}$, $\Gamma_b = \mathrm{closure}_X\{b(p) :\ p \in P\}$ $\subset \Pi_x\mathbb{A}$ are compact sets in X.

From $a_T(p) = \varphi(T, \theta_{-T}p)\alpha(p)$, $b_T(p) = \varphi(T, \theta_{-T}p)\beta(p)$, we deduce that these functions are continuous semi-equilibria. If $T_1 < T_2$ then $a_{T_2} \le a_{T_1}$, $b_{T_1} \le b_{T_2}$, and $a(p) = \lim_{T\to\infty} a_T(p)$, $b(p) = \lim_{T\to\infty} b_T(p)$ for every $p \in P$, showing that these functions are equilibria. Thus,

$$\{(p, x) :\ x \le a(p)\} = \bigcup_{T\ge 0} \{(p, x) :\ x \le a_T(p)\}$$

$$\{(p, x) :\ x \ge b(p)\} = \bigcup_{T\ge 0} \{(p, x) :\ x \ge b_T(p)\}$$

are closed. In consequence, the equilibria a, b are semi-continuous, so that, by Proposition 3.4 they admit a residual invariant set $P_c \subset P$ of continuity points.

For v), let $p_0 \in \mathrm{P}$ and $p_n \to p_0$. It is clear that there exists $a_0 \in X$ such that $(p_n, a(p_n)) \to (p_0, a_0) \in \mathbb{A}$ and so $a_0 \ge a(p)$. Similarly we get it for $b(p)$. Thus, from Aubin and Frankowska [4], we conclude the existence of a residual set $\mathrm{P}_c \subset \mathrm{P}$ of continuity points of a and b.

For v), suppose $p_0 \in \mathrm{P}$ and $p_1 \in \mathrm{P}_c$. Let t_n such that $\theta_{t_n} p_0 \to p_1$. Then, by continuity, we also have that $a(\theta_{t_n} p_0) \to a(p_1)$ and $b(\theta_{t_n} p_0) \to b(p_1)$. Thus, $\mathbb{K}_a \cap \Pi^{-1}(p_1) = \{(p_1, a(p_1))\}$ and $\mathbb{K}_b \cap \Pi^{-1}(p_1) = \{(p_1, b(p_1))\}$. This implies that $\mathbb{K}_a, \mathbb{K}_b$ are minimal semiflows and sections (in p) are singleton if $p \in \mathrm{P}_c$, so that they are almost automorphic extension of (P, θ). $\square$

Remark 3.6. *We want to study the continuity of the cocycle attractor* $A(p)$*. Note that, in this framework, the continuity of* $A(p)$ *requires continuity of functions* $a(\cdot), b(\cdot)$.

3.2. **Oscillatory functions on an almost periodic base.** In the following we consider (P, θ) minimal and almost periodic. Then, P is ergodic with a unique invariant measure ν given by Haar measure . Let

$$C_0(\mathrm{P}) = \{h \in C(\mathrm{P}) : \int h d\nu = 0\}.$$

From now on, and for a more clear writing, we will write $pt = \theta_t p$, for any $p \in P$.

The following result is classical and can be found in Gottschalk and Hedlund [20]

Proposition 3.7. *Let $h \in C_0(P)$. The following items are equivalent*

i) *There exists $k \in C(P)$ satisfying*

$$k(pt) - k(p) = \int_0^t h(ps)ds \tag{3.1}$$

for all $p \in P, t \in \mathbb{R}$.

ii) *For all $p \in P$ it holds*

$$\sup\{|\int_0^t h(ps)ds|,\ t \in \mathbb{R}\} < \infty.$$

iii) *There exists $p_0 \in P$ such that*

$$\sup\{|\int_0^t h(p_0 s)ds|,\ t \in \mathbb{R}\} < \infty.$$

iv) *There exists $p_0 \in P$ such that*

$$\sup_{t \geq 0}\{|\int_0^t h(p_0 s)ds|\} < \infty.$$

We denote by $B(P) = \{h \in C_0(P)$ satisfying (3.1)$\}$, i.e., the set of functions in $C_0(P)$ with bounded primitive. It is known that if P is almost-periodic but no periodic it holds that $C_0(P) \setminus B(P) \neq \emptyset$. Moreover, it is easy to see that

i) $B(P)$ is dense in $C_0(P)$.
ii) $U(P) = C_0(P) \setminus B(P)$ is residual in $C_0(P)$.

The following theorem comes from Johnson [24] (see also Jorba et al. [28]):

Theorem 3.8. *Let $h \in U(P)$. Then there exists a residual invariant set $P_o \subset P$ such that for all $p_0 \in P_o$ there exist sequences $\{t_n^i\}_{n \in \mathbb{N}}, \quad i = 1, \ldots, 4$ with*

$$\lim_{n\to\infty} t_n^i = \infty, \quad i = 1, 2, \quad \lim_{n\to\infty} t_n^i = -\infty, \quad i = 3, 4,$$

and

$$\lim_{n\to\infty}\int_0^{t_n^i} h(p_0 s)ds = \infty, \quad i = 1,3,$$
$$\lim_{n\to\infty}\int_0^{t_n^i} h(p_0 s)ds = -\infty, \quad i = 2,4.$$

3.3. **A 1-D linear model for $h \in U(P)$.** Consider the linear equation

$$y'(t) = h(pt)y(t), \quad p \in P,\ t, y \in \mathbb{R}. \tag{3.2}$$

with $h \in U(P)$. For each $p \in P$ and $y_0 \in \mathbb{R}$ we denote by $y(t)$ the solution through p with initial value y_0, i.e, $y(0, p; y_0) = y_0$. It is easy to check that Equation (3.2) has no exponential dichotomy in $C_0(P)$ (see, for instance, Sacker and Sell [42]). Thus, there exists a nontrivial bounded solution (Selgrade [43]), i.e., there exists $p_0 \in P \setminus P_c$, $y_0 \in \mathbb{R}$ with

$$y(t, p_0; y_0) = y_0 e^{\displaystyle\int_0^t h(p_0 s)ds} \text{ bounded}, \tag{3.3}$$

so that for $c_1 \in R$

$$\int_0^t h(p_0 s)ds \le c_1, \text{ for all } t \in \mathbb{R}.$$

For p_0 satisfying (3.3), we define

$$M_0 = \overline{\{(p_0 t, \pm y(t, p_0; y_0)),\ t \in \mathbb{R}\}}^{P\times X}$$

It is clear that M_0 is an invariant compact set in $P \times X$.

Lemma 3.9. *It holds that*

a) *If $(p, x) \in M_0$ then $(p, -x) \in M_0$.*
b) $(p_0, \pm 1) \in M_0$.
c) $\{p\} \times \{0\} \in M_0$ *for all $p \in P$.*
d) $M_0 \cap \Pi^{-1}(p) = \{p\} \times \{0\}$ *for all $p \in P_o$, where P_o comes from Theorem 3.5.*

Proof. We only need to prove d). If d) is not true, let $p_1 \in P_o$ and $y_1 \in \mathbb{R}^+ \setminus \{0\}$ with $(p_1, y_1) \in M_0$. Then $\{(p_1 t, y(t, p_1; y_1),\ t \in \mathbb{R}\} \subset M_0$, as it is a compact invariant set, but $y(t, p_1; y_1) = y_1 e^{\int_0^t h(p_1 s)ds}$ is unbounded in t, which is a contradiction. ☐

Remark 3.10. *Note that, if $p_0 \in P_o$, then $b(p_0) = 0$ implies $b(p_0 t) = 0$ for all $t \ge 0$. Moreover, we also have that if $b(p) = 0$ then $p \in P_c$, i.e., it is a continuity point of function $b(\cdot)$.*

The above lemma is showing that the set M_0 is pinched, since is the singleton $p \times \{0\}$ for $p \in P_0$ and strictly bigger (containing $(p_0, \pm 1)$) outside P_0. In what follows we will take advantage of this fact.

3.4. **A 1-D nonlinear equation for $h \in U(P)$.** Let

$$r_0 = 2 \sup\{x \in \mathbb{R} : \text{ such that } (p, x) \in M_0\}. \tag{3.4}$$

In the following model we will find a pullback attractor which is a pinched set by containing M_0. We define the family of linear-dissipative differential equations given by

$$x' = h(pt)x + g(x), \tag{3.5}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous function with $g(x) = 0$ if $x \in [-r_0, r_0]$, $-xg(x) \leq 0$ for all $x \in \mathbb{R}$, $\lim_{x\to\infty} g(x) = -\infty$ and $\lim_{x\to-\infty} g(x) = \infty$. For simplicity we take in what follows

$$g(x) = \begin{cases} -(x - r_0)^2 & x \geq r_0 \\ 0 & -r_0 \leq x \leq r_0 \\ (x + r_0)^2 & x \leq -r_0 \end{cases} \tag{3.6}$$

An alternative study of the structure of the set of bounded solutions for a convex or concave scalar ODE was given in Alonso and Obaya [1]. For each $p \in P$ and $x_0 \in \mathbb{R}$ we denote by $y(t)$ the solution through p with initial value x_0, i.e, $x(0, p; x_0) = x_0$. Note that if $r >> r_0$ it holds that $h(p)r + g(r) < 0$ and $-h(p)r + g(-r) > 0$, i.e., the functions $\beta(p) = r$ and $\alpha(p) = -r$ are continuous super and sub-equilibria respectively, i.e, if $x(t, p; x_0)$ is solution of (3.5)

$$x(t, p; r) \leq r, \text{ for all } t \geq 0, p \in P,$$

$$x(t, p; r) \geq -r, \text{ for all } t \geq 0, p \in P.$$

We define, for $T > 0$,

$$b_T(p) = x(T, p(-T); r)$$

and

$$a_T(p) = x(T, p(-T); -r).$$

Then b_T, a_T are respectively super and sub-equilibria satisfying

$$0 \leq b_{T_1}(p) \leq b_{T_2} \leq r,$$

$$-r \leq a_{T_2}(p) \leq a_{T_1} \leq 0,$$

for all $p \in P$, $0 < T_2 < T_1$.

From now on we fix r, b_T and a_T. Define

$$b(p) = \lim_{T\to\infty} b_T(p); \quad a(p) = \lim_{T\to\infty} a_T(p). \tag{3.7}$$

Proposition 3.11. *The following items hold:*

a) $a, b : P \to [-r, r]$ *are equilibria for* (3.5), *i.e., for all* $p \in P$ *and* $t \in \mathbb{R}$

$$x(t, p; a(p)) = a(pt), \quad x(t, p; b(p)) = b(pt).$$

b) $a(p) = -b(p)$, *for all* $p \in P$.

c) $M_0 \subset \bigcup_{p\in P}\{p\} \times [a(p), b(p)]$. *In particular,* $a(p_0) < 0, b(p_0) > 0$.

d) *There exists a residual set* P_s *such that, for all* $p \in P_s$ *it holds* $a(p) = 0 = b(p)$.

e) *For all* $p \in P$, $\sup_{t\in\mathbb{R}} b(pt) \geq r_0$.

Proof. a) is a consequence of Theorem 3.5. Note that $a_T(p) = -b_T(p)$ for all $T > 0, p \in P$, which implies b).

For c), define

$$b_0(p) = \sup\{x \in X : (p, x) \in M_0\}; \quad a_0(p) = \inf\{x \in X : (p, x) \in M_0\}.$$

It is clear that $b_0(p) = -a_0(p)$. Since $x(T, p(-T), b_0(p(-T)) = b_0(p)$, we have that

$$b(p) = \lim_{T\to\infty} b_T((p) = \lim_{T\to\infty} x(T, p(-T), r) \geq b_0(p),$$

and, similarly,

$$a(p) = \lim_{T\to\infty} a_T((p) = \lim_{T\to\infty} x(T, p(-T), r) \geq a_0(p).$$

In particular, $b(p_0) > 0$ and $a(p_0) < 0$.

For d), it follows from Theorem 3.4 that a, b possess a subset P_s of points of continuity. We will prove that $a(p) = b(p) = 0$ for all $p \in P_s$. Indeed, if there exists $p_1 \in P_s$ with $b(p_1) = 2\delta > 0$ for some $\delta > 0$ there exists $\tilde{r} > 0$ such that, for all $p \in P$ with $d(p, p_1) \leq \tilde{r}$ we have $b(p) > \delta$. From the minimality of (P, θ) there exists $T > 0$ such that if $p \in P$ we can find $0 \leq t \leq t(p) \leq T$

with $pt \in \overline{B}(p_1, \tilde{r})$. Moreover, $b(p) = x(-t, pt; b(pt))$.

Thus, the application

$$x : [-T, 0] \times \overline{B}(p_1, \tilde{r}) \times [\delta, r] \longrightarrow \mathbb{R}^+$$
$$(t, p; x_0) \longrightarrow x(t, p; x_0) > 0$$

is continuous and strictly positive on a compact set, so that there exists $\delta_1 > 0$ with $x(t, p; x_0) > \delta_1$ for all $(t, p; x_0) \in [-T, 0] \times \overline{B}(p_1, \tilde{r}) \times [\delta, r]$. In particular, as for all $p \in P$ there exists $t \in [0, T]$ with $b(p) = x(-t, pt; b(pt))$, $d(pt, p_1) \leq \tilde{r}$, then $b(p) \geq \delta_1 > 0$, for all $p \in P$. Moreover,

$$b'(tp) = h(tp)b(pt) + g(b(pt)) \leq h(tp)b(pt),$$

Thus, an standard argument of comparison provides

$$y(t, p; b(p)) \geq b(pt) \geq \delta_1 \text{ for all } p \in P, t \geq 0.$$

But, if $p_0 \in P_0$ (P_0 from Theorem 3.5) there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ and

$$\lim_{n\to\infty} y(t_n, p_0; b(p_0)) = \lim_{n\to\infty} b(p_0)e^{\int_0^{t_n} h(p_0 s)ds} = 0,$$

which implies $\lim_{n\to\infty} b(p_0 t_n) = 0$, a contradiction. As a consequence, $b(p) = 0$ for all $p \in P_s$. For the proof of e) we again argue by contradiction. Suppose $p \in P$ with $\sup_{t\in\mathbb{R}} b(pt) \leq \delta < r_0$. This means that the function $t \to \frac{r_0}{\delta} b(pt)$ is a bounded solution of the linear equation (3.2). The same argument in c) leads to e) □

We can now prove the main result for the attractor associated to (3.5).

Theorem 3.12. *In the above conditions,*

a) $\{A(p) = [a(p), b(p)]\}_{p\in P}$ *is the pullback cocycle attractor for* (3.5).

b) *The maps*

$$P \longrightarrow \mathbb{R}$$
$$p \longrightarrow b(p)$$
$$p \longrightarrow -b(p) = a(p)$$

are continuous in the residual set $P_s = \{p \in P : \, b(p) = 0\}$ *and discontinuous in* $P \setminus P_s$.

c) $\mathbb{A} = \cup_{p\in P}\{p\} \times [a(p), b(p)]$ *is the global attractor for* (3.5) *with respect to the associated skew-product semiflow* Π.

Proof. a) and b) follows directly from Proposition 3.11, and c) from Theorem 2.5. □

4. Recurrent and asymptotic points. Forwards versus pullback attraction

Let consider $h \in U(P)$ and the function $H(t,p) = \displaystyle\int_0^t h(ps)ds$, $p \in P, t \in \mathbb{R}$. We next introduce different possible properties of H with important dynamical consequences on the corresponding cocycle attractors. Precise examples of all these situations appear in the work of Poincaré (see [39]) and the references therein; such examples have been constructed in the quasi-periodic and limit-periodic cases.

Definition 4.1. a) *A point $p \in P$ is said (Poincaré) recurrent at ∞ for h if there exists a sequence $t_n \to \infty$ with $\displaystyle\int_0^{t_n} h(ps)ds \to 0$. Anagously, a point $p \in P$ is said (Poincaré) recurrent at $-\infty$ for h if there exists a sequence $t_n \to -\infty$ with $\displaystyle\int_0^{t_n} h(ps)ds \to 0$.*

b) *A point $p \in P$ is said asymptotic for h if $\displaystyle\int_0^{t} h(ps)ds \to -\infty$ as $t \to \infty$.*

Note that if $h \in B(P)$ all $p \in P$ is recurrent. We will denote by P_r^+ the set of recurrent points at ∞, by P_r^- the set of recurrent points at $-\infty$, and by $P_r = P_r^+ \cap P_r^-$.

The following result comes from Shneiberg [47].

Theorem 4.2. *Let $h \in C_0(P)$. The set $P_r \subset P$ of recurrent points is invariant and of full measure, i.e. $\nu(P_r) = 1$.*

It is immediate that the set of oscillatory points P_o satisfies $P_0 \subset P_r$. . As a consequence, P_r is residual and has full measure. The argument of Steinberg [47] proves that the set P_r has full measure. The invariance in the present conditions is a simple application of Fubbini's theorem.

Moreover, for the n-dimensional torus, we have that all the points are recurrent in the quasi periodic case if enough regularity is required (Kozlov [31], Konyagin [30], Moschevitin [34]):

Theorem 4.3. *Let $n \geq 1$. Then there exists $k_n \in \mathbb{N}$ such that, if $h \in C^k(\mathbb{T}^n) \cap C_0(\mathbb{T}^n)$ then every $p \in \mathbb{T}^n$ is recurrent for h.*

This result was deduced by Kozlov [31] for $n = 2$ and conjetures for the general case. It has been proved by Konyagin [30] for n odd and by Moshchevitin [34] for general $n \geq 1$. Last result leads us to the following definition

Definition 4.4. *A function $h \in C_0(P)$ is Kozlov if every $p \in P$ is recurrent for h.*

We consider $h \in C_0(p)$ and the above framework for (3.5). Then there exists P_f, invariant and of first category, and its complementary, the residual set P_s, such that the pullback attractor $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ if $p \in P_f$ and $b(p) = 0$ if $p \in P_s$. Let P_r be the recurrent points and P_a the asymptotic points. Recall that we denote by P_o the oscillatory points in P. We firstly have the following result

Proposition 4.5. *Let $p_0 \in P$.*

i)

$$\sup_{t\le 0} \int_0^t h(p_0 s)ds = \infty \text{ if and only if } b(p_0) = 0, \text{ i.e., } p_0 \in P_s,$$

and

ii)

$$\sup_{t\le 0} \int_0^t h(p_0 s)ds < \infty \text{ if and only if } b(p_0) > 0, \text{ i.e., } p_0 \in P_f.$$

iii) *If*

$$\limsup_{t\to-\infty} \int_0^t h(p_0 s)ds < \sup_{t\in\mathbb{R}} \int_0^t h(p_0 s)ds$$

then $p_0 \in P_s$ and there exists $t \in \mathbb{R}$ with $b(p_0 t) > r_0$.

Proof. i) Let $y_{p_0}(t) = e^{\int_0^t h(p_0 s)ds}$. Note that $y_{p_0}(t) = y(t, p_0; 1)$, $y_{p_0}(0) = 1$, with $y(t)$ the solution of (3.2). Then there exists $t_n \to \infty$ with $\lim_{n\to\infty} \int_0^{-t_n} h(p_0 s)ds \to \infty$. Suppose n big enough. For $T > 0$ $b_T(p_0) \le y(T, p_0(-T), r)$, and $b(p_0) = \lim T \to \infty b_T(p_0)$. We have

$$y(t_n, p_0(-t_n), r) = \frac{r}{y_{p_0}(-t_n)} y(t_n, p_0(-t_n), y_{p_0}(-t_n)) = \frac{r}{y_{p_0}(-t_n)}$$

converges to zero as $n \to \infty$, which implies the equivalence with $b(p_0) = 0$, for all $p_0 \in P_s$.

For ii), let $\rho > 0$ with

$$\sup_{t\le 0} \rho \int_0^t h(p_0 s)ds \le r_0.$$

Then, if $x(t; p_0; \rho)$ is the solution of (3.5) with $x(0) = \rho$ it holds

$$x(t, p_0; \rho) = \rho \int_0^t h(p_0 s)ds \text{ for all } t \le 0.$$

On the other hand, since $\{x(t, p_0; \rho) : \; t \in R\}$ is bounded, it is on the pullback attractor, i.e. $[0, \rho] \subset [0, b(p_0)]$ and then $b(p_0) > 0$.

For iii), there are $t_1 < t_2$ with $\int_0^t h(p_0 s)ds < \int_0^{t_2} h(p_0 s)ds = \rho$ for every $t \leq t_1$. Let $y(t) = y(t, p_0; \frac{r_0}{\rho}) = \frac{r_0}{\rho} \int_0^t h(p_0 s)ds$ solution of (3.2). Let $t_3 \in (-\infty, t_2]$ be the first point with $y(t_3) = r_0$. There exists $\gamma > 1$ with $\gamma y(t_1, p_0; \frac{r_0}{\rho}) = y(t_1, p_0; \frac{\gamma r_0}{\rho}) = \gamma r_0$. Hence the solution of the nonlinear equation (3.5) satisfies

$$x(t_3, p_0; \frac{\gamma r_0}{\rho}) > x(t_3, p_0; \frac{r_0}{\rho}) = r_0,$$

which implies that $b(p_0 t_3) \geq x(t_3, p_0; \frac{\gamma r_0}{\rho}) > r_0$. $\square$

By this last result we get

Corollary 4.6. *It holds that $P_0 \subset P_s$.*

The following result characterizes the forwards attraction in the pullback attractor.

Proposition 4.7. *Let $p_0 \in P$ and $x_0 \in \mathbb{R}$. Then it holds*

$$\lim_{t\to\infty} x(t, p_0; x_0) = 0 \text{ if and only if } p_0 \in P_a.$$

Proof. Suppose there exists t_0 such that if $t \geq t_0$ then $x(t, p_0; x_0) \leq r_0$. Then

$$x(t, p_0; x_0) = x(t - t_0, p_0 t_0; x(t_0, p_0; x_0)) = x(t_0, p_0; x_0) e^{\int_0^{t-t_0} h(p_0(t_0+s))ds},$$

which implies

$$\lim_{t\to\infty} \int_0^t h(p_0 s)ds = -\infty,$$

so that $p_0 \in P_a$.

On the other hand,

$$0 < x(t, p_0; x_0) \leq x_0 e^{\int_0^t h(p_0(s))ds}$$

which tends to zero if $p_0 \in P_a$. $\square$

Proposition 4.8. *Let $p_0 \in P$.*

i) *If $p_0 \in P_a$, the process $\varphi(t, p_0)$ has a forwards attractor defined by $\{0\}$.*

ii) *If $p_0 \in P_s \cap P_r^+$, the process $\varphi(t, p_0)$ has no forwards attractor.*

Proof. i) and ii) are clear from Proposition 4.7. Indeed, if we have forwards attraction to zero we have that $p_0 \in P_a$. □

Remark 4.9. *Note that if $p_0 \in P_a \cap P_f$ we have proved that $b(p_0) > 0$ and $\lim_{t\to\infty} b(p_0 t) = 0$. Thus, a proper definition of a forwards attractor $\{A(p)\}_{p\in P}$ for the cocycle should consider minimality for the family $A(p)$, in the sense that there is no proper invariant forwards attracting family included in $A(p)$. Thus, for $p_0 \in P_a \cap P_f$ the forwards attractor should be defined as the constant family $A(p) = 0$.*

The following results are immediate consequences of Proposition 4.8.

Corollary 4.10. *If h is Kozlov then there is no forwards attractor in P_s.*

Corollary 4.11. *If $h \in U(P)$ and $\nu(P_s(h)) = 1$ then there exists a residual set of full measure, P_s^* such that, if $p_0 \in P_s^*$ the process $\varphi(t, p_0)$ has no forwards attractor.*

5. The sets $R_s(P), R_f(P)$. Genericity of $\nu(P_s) = 1$.

We define the sets

$$R_s(P) = \{h \in C_0(P) :\ \nu(P_s(h)) = 1\}$$

and

$$R_f(P) = \{h \in C_0(P) :\ \nu(P_f(h)) = 1\}$$

It is clear that $B(P) \subset R_s(P)$ and that $R_s(P) \cup R_f(P) = C_0(P)$. In this section we analyze the topological size of these sets in $C_0(P)$.

5.1. $R_s(P)$ is residual in $C_0(P)$.

We consider the time reversed flow $\hat{\sigma}$ on $\mathbb{R} \times P$ defined as

$$\hat{\sigma}(t, p) = p(-t).$$

If $y(t)$ is a solution of (3.2) then $\hat{y}(t) = y(-t)$ satisfies $\hat{y}'(t) = -h(p_0(-t))\hat{y}(t)$. For simplicity we denote by $\hat{P}$ the base space with time reversed flow, i.e. $\hat{P} = (P, \hat{\sigma}, \mathbb{R})$. Note that the reverse of the flow $\hat{\sigma}$ is again σ.

Proposition 5.1. *It holds*

i) *For any $h \in C_0(P)$, either $h \in R_s(P)$ or $-h \in R_s(P)$.*

ii) *For the time-reversed flow,* $R_s(P) = R_s(\hat{P})$, $R_f(P) = R_f(\hat{P})$.

Proof. Let $h \in U(p)$ and fix $p_0 \in P$.
If $\sup_{t\leq 0} \int_0^t h(p_0 s)ds \leq \infty$ then it follows from Proposition 3.7 that $\inf_{t\leq 0} \int_0^t h(p_0 s)ds = -\infty$, so that $\sup_{t\leq 0} \int_0^t -h(p_0 s)ds = \infty$ and $p_0 \in P_s(-h)$. As a consequence of the ergodicity of ν we conclude that at least either $\nu(P_s(h)) = 1$ or $\nu(P_s(-h)) = 1$.
For ii), suppose that we are in the case with $\nu(P_s(h)) = 1$ and take $p_0 \in P_s(h)$, so that

$$\sup_{t\leq 0} \int_0^t h(p_0 s)ds = \infty.$$

Then there exist a sequence $t_n^1 \to -\infty$ with

$$\int_0^{t_n^1} h(p_0 s)ds = \infty.$$

As $\nu(P_r) = 1$ we can suppose that all the points of the sets $\{p_0 t : \ t \in \mathbb{R}\}$ are recurrent points. For each $n \in \mathbb{N}$ there exists $t_n^2 > 0$ such that the sequence

$$\int_{t_n^1}^{t_n^2} h(p_0 s)ds = \int_0^{t_n^2 - t_n^1} h(p_0(t_n^1 + s))ds \to 0,$$

this last property by the recurrence of path. But note that

$$\int_{t_n^1}^{t_n^2} h(p_0(s))ds = -\int_0^{t_n^1} h(p_0(s))ds + \int_0^{t_n^2} h(p_0(s))ds$$

so that, as $\int_0^{t_n^1} h(p_0(t_n^1 + s))ds \to -\infty$, it holds that $\sup_{t\geq 0} \int_0^t h(p_0 s)ds = \infty$.

We consider the time reversed flow $\hat{\sigma}$ on $\mathbb{R} \times P$. Since, for $t > 0$

$$\int_0^t h(p_0 s)ds = \int_0^{-t} -h(p_0(-s))ds,$$

it holds

$$\sup_{t\leq 0} \int_0^{-t} -h(p_0(-s))ds = \sup_{t\geq 0} \int_0^t h(p_0 s)ds\infty$$

hence $p_0 \in \hat{P}_s(-h)$ and then we have that $\nu(\hat{P}_s(-h)) = 1$. As a consequence $R_s(P) \subset R_s(\hat{P})$. A symmetric argument proves that $R_s(\hat{P}) \subset R_s(P)$ and thus equality. $\square$

Observe that this last result shows how big $R_s(P)$ is on $C_0(P)$, since shows that $C_0(P) = R_s(p)\bigcup(-R_s(P))$.

The references Johnson [27] and Novo and Obaya [35] provide examples of functions $h \in U(P)$ and $L : P \to \mathbb{R}$ measurable with $L(pt)-L(p) = \int_0^t h(ps)ds$ for almost every $p \in P$ and $t \in \mathbb{R}$. We say that L is a measurable primitive along the flow on h. The condition $h \in R_f(P)$ requires in addition that $e^L \in L^\infty(P)$. The example 3.2.1 in Johnson [27] uses methods, already suggested in Anosov [2], to construct quasi period flows in the 2D torus $\mathbb{T}^2$ and a function $h \in C_0(\mathbb{T}^2)$ with

$$L(p) = \sup_{t\in\mathbb{R}} \int_0^t h(ps)ds \leq L_0 < \infty \quad a.s.$$

In this case $h \in R_f(P)$ and, moreover, for a.a. $p \in P$

$$L(pt) - L(p) = \int_0^t h(ps)ds \text{ for all } t \in \mathbb{R}.$$

This method was improved in the Appendix of Ortega and Tarallo [38], which in particular implies that this kind of function h exists for every quasi-periodic flow.

Theorem 5.2. i) *$R_s(P)$ is a residual set in $C_0(P)$.*

ii) *The set $R_0(P) = \{h \in C_0(P) :\ \nu(P_o(h) = 1)\}$ is also residual in $C_0(P)$.*

Proof. For $h \in C_0(P)$, $k \in \mathbb{N}, k \geq 1$ we define

$$N_k(h) = \{p \in P :\ \limsup_{T\to\infty} \frac{1}{2T} l(\{t \in [-T,T] :\ |\int_0^t h(ps)ds| \leq k\}) = 0\},$$

with l the Lebesgue measure in $\mathbb{R}$. It is clear that $N_{k+1}(h) \subset N_k(h)$ for all $k \in \mathbb{N}$. In Johnson [26] it is proved that the set

$$D_0 = \{h \in C_0(P) :\ \nu(N_k(h) = 1 \text{ for all } k \in \mathbb{N}\}$$

is residual in $C_0(P)$. If $h \in R_f(P)$ then $\nu(P_f(h)) = \nu(\hat{P}_f(h) = 1$, i.e.

$$L(p) = \sup_{t\in\mathbb{R}} \int_0^t h(ps)ds < \infty \quad a.e.$$

and hence for a.e. $p \in P$ and all $t \in \mathbb{R}$

$$L(pt) - L(p) = \int_0^t h(ps)ds.$$

Thus, for k big enough

$$\limsup_{T\to\infty} \frac{1}{2T} l(\{t \in [-T,T] : \, |\int_0^t h(ps)ds| \leq k\}) > 0$$

for a.a. $p \in P$. Thus, for k big enough, $\nu(N_k(h)) = 0$, so that $h \in C_0(P) \setminus N$, i.e., $D_0 \subset R_s(P)$ and this set is residual.

For ii), Let R^* the set of functions $h \in U(P)$ satisfying

$$\sup_{t\leq 0} \int_0^t h(p_0 s)ds = \sup_{t\geq 0} \int_0^t h(p_0 s)ds = \infty \text{ for a.a. } p \in P,$$

$$\inf_{t\leq 0} \int_0^t h(p_0 s)ds = \inf_{t\geq 0} \int_0^t h(p_0 s)ds = -\infty \text{ for a.a. } p \in P.$$

Then $R^* = R_s(P) \bigcap (-R_s(P))$ is residual, which implies that

$$R_0(P) = \{h \in U(P) : \, \nu(P_0(h)) = 1\} \text{ is a residual set in } C_0(P).$$

□

5.2. **The case $R_f(h)) \neq \emptyset$.** In this section suppose there exists $h \in U(P)$ with $\nu(P_f(h)) = 1$. Then it holds

Proposition 5.3. *$R_f(P)$ is a dense first category set in $C_0(P)$.*

Proof. From the last result, it is clear that $R_f(P)$ is of first category. Fix $h^* \in R_f(P)$. Then

$$\{h + \rho h^* : \, h \in B(P), \, \rho > 0\} \subset R_f(P).$$

Fix $h \in C_0(P)$ and $\varepsilon > 0$. There exist $h_0 \in B(P)$, $\rho_0 > 0$ with $||h - h_0|| < \varepsilon/2$, and $\rho_0 ||h^*|| < \varepsilon/2$. Then $h_0 + \rho_0 h^* \in R_f(P)$ and $||h - h_0 - \rho_0 h^*|| < \varepsilon$. □

In Section 3.4 we have shown the existence of a pullback cocycle attractor which contains a pinched set, which is continuous in parameter p if $p \in P_s$ and which is not forward attracting in the residual set P_s.

The following result gives a forward attraction to the pullback cocycle attractor in a set of no continuity and of full measure. Note that, from the result in Cheban et al [13] one could tend to think that the forwards attraction in a pullback attractor is related to the continuity in the parameter for the cocycle attractor. The following result shows that the uniformity condition for the continuity in [13] is necessary.

Theorem 5.4. *Let $h \in C_0(P) \setminus B(P)$, with $\nu(P_s(h)) = 0$. Then there exists an invariant set $\Delta \subset P_f(h)$ with $\nu(\Delta) = 1$ such that if $p \in \Delta$ then $A(p)$ is the forwards attractor of the process $\varphi(t, \theta_s p)x_0 = x(t-s, ps; x_0)$ associated to* (3.5).

Proof. For $p \in P_f(h)$ we have that $A(p) = [a(p), b(p)]$ with $a(p) < 0 < b(p)$. Moreover, for r big enough,

$$b(p) = \lim_{T\to\infty} b_T(p) = \lim_{T\to\infty} x(T, p(-T); r); \quad a(p) = \lim_{T\to\infty} a_T(p) = \lim_{T\to\infty} x(T, p(-T); -r).$$

By Egorov theorem (Rudin [41]) there exists a compact set $\Delta_0 \subset P_f(h)$ with $\nu(\Delta_0) > 0$ (as close to one as desired) such that

$$b(p) = \lim_{T\to\infty} b_T(p) \text{ uniformly in } \Delta_0.$$

Thus, b is continuous in the compact set Δ_0 and then there exists $\delta > 0$ with $b_{|\Delta_0} \geq \delta > 0$. Let $\lambda \geq 1$. We now prove that $\lambda g(b(p)) \geq g(\lambda b(p))$. Indeed, if $g(b(p)) = 0$ is clear. If $g(b(p)) = -(b(p) - r_0)^2$ then $g(\lambda b(p)) = -(\lambda b(p) - r_0)^2 \leq -\lambda^2 (b(p) - r_0)^2 \leq -\lambda (b(p) - r_0)^2 = \lambda g(b(p))$. Thus,

$$(\lambda b(p))' = \lambda b(p) h(p) + \lambda g(b(p)) \geq h(p) \lambda b(p) + g(\lambda b(p)),$$

which means that $\lambda b(p)$ is a super-equlibrium for (3.5). Thus, if $\lambda > 1$ and $p \in P$

$$b(pt) \leq x(t, p; \lambda b(p)) \leq \lambda b(pt), \text{ for all } t \geq 0.$$

By Birkhoff's Ergodic Theorem (Nemytskii and Stepanov [37]) there exists an invariant set Δ with $\nu(\Delta) = 1$ such that for all $p \in \Delta$ there exists a sequence $\{t_n^*\}_{n\in\mathbb{N}}$ with $t_n^* \to \infty$ and $pt_n^* \in \Delta_0$. We will prove that for $p \in \Delta$ and $r > r_0$ big enough we have that

$$\lim_{t\to\infty} x(t, p; r) - b(pt) = 0.$$

Let $\varepsilon > 0$ and $\lambda > 1$ with $b(p)(\lambda - 1) \leq \varepsilon$ for all $p \in P$. For $p \in \Delta$, there exists a t_n^* with $pt_n^* \in \Delta_0$ satisfying $b_{t_n^*}(pt_n^*) \leq \lambda b(pt_n^*)$ by the uniform convergence in Δ_0, hence

$$0 \leq b_{t_n^*}(pt_n^*) - b(pt_n^*) \leq (\lambda - 1) b(pt_n^*) \leq \varepsilon.$$

Then, if $t \geq t_n^*$ it holds that

$$\begin{aligned} x(t,p;r) &= x(t-t_n^*, pt_n^*; x(t_n^*,p;r)) = x(t-t_n^*, pt_n^*; b_{t_n^*}(pt_n^*)) \\ &\leq x(t-t_n^*, pt_n^*; \lambda b(pt_n^*)) \\ &\leq \lambda b(t-t_n^*, pt_n^*; b(pt_n^*)) = \lambda b(pt). \end{aligned}$$

Then, for all $t \geq t_n^*$,

$$0 \leq x(t,p;r) - b(pt) \leq (\lambda - 1) b(pt) \leq \varepsilon,$$

which implies the forwards convergence in Δ.

□

Remark 5.5. *Note that in this case we have obtained that the cocycle attractor $A(p) \neq \{0\}$ with full measure (as $\nu(\Delta) = 1$ in a subset of no continuity points for the cocycle attractor, in which we also find forwards attraction. We see that is a natural fact not to get forwards convergence where the cocycle attractor is continuous.*

6. Chaotic dynamics on the attractor

In this last section we study in detail the dynamical complexity of cocycle attractors. We show the presence of different types of chaotic behaviour in our cocycle attractor. In particular, we prove that the attractor possesses chaotic dynamics in the Li-Yorke sense, and that there exists sensitive dependence on initial conditions.

6.1. **Chaotic cocycle attractors in the Li-Yorke sense.** In this final section we will study chaotic dynamics in the Li-Yorke sense on our cocycle attractors.

Definition 6.1. *Given (K, σ, d) a continuous flow in a compact metric space, a pair $\{x, y\} \in K$ is said a Li-Yorke pair if it holds*

$$\limsup_{t\to\infty} d(\sigma(t;x), \sigma(t;y)) > 0, \quad \liminf_{t\to\infty} d(\sigma(t;x), \sigma(t;y)) = 0.$$

A set $S \subset K$ is said scrambled if every $\{x, y\} \in S$ is a Li-Yorke pair. Finally, we say that the flow (K, σ, d) is chaotic in the Li-Yorke sense if there exists an uncountable scrambled $S \subset K$.

We will now consider our cocycle attractor $A(p) = [a(p), b(p)]$ associated to (3.5) and consider

$$K = \mathbb{A} = \bigcup_{p\in P} \{p\} \times [a(p), b(p)].$$

Since our flow on the base $(\Omega, \sigma, \mathbb{R})$ is almost-periodic it is obvious that if $(\xi_1, x_1) \in \Omega \times \mathbb{R}$, $(\xi_2, x_2) \in \Omega \times \mathbb{R}$ are a Li-Yorke pair then $\xi_1 = \xi_2$. Thus, if $S_0 \subset \Omega \times \mathbb{R}$ is a scrambled set there exists $P_0 \subset P$ such that $S_0 \subset A(p_0)$. This motivates the following definition:

Definition 6.2. *We say that $\mathbb{A}$ is fiber-chaotic in measure in the Li-Yorke sense if there exists an invariant set $P_{ch} \subset P$ with $\nu(P_{ch}) = 1$ such that $A(p)$ is scrambled for all $p \in P_c$.*

Note that $P_{ch} \subset P_f$ and it is a set of first category. Thus, our set is different from the residually Li-Yorke chaotic sets analyzed in Bjerklov [7] and Huand and Yi [23]. The arguments of this papr also shows that our fiber-chaotic compact set have zero topological entropy.

6.1.1. *Chaotic dynamics with full measure.* We consider the framework of the previous Section, that is, we have $\nu(P_f) = 1$ being $b(p) > 0$ for all $p \in P_f$.

We first need the following important result which guaranties that, with full measure, the pullback attractor is described from complete bounded trajectories of the linear system (3.2).

Proposition 6.3. *There exists $P_l \subset P_f$ invariant and with $\nu(P_l) = 1$ such that $0 < b(p) \leq r_0$ for all $p \in P_l$.*

Proof. Let us define $C_0 = \{p \in P : \text{ there exists } t \in \mathbb{R} \text{ with } b(pt) > r_0\}$. It is clear that C_0 is measurable and invariant. We argue by contradiction and assume that $\nu(C_0) = 1$. Take $m \in \mathbb{N}$ and $C_m = \{p \in P : \text{ there exists } t \in \mathbb{R} \text{ with } b(pt) > r_0 + \frac{1}{m}\}$. Note that $C_0 = \bigcup_{m=1}^{\infty} C_m$. Then there exists $m_0 \in \mathbb{N}$ with $\nu(C_{m_0}) > 0$. Define

$$C_m^+ = \{p \in P : \text{ there exists } t > 0 \text{ with } b(pt) > r_0 + \frac{1}{m}\}.$$

Let $E_0 \subset C_{m_0}$ compact with $\nu(E_0) > 0$, $b_{|E_0}$. Then there exists a compact set $E_1 \subset E_0$ with $0 < \nu(E_1) < \nu(E_0)$ such that for all $p \in E_1$ there exist sequences $s_n^1 \to \infty$, $s_n^2 \to -\infty$ (depending on p) such that $ps_n^1, ps_n^2 \in E_0$ for all $n \in \mathbb{N}$. Note that for all $p \in E_0$ there exists $t(p)$ with $b(pt(p)) > r_0 + \frac{1}{m_0}$. Since $b(pt) = x(t, p; b(p))$ for every $p \in P$ and b is continuous on E_0, it holds that $b(p_1 t(p)) > r_0 + \frac{1}{m_0}$ for all $p_1 \in B(p, \delta(p)) \cap E_0$.

Finally, $E_0 \subset \cup_{p \in E_0} B(p, \delta(p))$ admits a finite recovering, so that there exists $T_0 > 0$ such that, for all $p \in E_0$ we find $t(p)$ with $|t(p)| \leq T_0$ satisfying $b(pt(p)) > r_0 + \frac{1}{m_0}$.

If we now denote by $x(t,p;x_0)$ the solution of the linear equation (3.2), we will prove that, if we take and fix $p \in E_1$, then $\lim_{n\to\infty} x(s_n^1, p; b(p)) = \infty$. Denote $s_n = s_n^1$. We can suppose that $s_{n+1} - s_n \geq T_0 + 1$ for every $n \in \mathbb{N}$. Suppose also that $ps_n \in E_0$ tends to $p^* \in E_0$ and $\lim_{n\to\infty} x(s_n, p; b(p) = \gamma_0 b(p^*) < \infty$. For all $n \in \mathbb{N}$ there exists $|t_n| \leq T_0$ with $b(p(t_n + s_n)) > r_0 + \frac{1}{m_0}$ and $p(t_n + s_n) = p(s_{n-1} + (s_n - s_{n-1} + t_n))$, implying that $ps_n, p \in C^+_{m_0}$, and then $E_1 \subset C^+_{m_0}$.

Note that if $\lambda \geq 0$

$$(\lambda b(pt))' = h(pt)\lambda b(pt) + \lambda g(b(pt)) \leq h(pt)\lambda b(pt), \text{ for all } t \in \mathbb{R},$$

i.e., $\gamma_0 b(p)$ is a super-equilibrium for (3.5) and, for all $t \geq 0$, $p \in P$

$$x(t,p;\gamma_0 b(p)) \geq \gamma_0 b(pt).$$

Moreover, $\gamma_0 g(b(pt(p))) > 0$ and

then

$$\frac{d}{dt}(\gamma_0 b(pt))_{|t=t(p)} \leq h(pt)\gamma_0 b(pt)_{|t=t(p)},$$

implying that the super-equilibrium is strong.

Then there exist $\gamma_2 > \gamma_1 > \gamma_0$ and $t_0 > 0$ with

$$x(t,p^*;\gamma_0 b(p^*)) \geq \gamma_2 b(p^* t)$$

for every $t \geq t_0$. Moreover

$$b(p^* t_0) = \lim_{n\to\infty} b(p^*(s_n + t_0))$$

hence there exists $n_0 \in \mathbb{N}$ such that

$$x(s_{n_0} + t_0, p; b(p)) \geq \gamma_1 b(p(s_{n_0} + t_0))$$

and thus

$$x(s_n, p; b(p)) \geq \gamma_1 b(p(s_n))$$

if $s_n \geq s_{n_0} + t_0$, so that

$$\lim_{n\to\infty} x(s_n, p; b(p)) \geq \gamma_1 b(p^*),$$

which contradicts the definition of γ_0 and finishes the proof.

□

Let $\Delta \subset P_l$ a compact set such that $\nu(\Delta) > 0$ and

$$\Delta_\infty = \{p \in P_l : \text{ there exists } t_n \to \infty \text{ with } pt_n \in \Delta\}.$$

We know that $\nu(\Delta_\infty) = 1$. We will prove that

Theorem 6.4. *For all $p \in \Delta_\infty$, the sets $[-b(p), b(p)]$ are scrambled.*

Proof. Note that it is enough to prove it for $[0, b(p)]$. Take $p \in \Delta_\infty$. Then there exist sequences t_n^1, t_n^2 with $pt_n^1 \in \Delta$ for all $n \in \mathbb{N}$ and $pt_n^2 \to p_0 \in P_s$. Then, if $x_1, x_2 \in (0, b(p)]$ there exist $\gamma_1, \neq \gamma_2 \in (0,1)$ such that $x_1 = \gamma_1 b(p)$ and $x_2 = \gamma_2 b(p)$. It holds that

$$|x(t_n^1, p; \gamma_1 b(p)) - x(t_n^1, p; \gamma_2 b(p))| = |\gamma_1 - \gamma_2| b(p) e^{\displaystyle\int_0^{t_n^1} h(ps)ds} = |\gamma_1 - \gamma_2| b(pt_n^1)$$

which is between $\delta|\gamma_1 - \gamma_2|$ and $\gamma|\gamma_1 - \gamma_2|$ for some $\delta, \gamma > 0$ by the continuity of b on the compact set Δ.

In the same way

$$|x(t_n^2, p; \gamma_1 b(p)) - x(t_n^2, p; \gamma_2 b(p))| \leq b(pt_n^2) \to 0.$$

Note that the result is also true if $\gamma_1 = 0$.

□

6.1.2. *Chaotic dynamics in a fiber.* In this final section we prove the existence of chaotic dynamics in the Li-Yorke sense in the case of $\nu(P_f) = 0$.

Theorem 6.5. *Let $h \in C_0(P)$ a Kozlov function. Then the pullback attractor associated to* (3.5) *is chaotic in the Li-Yorke sense.*

Proof. There exists $p_0 \in P_f$ with

$$\sup_{t \in \mathbb{R}} \int_0^t h(p_0 s)ds = \rho < \infty.$$

If $0 \leq x_0 \leq \dfrac{r_0}{\rho}$ then

$$x(t, p_0; x_0) = x_0 e^{\displaystyle\int_0^t h(p_0 s)ds}.$$

It then holds that $[0, \frac{r_0}{\rho}] \subset [0, b(p_0)]$. We see that $[0, \frac{r_0}{\rho}]$ is scrambled. Let $0 < \lambda < \nu < \frac{r_0}{\rho}$. We have that

$$x(t, p_0; \nu) - x(t, p_0; \lambda) = (\nu - \lambda) e^{\displaystyle\int_0^t h(p_0 s) ds}.$$

As p_0 is recurrent, there exists a sequence $t_n^1 \to \infty$ with $\displaystyle\int_0^{t_n^1} b(p_0 s) ds \to 0$. Then

$$\lim_{n \to \infty} (x(t_n^1, p_0; \nu) - x(t_n^1, p_0; \lambda)) = (\nu - \lambda) > 0.$$

On the other hand, as P is minimal and P_s dense, given $p_1 \in P_s$ there exists $t_n^2 \to \infty$ such that $\displaystyle\int_0^{t_n^2} b(p_0 s) ds \to -\infty$ and

$$\lim_{n \to \infty} (x(t_n^2, p_0; \nu) - x(t_n^2, p_0; \lambda)) = 0.$$

□

Remark 6.6. *Observe that if h is a Kozlov function, we had proved the non-existence of forwards attractor with full measure. Now we have proved the Li-Yorke chaotic motion in this framework.*

6.2. **Sensitive dependence on initial conditions.** Let (K, d) a compact metric space with continuous flow σ and $M \subset K$ compact and invariant.

Definition 6.7. *We say that M is sensitive with respect to initial conditions (sensitive for brevity) in K if there exists $\rho > 0$ such that for all $x \in M, \delta > 0$ there exists $y \in K$ and $t > 0$ with*

$$d(x, y) \leq \delta \text{ and } d(\sigma(t, x), \sigma(t, y)) \geq \rho.$$

If $M = K$ we say that K is sensitive with respect to initial conditions.

Definition 6.8. *A dynamical system (K, σ) is called transitive if there exists a point $x \in K$ with dense orbit in K. Any such point is called transitive point.*

Definition 6.9. *We call dynamical system (K, σ) chaotic in the Auslander-Yorke sense if it is both sensitive and transitive.*

Now we consider the pullback cocycle attractor for (3.5).

Proposition 6.10. *The minimal $\mathbb{A}_0 = \cup_{p \in P} \{p\} \times 0$ is sensitive in $\mathbb{A}$.*

Proof. Let $p_0 \in P$ and $p_1 \in P_f$ with $b(p_1) = r_0$. Fix $\delta > 0$. Then there exists $p_2 \in P_s$ with $d(p_0, p_2) < \delta/2$ and a sequence $t_n \to -\infty$ such that $\lim_{n\to\infty}(p_1 t_n, b(p_1 t_n)) = (p_2, 0)$.

We consider the distance $\tilde{d}((p_1, x_1), (p_2, x_2)) = d(p_1, p_2) + |x_1 - x_2|$. Then there exists n_0 with $\tilde{d}((p_1 t_n, b(p_1 t_n)), (p_2, 0)) \le \delta/2$ for all $n \ge n_0$. Thus,

$$\tilde{d}((p_1 t_n, b(p_1 t_n)), (p_0, 0)) \le \delta$$

and

$$\tilde{d}((p_1, b(p_1)), (p_0(-t_n), 0)) \ge r_0,$$

which completes the proof. $\square$

We now consider the case in which $\nu(P_f) = 1$.

We know that $A(p) = [-b(p), b(p)]$ with $b(p) > 0$ if $p \in P_f$. For each $\lambda \in [0, 1]$ we define the measure μ_λ on $\mathbb{A}$ by Riesz theorem by

$$\int_{\mathbb{A}} f d\mu_\lambda = \int_P f(p, \lambda b(p)) d\nu, \text{ for all } f \in C^0(\mathbb{A}).$$

By Proposition 6.3 we can suppose that $b(p) \le r_0$ for all $p \in P_f$. Then, since $x(t, p; \lambda b(p)) = \lambda b(pt)$ for all $p \in P_f, t \in \mathbb{R}$ then for each $t \in \mathbb{R}$ and $f \in C^0(\mathbb{A})$

$$\int_{\mathbb{A}} \Pi(t) f d\mu = \int_P f(pt, \lambda b(pt)) d\mu = \int_P f(p, \lambda b(p)) d\mu = \int_{\mathbb{A}} f d\mu$$

so that μ is an invariant measure on $\mathbb{A}$ with $\mu(\mathbb{A}) = 1$, which is also ergodic. We now denote by $\sup\mu_\lambda$ the support of $\mathbb{A}$, which is a compact invariant set in $\mathbb{A}$. For each $\lambda \in [0, 1]$ we denote by $\mathbb{A}_\lambda = \sup\mu_\lambda \subset \mathbb{A}$. It is clear that $\mathbb{A}_\lambda = \{(p, \lambda x) : (p, x) \in \mathbb{A}_1\}$.

Theorem 6.11. *Suppose $\nu(P_f) = 1$. Then the compact invariant set $\mathbb{A}_\lambda$ is sensitive and chaotic in the Auslander-Yorke sense.*

Proof. Since μ_λ is ergodic there exists an invariant set $K_\lambda \subset \mathbb{A}$ of transitive points with $\mu_\lambda(K_\lambda) = 1$. Thus, $\mathbb{A}_\lambda$ is topologically transitive. Clearly, $(p, \lambda b(p)) \in \mathbb{A}_\lambda$ for a.a. $p \in P_f$ and $\mathbb{A}_0 subset \mathbb{A}_\lambda$. Then, the flow $\Pi(t)$ on $\mathbb{A}_\lambda$ is not equicontinuous. Thus, by Theorem 1.3 in Glasner and Weiss [18] v is sensitive, which finishes the proof. $\square$

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