A Note on the Stability of Fully Endogenous Growth with Increasing Returns and Exhaustible Resources[*]

Francisco Cabo[a†]   Guiomar Martín-Herrán[a‡]

María Pilar Martínez-García[b§]

[a]IMUVA, Universidad de Valladolid,

[b]Universidad de Murcia

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Abstract

We analyze a general R&D-based endogenous growth model with a growth-essential natural resource. The economy comprises two separate sectors: final output and R&D, both directly or indirectly dependent on the natural resource. Because the resource is exhaustible and it is an essential productive input, increasing returns to scale to man-made inputs are compatible with non-explosive sustained growth. The instability problem usually associated with increasing returns is overcome thanks to the existence of imperfect markets in a decentralized economy. We find an admisible range of values for the elasticity of capital in the R&D sector under which growth is fully endogenous and saddle-path stable with no need of exogenous population growth.

Corresponding author: Guiomar Martín-Herrán.

Dept. Economía Aplicada (Matemáticas).

Avda. Valle Esgueva 6, 47011 Valladolid, Spain.

Tel.: +34 983 423330. Fax.: +34 983 423299. e.mail: guiomar@eco.uva.es.

1 Introduction

The literature on R&D-based endogenous growth models and non-renewable natural resources assumes constant returns to scale (CRS) to man-made inputs in both the final output and the research sectors (see, for example, Grimaud and Rougé, 2003; Scholz and Ziemes, 1999; Schou 2002). As Solow (1994) points out, this knife-edge condition implies a lack of robustness. Slightly increasing returns to scale (IRS) would lead to explosive growth, while diminishing returns to scale (DRS) would lead to stagnation. Groth (2007) built a semi-endogenous growth model with DRS to man-made inputs (capital and knowledge) and a growth-essential non-renewable natural resource. The model qualifies as a hybrid model in the terminology of Eicher and Turnovsky (1999). Endogenous growth is feasible thanks to population growth. In this note, population growth is not needed and endogenous growth is still attained thanks to the combination of IRS to man-made inputs and a growth-essential resource.

The assumption of IRS is often linked with instability of the balanced path equilibrium. As Groth and Schou (2002) show, the use of an exhaustible resource as an essential input is not a strong enough stabilizer to overcome the instability problem when a one-sector economy is considered, unless the population is assumed to grow at a positive constant rate. Groth and Schou (2002, 2007) leave an open question: to what extent could the instability problem associated with IRS be overcome if a separate R&D sector were considered? As we prove here, the instability problem associated with IRS cannot be solved within a central planner setting, where all externalities are internalized. Conversely, a decentralized economy characterized by market inefficiencies (knowledge externalities and monopolistic competition) allows for a positive answer to this question.

This model is borrowed from Groth (2007) (section 5.2.2). The existence of a balanced growth path (BGP) requires a condition on the elasticity of final output with respect to knowledge, which depends on whether there is population growth or not. In a central planner setting, Groth (2007) focuses on the case where population growth is required. Conversely, we study the alternative case in a decentralized setting: fully endogenous growth can be attained without the aid of population growth. Saddle-path stability can be ensured under an additional assumption on the elasticity on the growth rate of knowledge with respect to capital. This condition is predominantly fulfilled under

the plausible assumption of an elasticity of capital in the R&D sector smaller than in the final output sector.

The note is organized as follows. Section 2 briefly recalls the general R&D-based growth model with a growth-essential exhaustible natural resource presented in Growth (2007). Conditions for existence and uniqueness of the balanced-path equilibrium are provided under the assumption of constant population. In Section 3 the saddle-path stability of the balanced path equilibrium is proved. The transitional dynamics is also analyzed. Section 4 concludes.

2   R&D-based growth model with a growth-essential resource

We consider the general R&D-based endogenous growth model presented in Groth (2007) with a non-renewable natural resource as a productive input:[1]

$$Y = A^\varepsilon K_Y^\alpha L_Y^\beta R^\gamma, \qquad \alpha,\beta,\gamma \in (0,1),\ \alpha+\beta+\gamma=1,\ \varepsilon>0, \tag{1}$$

$$\dot{A} = \mu A^\varphi K_A^{1-\eta} L_A^\eta, \qquad A(0)=A_0,\ \eta \in (0,1),\ \varphi \in \mathbb{R}, \tag{2}$$

$$\dot{K} = Y - C, \qquad K(0)=K_0, \tag{3}$$

$$\dot{S} = -R, \qquad S(0)=S_0, \tag{4}$$

where Y is final output (FO), a composite good used either for consumption or for investment in physical capital, and A is knowledge or technology. $K_Y (K_A)$ and $L_Y (L_A)$ are the capital and labor required to produce FO (R&D), where $K = K_Y + K_A$ and $L = L_Y + L_A = L(0)e^{nt}$, $n \geq 0$. C is aggregated consumption, S the stock of the non-renewable resource and R the total harvesting.

These equations stem from a quite general model of an expanding variety of intermediate goods (see, for example, Barro and Sala-i-Martin, 2004) with an exhaustible resource as a productive input. FO is produced using labor, an exhaustible natural resource and a set of durable intermediate goods. We analyze a decentralized setting[2] in which the non-renewable resource is harvested by competitive firms. In the intermediate-goods sector, monopoly firms transform capital into durable goods using the designs created in the R&D sector. The Ethier effect in the FO sector associated with a rise in the number of designs is assumed $\varepsilon \geq 1-\alpha$, encompassing increasing $(\varepsilon + \alpha > 1)$ or constant $(\varepsilon + \alpha = 1)$ returns to man-made inputs. Discoveries are non-rival goods which are non-excludable

within the innovative sector. The "standing on shoulders" externality does not necessarily imply the "knife-edge" assumption of $\varphi = 1$. Innovation depends on the existing stock of technology and the number of researchers, but also on the capital stock (or foregone output). Thus, this model does not consider research as independent of the capital stock (like the standard "knowledge-driven" models), nor does it require identical elasticities in the production of R&D and FO (like the "lab-equipment" model in the terminology of Rivera-Batiz and Romer (1991)). It is a hybrid model in which research requires labor and capital but not necessarily mimicking the FO sector, $\eta \in (0, 1)$. Possibly $1 - \eta < \alpha$, since the R&D sector is relatively intensive in human capital.

The optimal behaviour of all agents in this decentralized economy with imperfect competition in the intermediate-goods sector can be summarized by the Ramsey and the Hotelling rules:[3]

$$g_C = \frac{1}{\sigma}(r - \rho) + n, \qquad g_{p_R} = r, \tag{5}$$

with $1/\sigma > 0$ the elasticity of intertemporal substitution of consumption (assuming an isoelastic utility function), ρ the discount rate, r the rate of return, and p_R the market price of the natural resource. Moreover, due to monopolistic competition in the intermediate-good sector, the relationship between the rate of return on capital and its marginal productivity in the FO sector (or the social rate of return) must satisfy:

$$r = \alpha \frac{\partial Y}{\partial K_Y} = \alpha^2 \frac{Y}{K_Y}. \tag{6}$$

2.1 Fully endogenous growth: existence and uniqueness

A balanced growth equilibrium requires a continuous rise of knowledge which overcomes the decline in the exhaustible essential resource. This equilibrium is characterized by the standard conditions: $g_K^* = g_{K_A}^* = g_{K_Y}^* = g_Y^* = g_C^* \equiv g^*$, $g_{l_Y}^* = g_{k_Y}^* = g_r^* = g_u^* = 0$, $g_R^* = g_S^* < 0$, with $l_Y = L_Y/L$, $k_Y = K_Y/K \in (0, 1)$ and $u = R/S$.

Log-differentiating the production function (1) and taking into account the characterization of a BGP, the growth rate of FO, g, can be written as:

$$g = \frac{1 - \varphi}{1 - \eta} g_A - \frac{\eta}{1 - \eta} n, \qquad g = \frac{(1 - \varphi)\gamma}{(1 - \alpha)(1 - \varphi) - \varepsilon(1 - \eta)} g_R + \frac{\varepsilon - (1 - \varphi)\gamma}{(1 - \alpha)(1 - \varphi) - \varepsilon(1 - \eta)} n. \tag{7}$$

To have a BGP with positive growth and to avoid explosive growth condition $\varphi < 1$ is required. Moreover, as shown in Groth (2007), a BGP with $g > 0$ can be based on a sufficiently rapid growth in

population, $n > 0$, together with an upper bound on the elasticity of FO with respect to knowledge, $\varepsilon < (1 - \varphi)(1 - \alpha)/(1 - \eta)$. Conversely, we are interested in a BGP which does not require the aid of an exogenous source of growth. Therefore, here and henceforth we assume constant population, $n = 0$. In consequence, a BGP is only feasible under a new set of conditions:

Condition 1: $\qquad 1 - \dfrac{\varepsilon(1 - \eta)}{1 - \alpha} < \varphi < 1.$

Condition 1 states, on the one hand, that the knowledge externality in R&D is less than linear; and on the other, that returns to scale to man-made inputs (capital and technology) must be "globally" increasing, considering the R& D and FO sectors altogether. The assumption of globally IRS reverses the condition in Groth (2007). Which assumption better reflects reality is difficult to asses as they involve parameters associated with technological knowledge which are difficult to quantify. The elasticity of capital in the R&D sector, $1 - \eta$, is considered null in the "knowledge-driven" model, and equal to α in the "lab-equipment" model. Here, we will see that Condition 1 is compatible with the assumption $1 - \eta < \alpha$. Parameter ε, which collects the Ethier effect in the FO sector, can be settled equal to $1 - \alpha$ (or greater if a knowledge spillover also appeared in the intermediate-goods sector). Finally, parameter φ representing the net effect of knowledge spillovers and the "fishing-out" effect has been claimed to be close to one[4], while other authors assume a smaller value (for example, Steger 2005 states $\varphi = 0.5$).

For shortness, we focus on the standard model à la Romer (1990) with CRS to man-made inputs in the FO sector, $\varepsilon + \alpha = 1$. In consequence, Condition 1 implies IRS in the innovative sector and can be re-written as: $\eta < \varphi < 1$. The general case with $\varepsilon \geq 1 - \alpha$ can be handled similarly, see Cabo et al. (2013).

The growth rate of the economy along the BGP is given by:

$$g^* = \frac{\rho}{1 - \sigma + \frac{(1-\alpha)(\varphi-\eta)}{\gamma(1-\varphi)}}. \tag{8}$$

This growth rate is positive if consumers are not excessively inelastic:[5]

Condition 2: $\qquad \sigma < 1 + \dfrac{(1 - \alpha)(\varphi - \eta)}{\gamma(1 - \varphi)}.$

A BGP corresponds to a steady state of the following system of five differential equations in

variables $c = C/K$, l_Y, $\chi = g_A$, $y = Y/K$, and $u = R/S$:

$$g_c = \left(\frac{\alpha^2}{\sigma k_Y} - 1\right) y + c - \frac{\rho}{\sigma}, \tag{9}$$

$$g_{l_Y} = \Phi(l_Y)\left\{\left(1 - \frac{\alpha}{k_Y} - \frac{(1-\gamma)(1-\eta)}{\alpha}\right) y + \left(\frac{(1-\gamma)(1-\eta)}{\alpha} - 1\right) c + \Gamma(l_Y)\chi\right\}, \tag{10}$$

$$g_\chi = \Psi(l_Y)g_{l_Y} + (1-\eta)y - (1-\eta)c + (\varphi-1)\chi, \tag{11}$$

$$g_y = \Omega(l_Y)g_{l_Y} + \left\{\alpha - 1 - \frac{\alpha\gamma}{1-\gamma}\left(\frac{\alpha}{k_Y} - 1\right)\right\} y + \frac{1-\alpha-\gamma}{1-\gamma}c + \frac{1-\alpha}{1-\gamma}\chi, \tag{12}$$

$$g_u = \Omega(l_Y)g_{l_Y} + \frac{\alpha}{1-\gamma}\left(1 - \frac{\alpha}{k_Y}\right) y + \frac{1-\alpha}{1-\gamma}\chi + u - \frac{\alpha}{1-\gamma}c, \tag{13}$$

where

$$k_Y = l_Y \frac{\alpha^2\eta}{\{\alpha^2 - (1-\eta)[\alpha^2 + \beta]\}l_Y + \beta(1-\eta)}, \tag{14}$$

$$\Phi(l_Y) = \frac{\alpha(1-l_Y)}{[(1-\gamma)(1-\eta)-\alpha](l_Y-k_Y)}, \quad \Psi(l_Y) = -\frac{(1-\eta)k_Y+\eta l_Y}{1-l_Y}, \quad \Omega(l_Y) = \frac{\alpha(1-k_Y)+\beta(1-l_Y)}{(1-\gamma)(1-l_Y)},$$

$$\Gamma(l_Y) = \frac{\gamma-\alpha}{\alpha} - \frac{(\varphi-1)(1-\gamma)}{\alpha} + \frac{\eta(1-\gamma)(1-\alpha)}{\beta}\frac{l_Y}{1-l_Y}.$$

Proposition 1. *For $n = 0$, under Conditions 1 and 2, there exists a unique steady-state equilibrium*[6] *for the economy with $l_Y^* \in (0,1)$ and $c^*, y^*, \chi^*, u^* > 0$.*

Proof. See Cabo et al. (2013). □

3 Saddle-path stability of the balanced growth equilibrium

For an economy with separated R&D and FO sectors, in a decentralized setting with imperfect markets and knowledge externalities, the instability associated with IRS to man-made inputs is overcome under certain conditions.

Variables c, l_Y and u in (9)-(13) can be regarded as controls. Correspondingly, y and χ are determined by the state variables K, A and S. Therefore the stable manifold must be at least of dimension two. The transition path is unique when the stable manifold is exactly two-dimensional.

Proposition 2. *Under Conditions 1 and 2, the determinant of the Jacobian matrix at the steady state is positive if and only if:*

$$1 - \eta \in \left(\frac{\alpha^2}{\alpha^2 + \beta}, \frac{\alpha}{1-\gamma}\right). \tag{15}$$

Additionally, sufficient condition[7] *$\alpha > \gamma$, guarantees that the stable manifold is either two or four-dimensional.*

Proof. See Appendix A. □

Numerical simulations show that the stable manifold is always two-dimensional. This result is robust to changes in parameter values. Therefore the path converging to the steady state is unique.

Condition (15) establishes a range of variation for the elasticity of capital in the R&D sector. Because empirically α is several times the size of γ, condition (15) is compatible with $1 - \eta < \alpha$. For example, following Barro and Sala-i-Martin (2004), $\alpha = 0.4$ seems adequate for developed countries, and assuming $\gamma = 0.15$, condition (15) reads: $1 - \eta \in (0.26, 0.47)$. This illustrates that the conditions in Proposition 2 are more likely satisfied when the elasticity of capital in the R&D sector, $1 - \eta$, is lower than in FO the sector ($\alpha = .4$).

<u>The social planner's problem versus the decentralized solution:</u>

As stated above, an elasticity of capital in the FO sector greater than in the research sector seems realistic. Specifically, we assume that $1 - \eta$ remains below $\alpha/(1 - \gamma)$, the upper bound in (15), which can be interpreted as the elasticity of capital in the FO sector when the resource is harvested following the Hotelling rule. Stability further requires that $1 - \eta$ surpasses the lower bound in (15), which states that the final output sector employs labor more intensively than capital, $l_Y^* > k_Y^*$.

In the decentralized setting, the monopolistic competition in the industry which provides intermediate goods to FO producers implies (6). Due to this inefficiency a R&D sector with a low elasticity of capital, $1 - \eta < \alpha/(1 - \gamma)$, can be compatible with a FO sector less capital intensive, $l_Y^* > k_Y^*$. By contrast, inefficiencies are not present in a social optimum and following the same steps as in the decentralized setting, it can be proved that without population growth, the social planner's problem remains unstable (as Groth and Schou (2002) anticipated).

3.1 Transitional adjustment

We now turn to the transitional adjustment of the economy in response to some types of disturbances. The state of the economy is represented by $E_1 = K^{1-\eta}/A^{1-\varphi}$ and $E_2 = S^\gamma A^{1-\alpha}/K^{1-\alpha}$, which are totally determined by the state variables K, A and S, and remain constant along the steady state. By linearizing the dynamics of variables E_1, E_2, and the control variables l_Y, c and u in a neighborhood of the steady state, we can study the qualitative behaviour of the relevant variables

along the transition. Our numerical example considers: $\mu = L = 1$, $\rho = .05$, $\alpha = .4$, $\gamma = .15$, $\sigma = 1.1$, $1 - \eta = .32$, $\varphi = .7$.

Insert Figure 1 over here

The existence of a two-dimensional stable manifold, means that stabilizing policy functions $l_Y(E_1, E_2)$, $c(E_1, E_2)$ and $u(E_1, E_2)$, can be uniquely adopted (no indeterminancy). The graphs in Figure 1 show the contour lines of the surfaces $l_Y(E_1, E_2)$ and $c(E_1, E_2)$ (the graph for $u(E_1, E_2)$ mimics Figure 1 right). Lighter (darker) regions represent values above (below) the steady state. The contour lines corresponding with the steady-state values c^* and l_Y^* are highlighted.

Starting from the steady-sate position $P_0 = (E_1^*, E_2^*)$, a positive shock on the stock of the natural resource[8] would imply $E_2 > E_2^*$ and $E_1 = E_1^*$ in point P_1. Labor devoted to FO, l_Y (and correspondingly capital share, k_Y) should be initially lower than l_Y^* (and k_Y^*). Thus more labor and capital should be devoted to R&D, and so, $\chi > \chi^*$. The depletion rate should also start below its steady-state value. Numerically, this rate becomes greater immediately after the shock. This confirms that a more abundant stock of resource leads to greater harvesting, which counteracts the reduction in labor and capital, leading to a greater production of FO.

Insert Figure 2 over here

These policies will take the economy from P_1 to the long-run equilibrium, P_0, following the dashed lines. The time paths of the relevant variables along the transition are depicted in Figure 2. The main outcome after a positive shock on the stock of the resource is a greater growth rate of the economy along the transition, which enables a greater consumption per unit of capital.

4   Conclusions

Following the formulation in Groth (2007) we study a general R&D-based endogenous growth model in which the FO and the R&D sectors are considered separately. It is a hybrid model (in the terminology of Eicher and Turnovsky, 1999) because capital and knowledge enter the production function of FO and R&D. A non-renewable natural resource is a direct input in the production of FO, and it enters indirectly into the growth engine because the capital is used in the R&D sector (the

resource is growth-essential). Groth (2007) focuses on the case where endogenous growth requires population growth. This note focuses on the alternative case, balanced growth is feasible without an exogenous growth in population, but based on IRS in man-made inputs. IRS are compatible with a BGP because the use of the non-renewable natural resource has to diminish over time.

The existence and uniqueness of a BGP needs two conditions. Condition 1 states that the knowledge externality in the research sector must be less than linear and the returns to scale to man-made inputs must be "globally" increasing. That is, if man-made inputs show CRS in the FO sector, as in the standard model à la Romer, then IRS to man-made inputs in the research sector are needed. More generally, to guarantee sustained growth, the DRS or CRS in one sector must be more than offset by IRS in the other sector. Condition 2 requires consumers not excessively reluctant to substitute present for future consumption.

In a decentralized economy with constant population the assumption of globally IRS to man-made inputs does not necessarily lead to instability. The inefficiency from monopolistic competition in the sector which provides intermediate goods to FO producers allows us to escape the instability problem associated with IRS. Saddle-path stability requires the elasticity of capital in the R&D sector to be low enough (with respect to the FO sector) but greater than a lower bound. By contrast, a central planner setting shows no inefficiencies and is always unstable.

Finally, we have numerically characterized the policy functions that bring the economy towards the BGP. Specifically, the transitional dynamics after a positive shock on the stock of the resource.

Appendix A

Proof of Proposition 2

From the system (9)-(13), the Jacobian matrix in c, l_Y, χ, y, u, at the steady state reads:

$$J^* = (a_{ij}), \quad i,j \in \{1, \dots, 5\}$$

where

$$a_{11} = c^*, \quad a_{12} = -\frac{\alpha}{\sigma}\Delta(l_Y^*)y^*c^*, \quad a_{14} = \left(\frac{\alpha}{\sigma}\Pi(l_Y^*) - 1\right)c^*, \quad a_{13} = a_{15} = 0,$$

$$a_{21} = l_Y^*\Phi(l_Y^*)\left(\frac{(1-\gamma)(1-\eta)}{\alpha} - 1\right), \; a_{22} = l_Y^*\Phi(l_Y^*)\left(\Delta(l_Y^*)y^* + \Upsilon\frac{\chi^*}{(1-l_Y^*)^2}\right),$$

$$a_{23} = l_Y^*\Phi(l_Y^*)\Gamma(l_Y^*), \; a_{24} = l_Y^*\Phi(l_Y^*)\left(1 - \Pi(l_Y^*) - \frac{(1-\gamma)(1-\eta)}{\alpha}\right), \quad a_{25} = a_{35} = a_{45} = 0,$$

$$a_{31} = \left(\frac{\Psi(l_Y^*)}{l_Y^*}a_{21} - (1-\eta)\right)\chi^*, \; a_{32} = \frac{\Psi(l_Y^*)}{l_Y^*}a_{22}\chi^*, \; a_{33} = \left(\frac{\Psi(l_Y^*)}{l_Y^*}a_{23} + \varphi - 1\right)\chi^*, \; a_{34} = \left(\frac{\Psi(l_Y^*)}{l_Y^*}a_{24} + (1-\eta)\right)\chi^*,$$

$$a_{41} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{21} + \frac{\beta}{1-\gamma}\right)y^*, \quad a_{42} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{22} + \frac{\alpha\gamma}{1-\gamma}\Delta(l_Y^*)y^*\right)y^*,$$

$$a_{43} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{23} + \frac{1-\alpha}{1-\gamma}\right)y^*, \; a_{44} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{24} + \frac{\alpha\gamma}{1-\gamma}(1-\Pi(l_Y^*)) - (1-\alpha)\right)y^*,$$

$$a_{51} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{21} - \frac{\alpha}{1-\gamma}\right)u^*, \; a_{52} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{22} + \frac{\alpha}{1-\gamma}\Delta(l_Y^*)y^*\right)u^*,$$

$$a_{53} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{23} + \frac{1-\alpha}{1-\gamma}\right)u^*, \; a_{54} = \left(\frac{\Omega(l_Y^*)}{l_Y^*}a_{24} + \frac{\alpha}{1-\gamma}(1-\Pi(l_Y^*))\right)u^*, \; a_{55} = u^*,$$

and

$$\Delta(l_Y^*) = \frac{(1-\eta)\beta}{\alpha\eta(l_Y^*)^2} > 0, \; \Pi(l_Y^*) = \frac{\alpha}{k_Y^*} = \alpha + \frac{\beta}{\alpha}\frac{1-\eta}{\eta}\frac{1-l_Y^*}{l_Y^*}, \; \Upsilon = \frac{\eta(1-\gamma)(1-\alpha)}{\beta}.$$

Since $u^* > 0$, the sign of $\det J^*$ equals the sign of the 4th leading principal minor. Further, $c^*, l_Y^*, \chi^*, y^* > 0$, so after some simplifications which do not change the sign of the determinant[9] $(f_1/c^*, \; f_2/l_Y^*, \; f_3/\chi^* - \Psi(l_Y^*)f_2/l_Y^*, \; f_4/y^* - \Omega(l_Y^*)f_2/l_Y^*)$, this principal minor equals

$$\Phi(l_Y^*)\begin{vmatrix} 1 & -\frac{\alpha}{\sigma}\Delta(l_Y^*)y^* & 0 & \frac{\alpha^2}{\sigma k_Y^*} - 1 \\ \frac{(1-\gamma)(1-\eta)}{\alpha} - 1 & \Delta(l_Y^*)y^* + \Upsilon\frac{\chi^*}{(1-l_Y^*)^2} & \Gamma(l_Y^*) & 1 - \frac{\alpha}{k_Y^*} - \frac{(1-\gamma)(1-\eta)}{\alpha} \\ -(1-\eta) & 0 & \varphi - 1 & 1 - \eta \\ \frac{\beta}{1-\gamma} & \frac{\alpha\gamma}{1-\gamma}\Delta(l_Y^*)y^* & \frac{1-\alpha}{1-\gamma} & \frac{\alpha\gamma}{1-\gamma}\left(1-\frac{\alpha}{k_Y^*}\right) - (1-\alpha) \end{vmatrix}.$$

Expanding this determinant along the second column, it follows:

$$\Phi(l_Y^*)\left(\Delta(l_Y^*)y^*\Theta + \Upsilon\frac{\chi^*}{(1-l_Y^*)^2}\Xi\right), \tag{16}$$

$$\Theta = \begin{vmatrix} 1 & -\frac{\alpha}{\sigma} & 0 & \frac{\alpha^2}{\sigma k_Y^*} - 1 \\ \frac{(1-\gamma)(1-\eta)}{\alpha} - 1 & 1 & \Gamma(l_Y^*) & 1 - \frac{\alpha}{k_Y^*} - \frac{(1-\gamma)(1-\eta)}{\alpha} \\ -(1-\eta) & 0 & \varphi - 1 & 1 - \eta \\ \frac{\beta}{1-\gamma} & \frac{\alpha\gamma}{1-\gamma} & \frac{1-\alpha}{1-\gamma} & \frac{\alpha\gamma}{1-\gamma}\left(1-\frac{\alpha}{k_Y^*}\right) - (1-\alpha) \end{vmatrix}, \quad \Xi = \begin{vmatrix} 1 & 0 & 0 & \frac{\alpha^2}{\sigma k_Y^*} - 1 \\ \frac{(1-\gamma)(1-\eta)}{\alpha} - 1 & 1 & \Gamma(l_Y^*) & 1 - \frac{\alpha}{k_Y^*} \\ -(1-\eta) & 0 & \varphi - 1 & 1 - \eta \\ \frac{\beta}{1-\gamma} & 0 & \frac{1-\alpha}{1-\gamma} & \frac{\alpha\gamma}{1-\gamma}\left(1-\frac{\alpha}{k_Y^*}\right) - (1-\alpha) \end{vmatrix}.$$

Adding the first and fourth columns of Θ, it follows that this determinant is zero.

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For Ξ, after some algebra, taking into account Condition 2 and $\varphi < 1$, it follows:

$$\Xi = \frac{1-\varphi}{1-\gamma}\frac{\alpha^2\gamma}{\sigma k_Y^*}\left[\sigma - 1 - \frac{(1-\alpha)(\varphi-\eta)}{\gamma(1-\varphi)}\right] < 0.$$

Because at the steady state χ^*, y^*, $(1-l_Y^*)$, $\Delta(l_Y^*) > 0$, from (16) it follows that the sign of $\det J^*$ equals the sign of $-\Phi(l_Y^*)$, which is positive if and only if $[(1-\eta) - \alpha/(1-\gamma)](l_Y^* - k_Y^*) < 0$.

From equation (14), $l_Y^* > k_Y^* \Leftrightarrow 1 - \eta > \alpha^2/(\alpha^2 + \beta)$. And it is easy to see that:

$$\frac{\alpha^2}{\alpha^2 + \beta} < \frac{\alpha}{1-\gamma} < 1.$$

Therefore under condition (15), $\Phi(l_Y^*) < 0$ and $\det J^* > 0$.

In a second step we prove that there exists an even (not null) number of negative eigenvalues.

All the elements in the fifth column in the Jacobian matrix are zero except a_{55}, which is a (positive) eigenvalue. We focus on the addition of the remaining elements in the main diagonal.

The steady-state values c^*, χ^* and y^* satisfy:

$$\chi^* = \frac{1-\eta}{\varphi-1}\left(\frac{\rho}{\sigma} - \frac{\alpha^2}{\sigma k_Y^*}y^*\right) = \frac{1-\eta}{1-\varphi}(y^* - c^*), \quad y^* = \frac{k_Y^*}{\alpha^2}\frac{\rho}{1-\Lambda}, \quad y^* - c^* = \frac{\rho}{\sigma}\frac{\Lambda}{1-\Lambda}, \qquad (17)$$

where k_Y^* denotes the value of variable k_Y satisfying (14) when l_Y equals l_Y^* and

$$\Lambda = \frac{\sigma}{1 + \frac{(1-\alpha)(\varphi-\eta)}{\gamma(1-\varphi)}}. \qquad (18)$$

Taking into account expressions in (17),

$$a_{33} = \frac{\alpha y^*}{k_Y^*}\left\{\Psi(l_Y^*)\Phi(l_Y^*)\left(1 - \frac{\alpha}{\sigma}\left(1 - \frac{(1-\gamma)(1-\eta)}{\alpha}\right)\right) - \frac{(1-\eta)\alpha}{\sigma}\right\} + \frac{\rho}{\sigma}\left(1 - \eta + \Psi(l_Y^*)\Phi(l_Y^*)\left(1 - \frac{(1-\gamma)(1-\eta)}{\alpha}\right)\right),$$

$$a_{22} = \Phi(l_Y^*)l_Y^*\left\{\frac{(1-\eta)\beta}{\alpha\eta}\frac{y^*}{(l_Y^*)^2} - \frac{\eta(1-\gamma)(1-\alpha)(1-\eta)}{\beta(1-\varphi)\sigma(1-l_Y^*)^2}\left(\rho - \frac{\alpha^2 y^*}{k_Y^*}\right)\right\}.$$

Replacing y^* by its expression in (17), the term $\Sigma = a_{11} + a_{22} + a_{33} + a_{44}$, can be written as:

$$\frac{\rho}{1-\Lambda}\left\{-\frac{\Lambda}{\sigma} - \frac{k_Y^*}{\alpha^2}\frac{1-l_Y^*}{l_Y^* - k_Y^*} - \frac{\gamma}{1-\gamma} - \frac{\Lambda}{\sigma}\frac{l_Y^*}{l_Y^* - k_Y^*} - \frac{\Omega(l_Y^*) - \Psi(l_Y^*)}{(1-\gamma)(1-\eta) - \alpha}\frac{1-l_Y^*}{l_Y^* - k_Y^*}\right.$$
$$\left.+ \frac{\alpha(1-\eta)}{(1-\gamma)(1-\eta) - \alpha}\frac{1}{l_Y^* - k_Y^*}\left[\frac{k_Y^*}{\alpha^3}\frac{\beta}{\eta}\frac{1-l_Y^*}{l_Y^*} + \frac{\eta(1-\gamma)(1-\alpha)\Lambda}{\beta(1-\varphi)\sigma}\frac{l_Y^*}{1-l_Y^*}\right]\right\}.$$

Because $\Lambda < 1$, the sign of Σ coincides with the sign of the term in brackets. Rearranging terms and taking into account that:

$$\frac{\Omega(l_Y^*) - \Psi(l_Y^*)}{(1-\gamma)(1-\eta) - \alpha} = \frac{\alpha}{(1-\gamma)(1-\eta) - \alpha}\frac{1}{l_Y^* - k_Y^*} - \frac{\alpha}{1-\gamma},$$

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the term in brackets above can be rewritten as:

$$\frac{1}{l_Y^* - k_Y^*}\left\{-\frac{k_Y^*(1-l_Y^*)}{\alpha^2} - \frac{\Lambda l_Y^*}{\sigma} + \frac{\alpha}{(1-\gamma)(1-\eta)-\alpha}\left[-\frac{1}{\alpha} + \frac{k_Y^*}{\alpha^3}\frac{(1-\eta)\beta}{\eta}\frac{1-l_Y^*}{l_Y^*}\right.\right.$$
$$\left.\left.+\frac{\eta(1-\eta)(1-\gamma)(1-\alpha)\Lambda}{\beta(1-\varphi)\sigma}\frac{l_Y^*}{1-l_Y^*}\right]\right\} + 1 - \frac{\Lambda}{\sigma}.$$

From (14) and (15) it follows that $l_Y^* - k_Y^* > 0$. Therefore, the sign of the expression above is not modified when multiplied by $l_Y^* - k_Y^*$. Furthermore, replacing k_Y^* using expression (14), and rearranging terms, the sign of Σ is given by the sign of:

$$-\frac{1-l_Y^*}{\alpha^2 + \beta\frac{1-\eta}{\eta}\frac{1-l_Y^*}{l_Y^*}}\Sigma_1 - \frac{\Lambda}{\sigma}l_Y^* - \frac{\alpha}{(1-\gamma)(1-\eta)-\alpha}\Sigma_2,$$

where

$$\Sigma_1 = 1 - \left(\beta\frac{1-\eta}{\eta} - \alpha^2\right)\left(1-\frac{\Lambda}{\sigma}\right), \quad \Sigma_2 = \frac{\alpha}{\alpha^2 + \beta\frac{1-\eta}{\eta}\frac{1-l_Y^*}{l_Y^*}} - \frac{\Lambda}{\sigma}\frac{(1-\gamma)(\gamma(1-\eta)+(1-\alpha)(\varphi-\eta))}{\alpha\gamma(1-\varphi)}.$$

Replacing Λ by (18), and after some manipulations, the sign of Σ_1 is given by the sign of:

$$1 + \alpha^2 - \beta\frac{1-\eta}{\eta} + \frac{\gamma(1-\varphi)}{(1-\alpha)(\varphi-\eta)}.$$

Under condition (15), the expression above and hence Σ_1 are positive.

Similarly, after some simplifications, the sign of Σ_2 coincides with the sign of:

$$\alpha^2[(\eta-\varphi)] + (1-\eta)[\alpha\gamma - (1-\gamma)(1-\alpha)(1-\eta)].$$

Therefore, a sufficient condition for $\Sigma_2 \leq 0$ is $\alpha\gamma - (1-\gamma)(1-\alpha)(1-\eta) \leq 0$. And this inequality holds under condition (15), assuming additionally that $\alpha > \gamma$.

Notes

[1] All variables depend on the time argument, which is omitted when no confusion can arise.

[2] For the micro foundations see Cabo et al. (2013).

[3] Here, and henceforth, g_x denotes the growth rate of variable x.

[4] Evidence in favor of a R&D productivity increasing nearly proportionally with the stock of ideas already discovered in a country-level can be found in Porter and Stern (2000), who estimate the shape of the ideas production function by using a panel data set of patents. They also find, nonetheless, that the stock of patents has a quite modest effect on total factor productivity.

[5] Under Condition 1 the upper bound for σ is greater than one, and Condition 2 is not excessively restrictive. In Groth (2007) this inequality is reversed, although due to the DRS assumption the lower bound for σ is lower than one.

[6] The unique steady-state equilibrium satisfies the transversality conditions, and functions are concave.

[7] Sufficient condition $\alpha > \gamma$ agrees with the empirical data which considers α to be several times γ (see Groth, 2007 and references therein).

[8] A similar analysis can be done starting from any other initial situation of unbalanced growth.

[9] Here f_i denotes the i-th file.

References

[1] Barro, Robert and Xavier Sala-i-Martin (2004) *Economic Growth, Second ed.* Cambridge, Ma: MIT Press.

[2] Cabo, Francisco, Guiomar Martín-Herrán, and María-Pilar Martínez-García (2013) Fully endogenous growth with increasing returns and exhaustible resources: existence and stability. *Cahier du GERAD* G-2010-33, May 2010, Revised May 2013.

[3] Eicher, Theo S. and Stephen J. Turnovsky (1999) Non-scale models of economic growth. *Economic Journal* 109, 394–415.

[4] Grimaud, André and Luc Rougé (2003) Non-renewable resources and growth with vertical innovations: optimum, equilibrium and economic policies. *Journal of Environmental Economics and Management* 45, 433–453.

[5] Groth, Christian (2007) A new-growth perspective on non-renewable resources. In Lucas Bretschger and Sjak Smulders (eds.), *Sustainable Resource Use and Economic Dynamics*, pp. 127-163. Dordrecht: Springer.

[6] Groth, Christian and Poul Schou (2002) Can non-renewable resources alleviate the knife-edge character of endogenous growth? *Oxford Economic Papers* 54, 386–411.

[7] Groth, Christian and Poul Schou (2007) Growth and non-renewable resources: The different roles of capital and resource taxes. *Journal of Environmental Economics and Management* 53, 80–98.

[8] Porter, Michael E and Scott Stern (2000) Measuring the "Ideas" production function: evidence from international patent output. *NBER Working Papers* 7891.

[9] Rivera-Batiz, Luis A. and Paul M. Romer (1991) Economic integration and endogenous growth. *Quarterly Journal of Economics* 106, 531–555.

[10] Romer, Paul M. (1990) Endogenous technical change. *Journal of Political Economy* 98, 71–102.

[11] Scholz, Christian M. and Georg Ziemes (1999) Exhaustible resources, monopolistic competition, and endogenous growth. *Environmental and Resource Economics* 37, 169–185.

[12] Schou, Poul (2002) When environmental policy is superfluous: Growth and polluting resources. *Scandinavian Journal of Economics* 104, 605–620.

[13] Solow, Robert M. (1994) Perspectives on growth. *Journal of Economic Perspectives* 8, 45–54.

[14] Steger, Thomas M. (2005) Welfare implications of Non-Scale R&D-based growth models. *Scandinavian Journal of Economics* 107, 737–757.
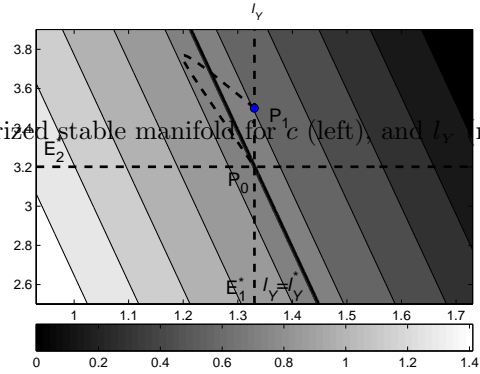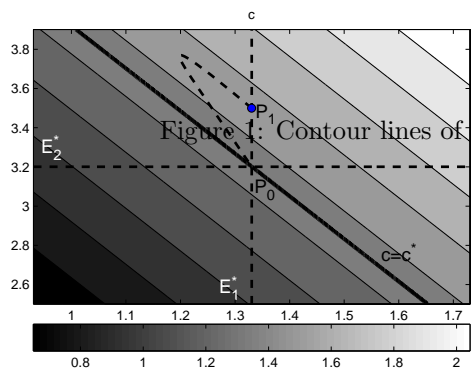
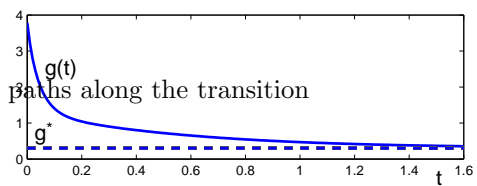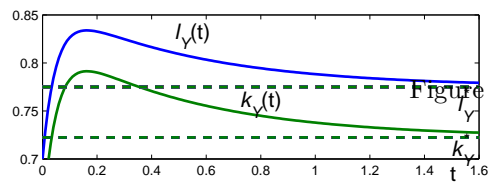Figure 1: Contour lines of the linearized stable manifold for c (left), and l_Y (right)

Figure 2: Time paths along the transition