# Characterization of fuzzy preference structures through Łukasiewicz triplets

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**Abstract**

In this paper we characterize the reconstruction of a fuzzy set from its subsets by means of Łukasiewicz triplets. This result allows us to introduce a new definition of fuzzy strict preference, which is also satisfied in the crisp framework. The usual definitions of fuzzy indifference and fuzzy incomparability together with this one enable to construct and to characterize fuzzy preference structures from a reflexive fuzzy binary relation.

*Key words:* fuzzy relations, fuzzy preference structures, Łukasiewicz triplets, reciprocal order automorphisms.

## 1 Introduction

It is usual in the theory of preference modelling to represent the agents' preferences on a set of alternatives $A$ by means of three binary relations: the strict preference relation $P$, the indifference relation $I$ and the incomparability relation $J$. The relation $P$ shows whether an alternative is preferred to another or vice versa. $I$ is used for representing the indifference between two alternatives. Finally, $J$ shows the pairs of alternatives which cannot be compared in terms of preference or indifference.

In the crisp theory, starting from the relation $R = P \cup I$, it is possible to reconstruct $P$, $I$ and $J$ by means of $P = R \cap R^d$, $I = R \cap R^{-1}$ and

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$J = R^c \cap R^d$. However, in the fuzzy framework, it has been proven that if we represent the complement, the intersection and the union of fuzzy sets through De Morgan triplets then the equality $B = (B \cap C) \cup (B \cap C^c)$ is not satisfied for all two fuzzy subsets $B$ and $C$ of $A$ (see Alsina [1] and Fodor and Roubens [12]). Consequently, the relationships $P = R \cap R^d$, $I = R \cap R^{-1}$ and $R = P \cup I$ are inconsistent for some reflexive fuzzy binary relation $R$.

Due to this result, some authors, such as Fodor [8] and [9], Fodor and Roubens [10], [11] and [12], and Ovchinnikov and Roubens [14] and [15], have proposed and developed an axiomatic model. A survey of this development can be found in De Baets and Fodor [5] and in Perny and Roubens [16].

On the other hand, Van de Walle, De Baets and Kerre [21] have proven that if we use De Morgan triplets with strong negations for defining fuzzy preference structures, then $\phi$-transforms of Łukasiewicz t-norm are the most suitable candidates. Consequently, De Morgan triplets have two automorphisms: the first one corresponds to the $\phi$-transform of Łukasiewicz t-norm and the second one corresponds to the strong negation. However, in many instances, Łukasiewicz triplets are used in the literature without being justified. On this, De Baets and Fodor [5] give the following argumentation in order to use Łukasiewicz triplets:

> *To avoid unnecessarily complicated notations and definitions we restrict ourselves to the case* $\phi_1 = \phi_2$. $(\dots)$ *A second reason for sticking to a single* $[0,1]$*-automorphism is given by the following relationships between the completeness conditions in case of a Łukasiewicz triplet.*

A similar assertion is given in Van de Walle, De Baets and Kerre [22]. Nevertheless, this argumentation is weak and the use of Łukasiewicz triplets can condition the relations between the completeness conditions.

Moreover, Van de Walle, De Baets and Kerre [22] give a non-trivial condition under which the fuzzy binary relations $P$, $I$ and $J$ are uniquely determined from $R$. However, they express the intersection of fuzzy sets through two different t-norms, which belong to the Frank t-norm family. Thus, they obtain similar expressions to the crisp theory ( $R = P \cup_\phi^\infty I$, $P = R \cap_\phi^{1/s} R^d$, $I = R \cap_\phi^s R^{-1}$, $J = R^c \cap_\phi^s R^d$ ) but these relations are expressed by means of two different t-norms instead of utilizing Łukasiewicz triplets for their representation.

In this paper we give a solution to the previous problems. Firstly, although there is no De Morgan triplet such that the equality $B = (B \cap C) \cup (B \cap C^c)$ holds for any two fuzzy subsets $B$ and $C$ of $A$, since $(B \cap C) \subseteq B$, the problem is to reconstruct a fuzzy set $B$ from its subsets. Therefore, if we consider $C \subseteq B$, the problem can be formulated as $B = C \cup (B \setminus C)$ or, equivalently, $B = C \cup (B \cap C^c)$. In Theorem 12 we characterize the t-norms, the t-conorms and the strict negations for which the previous equality is satisfied for any two fuzzy subsets $B$ and $C$ of $A$ such that $C \subseteq B$. These t-norms, t-conorms and strict negations form Łukasiewicz triplets.

From this result, since $I = R \cap R^{-1}$, we introduce a new definition of fuzzy strict preference, $P = R \cap I^c = R \cap (R \cap R^{-1})^c$. This definition is also satisfied in the crisp framework; in fact, it is equivalent to $P = R \cap R^d$, and therefore, there are no reason for using $P = R \cap R^d$ and not $P = R \cap (R \cap R^{-1})^c$. Corollary 13 guarantees that we can reconstruct $R$ from $P$ and $I$ if and only if we use Łukasiewicz triplets. For obtaining this result, it is not necessary to take De Morgan triplets like starting point.

Furthermore, the fuzzy preference structure obtained in Theorem 28, given by $P = R \cap (R \cap R^{-1})^c$, $I = R \cap R^{-1}$ and $J = R^c \cap R^d$, satisfies a factorization of $R$ into $P$ and $I$ that it is also satisfied in the crisp theory and, consequently, it seems that this fuzzy preference structure is the best-suited. As Bufardi [3] stand out:

> *In any method of construction of a fuzzy preference structure from a reflexive fuzzy relation, it is very important to preserve as much as possible the features of the classical method of the construction of preference structures.*

We also extend the results obtained by Van de Walle, De Baets and Kerre [21] to any t-conorm. Moreover, although we obtain the same minimal formulation for fuzzy preference structures that De Baets and Van de Walle [6] and Bufardi [4], the construction of fuzzy preference structures by means of fuzzy partitions is more straightforward than the use of completeness conditions and it allows to have the same definition for crisp and fuzzy preference structures (Remark 2 and Definition 19, respectively). Lastly, in Theorem 22 we give a representation of all fuzzy preference structures and we show that the strict preference relation and the incomparability relation depend on the indifference relation.

The paper is organized as follows. In Section 2 we show notation and basic definitions. In Section 3, crisp and fuzzy preference structures are formally introduced. Finally, Section 4 contains the main results of the paper.

## 2 Notation and basic concepts

Let $A$ be a not empty set of alternatives with $|A| \geq 2$. Subsets of $A$ will be called *crisp* subsets of $A$. A *partition* of $A$ is a family of crisp subsets $\{A_1, \ldots, A_n\}$ of $A$ such that

(1) $A_i \bigcap A_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$ such that $i \neq j$.

(2) $\bigcup_{i=1}^{n} A_i = A$.

A *crisp binary relation* $Q$ *on* $A$ is a crisp subset of $A^2$. We will use $aQb$ to denote $(a,b) \in Q$. Given a crisp binary relation $Q$ on $A$, the relations $Q^{-1}$, $Q^c$ and $Q^d$ are defined by

$$\begin{aligned}
Q^{-1} &= \{(a,b) \in A^2 \mid (b,a) \in Q\}, \\
Q^c &= \{(a,b) \in A^2 \mid (a,b) \notin Q\}, \\
Q^d &= \{(a,b) \in A^2 \mid (b,a) \notin Q\} = (Q^{-1})^c.
\end{aligned}$$

A crisp binary relation $Q$ on $A$ is:

(1) *reflexive* if $aQa$ for all $a \in A$.
(2) *irreflexive* if not $aQa$ for all $a \in A$.
(3) *symmetric* if $Q \subseteq Q^{-1}$ $\quad (aQb \Rightarrow bQa,$ for all $a,b \in A)$.
(4) *asymmetric* if $Q \cap Q^{-1} = \emptyset$ $\quad (aQb \Rightarrow$ not $bQa,$ for all $a,b \in A)$.
(5) *complete* if $Q \cup Q^{-1} = A^2$ $\quad (aQb$ or $bQa,$ for all $a,b \in A)$.

A function $\phi : [0,1] \longrightarrow [0,1]$ is an *order automorphism* if it is bijective and increasing. Any order automorphism $\phi$ is strictly increasing, continuous and satisfies $\phi(0) = 0$, $\phi(1) = 1$. Furthermore, the function $\phi^{-1}$ is also an order automorphism. An order automorphism $\phi$ is *reciprocal* if $\phi(1-x) = 1-\phi(x)$ for all $x \in [0,1]$. It is easy to check that $\phi$ is reciprocal if and only if $\phi^{-1}$ is reciprocal. On this, see García-Lapresta and Llamazares [13].

A *fuzzy subset* $B$ *of* $A$ is defined through its *membership function*, $\mu_B : A \longrightarrow [0,1]$, where $\mu_B(a)$ is the grade of membership of $a$ in $B$. The value $\mu_B(a)$ will be denoted by $B(a)$. Given two fuzzy subsets $B$ and $C$ of $A$, $C \subseteq B$ if $C(a) \leq B(a)$ for all $a \in A$.

The complement of a fuzzy set is defined through negations. A function $\mathcal{N} : [0,1] \longrightarrow [0,1]$ is a *negation* if it is decreasing and satisfies $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. A negation $\mathcal{N}$ is called *strict* if it is bijective. Consequently, a strict negation $\mathcal{N}$ is continuous and strictly decreasing. Moreover, its inverse, $\mathcal{N}^{-1}$, is also a strict negation. A negation $\mathcal{N}$ is called *strong* if $\mathcal{N}(\mathcal{N}(x)) = x$ for all $x \in [0,1]$. If $\mathcal{N}$ is a strong negation then it is also a strict negation and $\mathcal{N}^{-1} = \mathcal{N}$. Trillas [20] has proven that $\mathcal{N}$ is a strong negation if and only if there exists an order automorphism $\phi$ such that $\mathcal{N}(x) = \phi^{-1}(1 - \phi(x))$ for all $x \in [0,1]$. $\mathcal{N}(x) = 1 - x$ is called the *standard* negation. Given a negation $\mathcal{N}$, the complement of a fuzzy subset $B$ of $A$ is defined by $B^c(a) = \mathcal{N}(B(a))$ for all $a \in A$.

The intersection and the union of fuzzy sets are defined by means of triangular norms and conorms, respectively. These functions satisfy the following properties: commutativity, monotonicity, associativity and a boundary condition. Triangular norms and conorms were widely studied by Schweizer and Sklar [19] in the context of probabilistic metric spaces.

A function $\mathcal{T} : [0,1]^2 \longrightarrow [0,1]$ is a *triangular norm* (*t-norm*) if it satisfies the following conditions:

(1) $\mathcal{T}(1,x) = x$ for all $x \in [0,1]$.
(2) $\mathcal{T}(x,y) = \mathcal{T}(y,x)$ for all $x,y \in [0,1]$.
(3) $\mathcal{T}(x,y) \leq \mathcal{T}(u,v)$ for all $x,y,u,v \in [0,1]$ such that $x \leq u,\ y \leq v$.
(4) $\mathcal{T}(x,\mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z)$ for all $x,y,z \in [0,1]$.

A function $\mathcal{S} : [0,1]^2 \longrightarrow [0,1]$ is a *triangular conorm* (*t-conorm*) if it satisfies the following conditions:

(1) $\mathcal{S}(0,x) = x$ for all $x \in [0,1]$.
(2) $\mathcal{S}(x,y) = \mathcal{S}(y,x)$ for all $x,y \in [0,1]$.
(3) $\mathcal{S}(x,y) \leq \mathcal{S}(u,v)$ for all $x,y,u,v \in [0,1]$ such that $x \leq u,\ y \leq v$.
(4) $\mathcal{S}(x,\mathcal{S}(y,z)) = \mathcal{S}(\mathcal{S}(x,y),z)$ for all $x,y,z \in [0,1]$.

It is easy to check that $\mathcal{T}(x,0) = 0$ and $\mathcal{S}(x,1) = 1$ for all $x \in [0,1]$.

Given a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$, the intersection and the union of two fuzzy subsets $B$ and $C$ of $A$ are defined as follows:

(1) $(B \cap C)(a) = \mathcal{T}(B(a), C(a))$ for all $a \in A$.
(2) $(B \cup C)(a) = \mathcal{S}(B(a), C(a))$ for all $a \in A$.

A t-norm $\mathcal{T}$ is *Archimedean* if $\mathcal{T}(x,x) < x$ for all $x \in (0,1)$. A t-norm $\mathcal{T}$ has *zero divisors* if there exist $x,y \in (0,1)$ such that $\mathcal{T}(x,y) = 0$.

Given a t-norm $\mathcal{T}$ and a strict negation $\mathcal{N}$, the function $\mathcal{T}^{\mathcal{N}} : [0,1]^2 \longrightarrow [0,1]$ defined by $\mathcal{T}^{\mathcal{N}}(x,y) = \mathcal{N}^{-1}(\mathcal{T}(\mathcal{N}(x), \mathcal{N}(y)))$ is a t-conorm. If $\mathcal{N}$ is the standard negation, then $\mathcal{T}^{\mathcal{N}}$ is denoted by $\mathcal{T}^*$ and it is called the *dual t-conorm of* $\mathcal{T}$. Therefore, $\mathcal{T}^*(x,y) = 1 - \mathcal{T}(1-x, 1-y)$ for all $x,y \in [0,1]$.

We next show the *Lukasiewicz t-norm* and its dual t-conorm, which will play a key role in this paper:

$$W(x,y) = \max(x+y-1, 0),$$

$$W^*(x,y) = \min(x+y, 1).$$

Given a t-norm $\mathcal{T}$ and an order automorphism $\phi$, the *$\phi$-transform of $\mathcal{T}$* is the t-norm $\mathcal{T}_\phi$ defined by $\mathcal{T}_\phi(x,y) = \phi^{-1}(\mathcal{T}(\phi(x),\phi(y)))$ for all $x,y \in [0,1]$. Analogously, if $\mathcal{S}$ is a t-conorm, the *$\phi$-transform of $\mathcal{S}$* is the t-conorm $\mathcal{S}_\phi$ defined by $\mathcal{S}_\phi(x,y) = \phi^{-1}(\mathcal{S}(\phi(x),\phi(y)))$ for all $x,y \in [0,1]$. For instance, the $\phi$-transforms of Łukasiewicz t-norm and t-conorm are given by

$$W_\phi(x,y) = \phi^{-1}(\max(\phi(x)+\phi(y)-1,0)),$$
$$(W^*)_\phi(x,y) = \phi^{-1}(\min(\phi(x)+\phi(y),1)).$$

If $\mathcal{T}$ is a t-norm and $\mathcal{N}$ is a strict negation then $(\mathcal{T},\mathcal{T}^{\mathcal{N}},\mathcal{N})$ is a *De Morgan triplet*. A De Morgan triplet $(\mathcal{T},\mathcal{T}^{\mathcal{N}},\mathcal{N})$ is *continuous* if the t-norm $\mathcal{T}$ is continuous. If $\phi$ is an order automorphism and $\mathcal{N}(x) = \phi^{-1}(1-\phi(x))$ then $(W_\phi,(W_\phi)^{\mathcal{N}},\mathcal{N})$ is called the *$\phi$-Łukasiewicz triplet*. In this case, it is easy to check that $(W_\phi)^{\mathcal{N}} = (W^*)_\phi$.

A *fuzzy binary relation $Q$ on $A$* is a fuzzy subset of $A^2$. The value $\mu_Q(a,b)$ will be denoted by $Q(a,b)$. If $Q(a,b) \in \{0,1\}$ for all $a,b \in A$ then $Q$ is a crisp binary relation. In this case, $aQb$ denotes $Q(a,b)=1$.

If $\mathcal{N}$ is a strict negation and $Q$ is a fuzzy binary relation on $A$, the fuzzy relations $Q^{-1}$, $Q^c$ and $Q^d$ are defined by $Q^{-1}(a,b) = Q(b,a)$, $Q^c(a,b) = \mathcal{N}(Q(a,b))$ and $Q^d(a,b) = \mathcal{N}(Q(b,a))$, for all $a,b \in A$.

Given a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$, a fuzzy binary relation $Q$ on $A$ is:

(1) *reflexive* if $Q(a,a)=1$ for all $a \in A$.
(2) *irreflexive* if $Q(a,a)=0$ for all $a \in A$.
(3) *symmetric* if $Q(a,b)=Q(b,a)$ for all $a,b \in A$.
(4) *asymmetric* if $\mathcal{T}(Q(a,b),Q(b,a))=0$ for all $a,b \in A$.
(5) *complete* if $\mathcal{S}(Q(a,b),Q(b,a))=1$ for all $a,b \in A$.

## 3 Preference structures

In this section we present crisp and fuzzy preference structures. Preference structures are formed by three binary relations: a strict preference relation, $P$, an indifference relation, $I$, and an incomparability relation, $J$. The definition of crisp preference structure on $A$ was introduced by Roubens and Vincke [17].

**Definition 1.** *A crisp preference structure on $A$ is a triplet $(P,I,J)$ of crisp* *binary relations on $A$ that satisfy the following properties:*

*(1) $P$ is asymmetric.*
*(2) $I$ is reflexive and symmetric.*
*(3) $J$ is symmetric.*
*(4) $P \cap I = \emptyset$, $P \cap J = \emptyset$, $I \cap J = \emptyset$.*
*(5) $P \cup P^{-1} \cup I \cup J = A^2$.*

In the crisp framework, asymmetry implies irreflexivity. Therefore, $P$ is irreflexive. Moreover, $J$ is irreflexive since $I \cap J = \emptyset$ and $I$ is reflexive. It is important to emphasize that a crisp preference structure on $A$ is a partition of $A^2$ that satisfies some conditions. This fact is pointed out in the following remark.

**Remark 2.** *A triplet $(P, I, J)$ of crisp binary relations on $A$ is a crisp preference structure on $A$ if and only if the following statements hold:*

*(1) $I$ is reflexive and symmetric.*
*(2) $\{P, P^{-1}, I, J\}$ is a partition of $A$.*

From a strict preference relation $P$ and an indifference relation $I$ it is possible to obtain a reflexive crisp binary relation $R$ on $A$ by means of $R = P \cup I$. This relation shows whether an alternative is at least as good as another or vice versa.

**Definition 3.** *Let $R$ be a reflexive crisp binary relation on $A$. A crisp preference structure on $A$ associated to $R$ is a triplet $(P, I, J)$ of crisp binary relations that satisfy the following properties:*

*(1) $(P, I, J)$ is a crisp preference structure on $A$.*
*(2) $R = P \cup I$.*

It is well-known that, in the crisp framework, the preference structures on $A$ associated to $R$ are uniquely determined. Thus, $P = R \cap R^d$, $I = R \cap R^{-1}$ and $J = R^c \cap R^d$.

In the fuzzy framework, the definition of fuzzy preference structure on $A$ was first introduced by De Baets, Van de Walle and Kerre [7]. Here, we give a similar definition.

**Definition 4.** *Let $\mathcal{T}$ be a t-norm and $\mathcal{S}$ a t-conorm. A fuzzy preference structure on $A$ (FPS) is a triplet $(P, I, J)$ of fuzzy binary relations that satisfy the following properties:*

*(1) $P$ is irreflexive and asymmetric.*
*(2) $I$ is reflexive and symmetric.*

*(3) $J$ is irreflexive and symmetric.*
*(4) $P \cap I = \emptyset$, $P \cap J = \emptyset$, $I \cap J = \emptyset$.*
*(5) $P \cup P^{-1} \cup I \cup J = A^2$.*

## 4 The results

This section is devoted to characterizing FPS's. Van de Walle, De Baets and Kerre [21] have studied FPS's defined from De Morgan triplets with strong negations. Their outcomes are also satisfied for any t-conorm.

**Theorem 5.** *Let $\mathcal{T}$ be a t-norm without zero divisors and $\mathcal{S}$ a t-conorm. If $(P, I, J)$ is a FPS on $A$ then $P$, $I$ and $J$ are crisp binary relations on $A$.*

**PROOF.** It is similar to the provided by Van de Walle, De Baets and Kerre [21]. $\square$

The following result, given by Van de Walle, De Baets and Kerre [21], characterize the values for which a continuous non-Archimedean t-norm with zero divisors is null.

**Proposition 6.** *Let $\mathcal{T}$ be a continuous non-Archimedean t-norm with zero divisors. Then there exist $\theta \in (0,1)$ and an order automorphism $\phi$ such that for all $(x, y) \in [0,1]^2$*

$$\mathcal{T}(x,y) = 0 \;\Leftrightarrow\; \begin{cases} x = 0, \quad \textit{or} \\ y = 0, \quad \textit{or} \\ (x,y) \in (0,\theta)^2 \;\; \textit{and} \;\; \phi(\frac{x}{\theta}) + \phi(\frac{y}{\theta}) \leq 1. \end{cases}$$

This result allows us to prove that FPS's based on continuous non-Archimedean t-norms with zero divisors cannot take all the values in $[0,1]$.

**Theorem 7.** *Let $\mathcal{T}$ be a continuous non-Archimedean t-norm with zero divisors and $\mathcal{S}$ a t-conorm. There exists $\theta \in (0,1)$ such that if $(P, I, J)$ is a FPS on $A$ then the fuzzy binary relations $P$, $I$ and $J$ cannot take values in $[\theta, 1)$.*

**PROOF.** Analogously to Van de Walle, De Baets and Kerre [21], we only give the proof for $P$, since the proofs for $I$ and $J$ are similar. Let $\theta \in (0,1)$ be the value obtained in Proposition 6 and $(a,b) \in A^2$ such that $P(a,b) \geq \theta$. Since

$\mathcal{T}(P(a,b),P(b,a)) = 0$, $\mathcal{T}(P(a,b),I(a,b)) = 0$ and $\mathcal{T}(P(a,b),J(a,b)) = 0$, by Proposition 6 we have $P(b,a) = 0$, $I(a,b) = 0$ and $J(a,b) = 0$. Lastly, from $P \cup P^{-1} \cup I \cup J = A^2$ we obtain $P(a,b) = 1$. □

It is important to emphasize that these results do not depend on the t-conorm that we use for representing the union of fuzzy sets. So, if the t-norm has not zero divisors, then $P$, $I$ and $J$ are crisp binary relations. On the other hand, if the t-norm is continuous, non-Archimedean and it has zero divisors, then there exists a threshold, $\theta$, such that $P$, $I$ and $J$ take the value 1 or they are bounded from above by $\theta$. Therefore, if we consider continuous t-norms, it seems natural the use of Archimedean t-norms with zero divisors. These t-norms have been characterized by Ovchinnikov and Roubens [14] by means of $\phi$-transforms of Łukasiewicz t-norm.

**Theorem 8.** *$\mathcal{T}$ is a continuous Archimedean t-norm with zero divisors if and only if there exists an order automorphism $\phi$ such that $\mathcal{T} = W_\phi$.*

However, the preceding results impose no conditions about the t-conorm. Next we give a reason for the use of Łukasiewicz triplets. Previously, we point up the result given by Fodor and Roubens [12] (see also Alsina [1]).

**Theorem 9.** *There exists no De Morgan triplet $(\mathcal{T}, \mathcal{T}^{\mathcal{N}}, \mathcal{N})$ such that*

$$\mathcal{T}^{\mathcal{N}}(\mathcal{T}(x,y), \mathcal{T}(x,\mathcal{N}(y))) = x$$

*for all $x, y \in [0,1]$.*

This theorem shows us that there is no De Morgan triplet such that the equality $B = (B \cap C) \cup (B \cap C^c)$ holds for any two fuzzy subsets $B$ and $C$ of $A$. Consequently, there is no De Morgan triplet such that $R = P \cup I$ holds with $P = R \cap (R^{-1})^c$ and $I = R \cap R^{-1}$, for any reflexive fuzzy binary relation $R$.

However, it is possible to reconstruct any fuzzy subset $B$ from a fuzzy subset $C$ if $C \subseteq B$, i.e. there exist continuous De Morgan triplets such that $B = C \cup (B \setminus C)$ or, equivalently, $B = C \cup (B \cap C^c)$ for any two fuzzy subsets $B$ and $C$ of $A$ such that $C \subseteq B$. In the following remark we give a necessary and sufficient condition that the t-norm, the t-conorm and the strict negation have to fulfill in order that the previous equality may be satisfied.

**Remark 10.** *Let $\mathcal{T}$ be a t-norm, $\mathcal{S}$ a t-conorm and $\mathcal{N}$ a strict negation. The equality $B = C \cup (B \cap C^c)$ holds for any two fuzzy subsets $B$ and $C$ of $A$ such that $C \subseteq B$ if and only if $\mathcal{S}(y, \mathcal{T}(x, \mathcal{N}(y))) = x$ for all $x, y \in [0,1]$ such that $y \leq x$.*

In the characterization of continuous t-norms, continuous t-conorms and strict negations that satisfy this condition we will use the following result, given by Fodor and Roubens [12].

**Proposition 11.** *Let $\mathcal{S}$ be a continuous t-conorm and $\mathcal{N}$ a strict negation. Then the following conditions are equivalent:*

*(1)* $\mathcal{S}(x, \mathcal{N}(x)) = 1$ *for all* $x \in [0, 1]$.
*(2) There exists an order automorphism* $\phi$ *such that* $\mathcal{S} = (W^*)_\phi$ *and* $\mathcal{N}(x) \geq \phi^{-1}(1 - \phi(x))$ *for all* $x \in [0, 1]$.

**Theorem 12.** *Let $\mathcal{T}$ be a continuous t-norm, $\mathcal{S}$ a continuous t-conorm and $\mathcal{N}$ a strict negation. Then the following conditions are equivalent:*

*(1)* $\mathcal{S}(y, \mathcal{T}(x, \mathcal{N}(y))) = x$ *for all* $x, y \in [0, 1]$ *such that* $y \leq x$.
*(2) There exists an order automorphism* $\phi$ *such that* $\mathcal{N}(x) = \phi^{-1}(1 - \phi(x))$, $\mathcal{T} = W_\phi$ *and* $\mathcal{S} = (W_\phi)^{\mathcal{N}}$, *i.e.* $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ *is the* $\phi$*-Łukasiewicz triplet.*

**PROOF.**
(1) ⇒ (2): If $x = 1$ then for all $y \in [0, 1]$ we have

$$\mathcal{S}(y, \mathcal{N}(y)) = \mathcal{S}(y, \mathcal{T}(1, \mathcal{N}(y))) = 1.$$

By Proposition 11 there exists an order automorphism $\phi$ such that $\mathcal{S} = (W^*)_\phi$ and $\mathcal{N}(x) \geq \phi^{-1}(1 - \phi(x))$ for all $x \in [0, 1]$. Therefore,

$$\phi^{-1}(\min(\phi(y) + \phi(\mathcal{T}(x, \mathcal{N}(y))), 1)) = x$$

for all $x, y \in [0, 1]$ such that $y \leq x$. Hence, $\mathcal{T}(x, \mathcal{N}(y)) = \phi^{-1}(\phi(x) - \phi(y))$ for all $x, y \in [0, 1]$ such that $y \leq x < 1$. By the continuity of $\mathcal{T}$ and $\phi$ we have

$$\begin{aligned}\mathcal{N}(y) &= \mathcal{T}(1, \mathcal{N}(y)) = \lim_{x \to 1} \mathcal{T}(x, \mathcal{N}(y)) = \lim_{x \to 1} \phi^{-1}(\phi(x) - \phi(y)) \\ &= \phi^{-1}(1 - \phi(y)),\end{aligned}$$

for all $y \in [0, 1)$. It is obvious that this relationship is also satisfied for $y = 1$. Lastly, we are going to prove that $\mathcal{T} = W_\phi$. Given $x, y \in [0, 1)$, we consider $z = \mathcal{N}^{-1}(y)$, i.e. $y = \mathcal{N}(z) = \phi^{-1}(1 - \phi(z))$. Then, $\phi(x) + \phi(y) - 1 = \phi(x) - \phi(z)$. We distinguish two cases:

(i) If $z \leq x$, then $\phi(x) + \phi(y) - 1 \geq 0$ and

$$\begin{aligned}\mathcal{T}(x, y) &= \mathcal{T}(x, \mathcal{N}(z)) = \phi^{-1}(\phi(x) - \phi(z)) = \phi^{-1}(\phi(x) + \phi(y) - 1) \\ &= W_\phi(x, y).\end{aligned}$$

(ii) If $x < z$, then $\mathcal{N}(z) < \mathcal{N}(x)$, $\phi(x) + \phi(y) - 1 < 0$ and

$$\begin{aligned}\mathcal{T}(x, y) &= \mathcal{T}(x, \mathcal{N}(z)) \leq \mathcal{T}(x, \mathcal{N}(x)) = \phi^{-1}(\phi(x) - \phi(x)) = 0\\ &= W_\phi(x, y).\end{aligned}$$

If $x = 1$ or $y = 1$, then the equality is obvious.

(2) $\Rightarrow$ (1): If $y \leq x$ then $\phi(y) \leq \phi(x)$. Therefore,

$$W_\phi(x, \mathcal{N}(y)) = \phi^{-1}(\max(\phi(x) + \phi(\mathcal{N}(y)) - 1, 0)) = \phi^{-1}(\phi(x) - \phi(y)).$$

Hence, for all $x, y \in [0, 1]$ such that $y \leq x$ the following holds

$$(W_\phi)^{\mathcal{N}}(y, \mathcal{T}(x, \mathcal{N}(y))) = \phi^{-1}(\min(\phi(y) + \phi(x) - \phi(y), 1)) = x. \qquad \square$$

Given a reflexive fuzzy binary relation $R$, since $I = R \cap R^{-1} \subseteq R$, the previous theorem suggests us to consider $P = R \cap I^c = R \cap (R \cap R^{-1})^c$. Is is easy to see that this definition of $P$ is also satisfied in the crisp framework.

**Corollary 13.** *Let $\mathcal{T}$ be a continuous t-norm, $\mathcal{S}$ a continuous t-conorm and $\mathcal{N}$ a strict negation. For any reflexive fuzzy binary relation on $A$, $R$, the relations $P = R \cap (R \cap R^{-1})^c$ and $I = R \cap R^{-1}$ satisfy $R = P \cup I$ if and only if there exists an order automorphism $\phi$ such that $(\mathcal{T}, \mathcal{S}, \mathcal{N})$ is the $\phi$-Łukasiewicz triplet.*

**PROOF.** It is sufficient to take into account that $R = I \cup (R \cap I^c)$ and Theorem 12. $\square$

Notice that if we consider the standard negation, then the $\phi$-Łukasiewicz triplet is given by a reciprocal order automorphism.

**Corollary 14.** *Let $\mathcal{T}$ be a continuous t-norm and $\mathcal{S}$ a continuous t-conorm. Then the following conditions are equivalent:*

*(1) $\mathcal{S}(y, \mathcal{T}(x, 1 - y)) = x$ for all $x, y \in [0, 1]$ such that $y \leq x$.*
*(2) There exists a reciprocal order automorphism $\phi$ such that $\mathcal{T} = W_\phi$ and $\mathcal{S} = (W^*)_\phi$.*

**PROOF.** It is sufficient to consider $\mathcal{N}(x) = 1 - x$ in Theorem 12. $\square$

As we have mentioned in the introduction, some authors use the same order automorphism for $\phi$-transforms of Łukasiewicz t-norm and strong negations, i.e. they use $\phi$-Łukasiewicz triplets, because the following statements holds:

$$(W_\phi)^{\mathcal{N}}(x,y) = \phi^{-1}(\min(\phi(x)+\phi(y),1)) = (W^*)_\phi(x,y).$$

However, when we utilize the standard negation we can obtain the same relationship if we suppose that the order automorphism is reciprocal. In fact, this condition allows us to characterize reciprocal order automorphisms.

**Theorem 15.** *Let $\phi$ be an order automorphism. Then the following statements are equivalent:*

*(1) $\phi$ is reciprocal.*
*(2) $(W_\phi)^* = (W^*)_\phi$.*

**PROOF.**
(1) $\Rightarrow$ (2): If $\phi$ is reciprocal, then for all $x, y \in [0,1]$ we have

$$\begin{aligned}(W_\phi)^*(x,y) &= 1 - W_\phi(1-x,1-y) = 1-\phi^{-1}(W(\phi(1-x),\phi(1-y))) \\ &= \phi^{-1}(1-W(1-\phi(x),1-\phi(y))) = \phi^{-1}(W^*(\phi(x),\phi(y))) \\ &= (W^*)_\phi(x,y).\end{aligned}$$

(2) $\Rightarrow$ (1): This is proven by contradiction. Suppose, it were otherwise. Then there exists $x \in (0,1)$ such that $\phi(1-x)+\phi(x) \neq 1$. We distinguish two cases:

(i) If $\phi(1-x)+\phi(x) < 1$, then

$$(W_\phi)^*(x,1-x) = 1-\phi^{-1}(W(\phi(1-x),\phi(x))) = 1-\phi^{-1}(0) = 1.$$

On the other hand,

$$\begin{aligned}(W^*)_\phi(x,1-x) &= \phi^{-1}(1-W(1-\phi(x),1-\phi(1-x))) \\ &= \phi^{-1}(\phi(x)+\phi(1-x)) < 1,\end{aligned}$$

which contradicts the hypothesis.

(ii) If $\phi(1-x)+\phi(x) > 1$, then

$$\begin{aligned}(W_\phi)^*(x,1-x) &= 1-\phi^{-1}(W(\phi(1-x),\phi(x))) \\ &= 1-\phi^{-1}(\phi(1-x)+\phi(x)-1) < 1.\end{aligned}$$

On the other hand,

$$(W^*)_\phi(x,1-x) = \phi^{-1}(1-W(1-\phi(x),1-\phi(1-x))) = \phi^{-1}(1) = 1,$$

which contradicts the hypothesis. $\square$

In the sequel we only consider $\phi$-Łukasiewicz triplets. However, if $(P, I, J)$ is a FPS on $A$, it can happen that $(P \cup I) \cap J \neq \emptyset$ although $P \cap J = \emptyset$ and $I \cap J = \emptyset$. In order to avoid this situation, we introduce the definition of fuzzy partition of $A$.

**Definition 16.** *Let $\phi$ be an order automorphism and $(W_\phi, (W_\phi)^{\mathcal{N}}, \mathcal{N})$ the $\phi$-Łukasiewicz triplet. A $\phi$-fuzzy partition of $A$ is a family of fuzzy subsets $\{A_1, \ldots, A_n\}$ on $A$ such that*

*(1)* $A_i \cap \Big( \bigcup_{\substack{j=1 \\ j\neq i}}^{n} A_j \Big) = \emptyset$ *for all* $i \in \{1, \ldots, n\}$.

*(2)* $\bigcup_{i=1}^{n} A_i = A$.

In the following proposition we give a characterization of a $\phi$-fuzzy partition of $A$.

**Proposition 17.** *Let $\phi$ be an order automorphism and $(W_\phi, (W_\phi)^{\mathcal{N}}, \mathcal{N})$ the $\phi$-Łukasiewicz triplet. Then the following statements are equivalent:*

*(1)* $\{A_1, \ldots, A_n\}$ *is a $\phi$-fuzzy partition of $A$.*

*(2)* $\phi(A_1(a)) + \cdots + \phi(A_n(a)) = 1$ *for all* $a \in A$.

**PROOF.**

(1) $\Rightarrow$ (2): Given $a \in A$, if $x_i = A_i(a)$ for all $i \in \{1, \ldots, n\}$, then we have

$$1 = \Big( \bigcup_{i=1}^{n} A_i \Big)(a) = \phi^{-1}\Big( \min \Big( \sum_{i=1}^{n} \phi(x_i), 1 \Big) \Big).$$

Therefore, $\sum_{i=1}^{n} \phi(x_i) \geq 1$.

Consequently, there exists $i \in \{1, \ldots, n\}$ such that $\phi(x_i) > 0$. Hence,

$$\begin{aligned} 0 = \Big( A_i \cap \Big( \bigcup_{\substack{j=1 \\ j\neq i}}^{n} A_j \Big) \Big)(a) &= \phi^{-1}\Big( \max \Big( \phi(x_i) + \min \Big( \sum_{\substack{j=1 \\ j\neq i}}^{n} \phi(x_j), 1 \Big) - 1, 0 \Big) \Big) \\ &= \phi^{-1}\Big( \max \Big( \min \Big( \sum_{j=1}^{n} \phi(x_j) - 1, \phi(x_i) \Big), 0 \Big) \Big). \end{aligned}$$

Therefore, $\min\left(\sum_{j=1}^{n}\phi(x_j)-1,\phi(x_i)\right)\leq 0$. Since $\phi(x_i)>0$, then we have $\sum_{j=1}^{n}\phi(x_j)\leq 1$.

(2) $\Rightarrow$ (1): It is a simple checking. $\square$

**Remark 18.** *A similar condition of orthogonality was introduced by Ruspini [18] for defining fuzzy partitions.*

In Remark 2 we have emphasized the relationship between crisp preference structures and partitions of $A$. According to this result, we now give the following definition.

**Definition 19.** *Let $\phi$ be an order automorphism. A $\phi$-fuzzy preference structure on $A$ ($\phi$-FPS) is a triplet $(P,I,J)$ of fuzzy binary relations that satisfy the following properties:*

*(1) $I$ is reflexive and symmetric.*
*(2) $\{P,P^{-1},I,J\}$ is a $\phi$-fuzzy partition of $A$.*

Note that if $(P,I,J)$ is a $\phi$-FPS on $A$ then $P$ is irreflexive and asymmetric and $J$ is irreflexive and symmetric.

**Remark 20.** *Notice that Definition 19 coincides with the definition of $\phi$-fuzzy preference structure given by De Baets and Van de Walle [6] and Bufardi [4].*

As in the crisp framework, we now consider $\phi$-FPS's on $A$ associated to reflexive fuzzy binary relations.

**Definition 21.** *Let $\phi$ be an order automorphism and $R$ a reflexive fuzzy binary relation on $A$. A $\phi$-FPS on $A$ associated to $R$ is a triplet $(P,I,J)$ of fuzzy binary relations that satisfy the following properties:*

*(1) $(P,I,J)$ is a $\phi$-FPS on $A$.*
*(2) $R=P\cup I$.*

In the following theorem we obtain a representation of all fuzzy preference structures and we establish the lower and upper bounds for $I$ when $(P,I,J)$ is a $\phi$-FPS on $A$ associated to $R$. Moreover, the relations $P$ and $J$ are uniquely determined from $R$ and $I$. Similar relationships have been given by Bufardi [4] in the axiomatic model.

**Theorem 22.** *Let $R$ be a reflexive fuzzy binary relation on $A$ and $\phi$ an order automorphism. If $(P,I,J)$ is a $\phi$-FPS on $A$ associated to $R$, then for* *all* $a, b \in A$

$$P(a,b) = \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b))),$$
$$J(a,b) = \phi^{-1}(1 - \phi(R(a,b)) - \phi(R(b,a)) + \phi(I(a,b))),$$
$$\phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)) \leq I(a,b) \leq \min(R(a,b), R(b,a)).$$

**PROOF.** Given $a, b \in A$, we have

$$R(a,b) = \phi^{-1}(\min(\phi(P(a,b)) + \phi(I(a,b)), 1)).$$

Since $\phi(P(a,b)) + \phi(I(a,b)) \leq 1$, then $R(a,b) = \phi^{-1}(\phi(P(a,b)) + \phi(I(a,b)))$, and, consequently,

$$P(a,b) = \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b))).$$

Moreover, by Proposition 17, we have

$$\phi(P(a,b)) + \phi(I(a,b)) + \phi(P(b,a)) + \phi(J(a,b)) = 1.$$

Therefore, $J(a,b) = \phi^{-1}(1 - \phi(R(a,b)) - \phi(R(b,a)) + \phi(I(a,b)))$.

On the other hand, since $\phi(R(a,b)) = \phi(P(a,b)) + \phi(I(a,b))$, then $\phi(I(a,b)) \leq \phi(R(a,b))$, or equivalently $I(a,b) \leq R(a,b)$. Analogously, from

$$\phi(R(b,a)) = \phi(P(b,a)) + \phi(I(b,a)),$$

we have $I(a,b) = I(b,a) \leq R(b,a)$. Therefore, $I(a,b) \leq \min(R(a,b), R(b,a))$. Lastly, as $J(a,b) \geq 0$, we have

$$\phi(I(a,b)) \geq \phi(R(a,b)) + \phi(R(b,a)) - 1,$$

or, equivalently, $I(a,b) \geq \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0))$. $\square$

**Remark 23.** *In Theorem 22, if $\phi$ is the identity automorphism, then we obtain the fuzzy binary relations given by Barrett and Pattanaik [2].*

In the axiomatic model it is usual to suppose the condition $R^d = P \cup J$ together with $R = P \cup I$. However, Bufardi [3] has proven that both conditions are equivalent when we consider $\phi$-FPS's.

**Remark 24.** *It is easy to check that if $(P, I, J)$ is a $\phi$-FPS on $A$ associated to $R$, then the condition $R^d = P \cup J$ is also satisfied.*

The fuzzy binary relations $P$, $I$ and $J$ obtained in Theorem 22 are not uniquely determined, so that we can impose additional conditions. Thus, in the following theorems we also consider $P = R \cap R^d$, $P = R \cap (R \cap R^{-1})^c$, $I = R \cap R^{-1}$ and $J = R^c \cap R^d$, which are satisfied in the crisp framework. The conditions $P = R \cap R^d$ and $I = R \cap R^{-1}$ have also been used by Bufardi [3]. Moreover, under other hypothesis, similar FPS's can be found in Fodor and Roubens [10] and [12] and in Van de Walle, De Baets and Kerre [22]. The following result has been given by Bufardi [3].

**Theorem 25.** *Let $R$ be a reflexive fuzzy binary relation on $A$ and $\phi$ an order automorphism. Then the following statements are equivalent:*

*(1) $(P, I, J)$ is a $\phi$-FPS on $A$ associated to $R$ and $P = R \cap R^d$.*
*(2) For all $a, b \in A$*

$$\begin{aligned} P(a,b) &= \phi^{-1}(\max(\phi(R(a,b)) - \phi(R(b,a)), 0)), \\ I(a,b) &= \min(R(a,b), R(b,a)), \\ J(a,b) &= \phi^{-1}(\min(1 - \phi(R(a,b)), 1 - \phi(R(b,a)))). \end{aligned}$$

**Remark 26.** *In the $\phi$-FPS on $A$ obtained in Theorem 25, the fuzzy relation $I$ takes the maximum possible value. Consequently, $J$ also takes its maximum value while $P$ takes the minimum. Moreover, $\min(P(a,b), P(b,a)) = 0$ for all $a, b \in A$.*

**Remark 27.** *Bufardi [3] has proven that if we consider the fuzzy preference structure given in Theorem 25 then $R$ is symmetrical if and only if $P = \emptyset$. From Theorem 22, it is easy to check that if $R$ is symmetrical then the only fuzzy preference structure that satisfies $P = \emptyset$ is given by Theorem 25.*

**Theorem 28.** *Let $R$ be a reflexive fuzzy binary relation on $A$ and $\phi$ an order automorphism. Then the following statements are equivalent:*

*(1) $(P, I, J)$ is a $\phi$-FPS on $A$ associated to $R$ and it is satisfied one of the following relationships: $P = R \cap (R \cap R^{-1})^c$, $I = R \cap R^{-1}$ or $J = R^c \cap R^d$.*
*(2) $P = R \cap (R \cap R^{-1})^c$, $I = R \cap R^{-1}$ and $J = R^c \cap R^d$, i.e.*

$$\begin{aligned} P(a,b) &= \min(R(a,b), \phi^{-1}(1 - \phi(R(b,a)))), \\ I(a,b) &= \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)), \\ J(a,b) &= \phi^{-1}(\max(1 - \phi(R(a,b)) - \phi(R(b,a)), 0)), \end{aligned}$$

*for all $a, b \in A$.*

**PROOF.**

(1) $\Rightarrow$ (2): Firstly, we suppose that $I = R \cap R^{-1}$ is satisfied. In this case, for all $a, b \in A$ the following holds

$$I(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)).$$

By Theorem 22 we have

$$\begin{aligned} P(a,b) &= \phi^{-1}(\phi(R(a,b)) - \phi(I(a,b))) \\ &= \phi^{-1}(\phi(R(a,b)) - \max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)) \\ &= \phi^{-1}(\phi(R(a,b)) + \min(1 - \phi(R(a,b)) - \phi(R(b,a)), 0)) \\ &= \phi^{-1}(\min(1 - \phi(R(b,a)), \phi(R(a,b)))) \\ &= \min(\phi^{-1}(1 - \phi(R(b,a))), R(a,b)), \end{aligned}$$

and

$$\begin{aligned} J(a,b) &= \phi^{-1}(1 - \phi(R(a,b)) - \phi(R(b,a)) + \phi(I(a,b))) \\ &= \phi^{-1}(\max(0, 1 - \phi(R(a,b)) - \phi(R(b,a)))). \end{aligned}$$

We now suppose that $P = R \cap (R \cap R^{-1})^c$ is satisfied. Then, for all $a, b \in A$

$$P(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi((R \cap R^{-1})^c(a,b)) - 1, 0))$$

Since $(R \cap R^{-1})^c(a,b) = \phi^{-1}(1 - \max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0))$, we have

$$\begin{aligned} P(a,b) &= \phi^{-1}(\max(\phi(R(a,b)) - \max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0), 0)) \\ &= \phi^{-1}(\max(\min(1 - \phi(R(b,a)), \phi(R(a,b))), 0)) \\ &= \phi^{-1}(\min(1 - \phi(R(b,a)), \phi(R(a,b)))) \\ &= \min(\phi^{-1}(1 - \phi(R(b,a))), R(a,b)), \end{aligned}$$

By Theorem 22 the following holds

$$\begin{aligned} I(a,b) &= \phi^{-1}(\phi(R(a,b)) - \phi(P(a,b))) \\ &= \phi^{-1}(\phi(R(a,b)) - \min(1 - \phi(R(b,a)), \phi(R(a,b)))) \\ &= \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)). \end{aligned}$$

Therefore, $I = R \cap R^{-1}$, and, consequently

$$J(a,b) = \phi^{-1}(\max(1 - \phi(R(a,b)) - \phi(R(b,a)), 0)).$$

Finally, if $J = R^c \cap R^d$, then for all $a, b \in A$ the following holds

$$J(a,b) = \phi^{-1}(\max(1 - \phi(R(a,b)) - \phi(R(b,a)), 0)).$$

By Theorem 22 we have

$$\begin{aligned} I(a,b) &= \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1 + \phi(J(a,b))) \\ &= \phi^{-1}(\max(0, \phi(R(a,b)) + \phi(R(b,a)) - 1)). \end{aligned}$$

Therefore, $I = R \cap R^{-1}$, and, consequently

$$P(a,b) = \min(R(a,b), \phi^{-1}(1 - \phi(R(b,a)))).$$

(2) $\Rightarrow$ (1): It is obvious that $I$ is reflexive and symmetric. On the other hand, for all $a, b \in A$ we have

$$\phi(P(a,b)) + \phi(I(a,b)) + \phi(J(a,b)) = \max(\phi(R(a,b)), 1 - \phi(R(b,a))).$$

Since $\phi(P(b,a)) = \min(\phi(R(b,a)), 1 - \phi(R(a,b)))$, we have

$$\phi(P(a,b)) + \phi(P(b,a)) + \phi(I(a,b)) + \phi(J(a,b)) = 1,$$

for all $a, b \in A$, and, consequently, $\{P, P^{-1}, I, J\}$ is a $\phi$-fuzzy partition of $A$. Moreover,

$$\begin{aligned} \phi(P(a,b)) + \phi(I(a,b)) &= \min(\phi(R(a,b)), 1 - \phi(R(b,a))) \\ &\quad + \max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0) \\ &= \phi(R(a,b)). \end{aligned}$$

Therefore, $(P \cup I)(a,b) = \phi^{-1}(\phi(P(a,b)) + \phi(I(a,b))) = R(a,b)$ for all $a, b \in A$. $\square$

**Remark 29.** *Under this $\phi$-FPS on $A$ associated to $R$, the fuzzy relation $I$ takes the minimum possible value. Consequently, $J$ also takes its minimum value while $P$ takes the maximum. Moreover,* $\min(I(a,b), J(a,b)) = 0$ *for all $a, b \in A$.*

It is important to emphasize that the fuzzy preference structures obtained in Theorem 25 and Theorem 28 correspond to the two extreme solutions of the system of functional equations obtained in the axiomatic model. Remarks 26 and 29 show that these fuzzy preference structures are also extreme solutions in this model.

**Remark 30.** *Bufardi [3] has proven that if we consider the fuzzy preference structure given in Theorem 28 then* $R$ *is complete if and only if* $J = \emptyset$. *From Theorem 22, it is easy to check that if* $R$ *is complete then the only fuzzy preference structure that satisfies* $J = \emptyset$ *is given by Theorem 28.*

**Remark 31.** *From Theorem 22 we can obtain a geometrical relationship between the relations* $R$ *and* $P$. *Given* $a$ *and* $b$ *two different elements of* $A$, *since* $\phi(P(a,b)) = \phi(R(a,b)) - \phi(I(a,b))$, *when* $\phi(I(a,b))$ *ranges between* $\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0)$ *and* $\min(\phi(R(a,b)), \phi(R(b,a)))$ *we obtain the possible values for* $\phi(P(a,b))$ *and* $\phi(P(b,a))$. *We distinguish two cases:*

*(1) If* $\phi(R(a,b)) + \phi(R(b,a)) > 1$ *then the points* $(\phi(P(a,b)), \phi(P(b,a)))$ *are in the parallel segment to the bisecting line of the first quadrant with endpoints* $X_1$ *and* $X_2$, *where*

$$X_1 = (1 - \phi(R(b,a)), 1 - \phi(R(a,b))),$$

$$X_2 = (\max(\phi(R(a,b)) - \phi(R(b,a)), 0), \max(\phi(R(b,a)) - \phi(R(a,b)), 0)).$$

*The points* $X_1$ *and* $X_2$ *correspond to the fuzzy preference structures given in Theorem 28 and 25, respectively. Moreover,* $X_1$ *is simetric to* $Y = (\phi(R(a,b)), \phi(R(b,a)))$ *with respect to the straight line* $\phi(P(a,b)) + \phi(P(b,a)) = 1$ *(see Figure 1).* [Insert Figure 1 about here]

*(2) If* $\phi(R(a,b)) + \phi(R(b,a)) \leq 1$ *then the points* $(\phi(P(a,b)), \phi(P(b,a)))$ *are in the parallel segment to the bisecting line of the first quadrant with endpoints* $X_1$ *and* $X_2$, *where*

$$X_1 = (\phi(R(a,b)), \phi(R(b,a))),$$

$$X_2 = (\max(\phi(R(a,b)) - \phi(R(b,a)), 0), \max(\phi(R(b,a)) - \phi(R(a,b)), 0)).$$

*The points* $X_1$ *and* $X_2$ *correspond to the fuzzy preference structures given in Theorem 28 and 25, respectively. Moreover,* $X_1$ *coincides with* $Y = (\phi(R(a,b)), \phi(R(b,a)))$ *(see Figure 2).* [Insert Figure 2 about here]

*Since* $(\phi(R(a,b)), \phi(R(b,a)))$ *and* $(1 - \phi(R(b,a)), 1 - \phi(R(a,b)))$ *are simetric with respect to the straight line* $\phi(P(a,b)) + \phi(P(b,a)) = 1$, *Figures 1 and 2 show that the possible values of* $P$ *are the same for the relations* $R$ *and* $R^d \cup \triangle$, *where* $\triangle$ *is the minimal reflexive relation (notice that* $R^d$ *is irreflexive).*

The $\phi$-FPS on $A$ associated to $R$ given in Theorem 28 can also be obtained under the assumption $P \cup I \cup P^{-1} = R \cup R^{-1}$. This condition has also been used by Fodor and Roubens [12] in the axiomatic model for characterizing the same $\phi$-FPS.

**Theorem 32.** *Let $R$ be a reflexive fuzzy binary relation on $A$ and $\phi$ an order automorphism. Then the following statements are equivalent:*

*(1) $(P, I, J)$ is a $\phi$-FPS on $A$ associated to $R$ and $P \cup I \cup P^{-1} = R \cup R^{-1}$.*
*(2) $P = R \cap (R \cap R^{-1})^c$, $I = R \cap R^{-1}$ and $J = R^c \cap R^d$.*

**PROOF.**
(1) $\Rightarrow$ (2): Given $a, b \in A$, since $(P, I, J)$ is a $\phi$-FPS on $A$ we have

$$\phi(P(a,b)) + \phi(I(a,b)) + \phi(P(b,a)) \leq 1.$$

Then, by Theorem 22

$$(P \cup I \cup P^{-1})(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - \phi(I(a,b))).$$

On the other hand, $(R \cup R^{-1})(a,b) = \phi^{-1}(\min(\phi(R(a,b)) + \phi(R(b,a)), 1))$. Since $P \cup I \cup P^{-1} = R \cup R^{-1}$, we distinguish two cases:

(i) If $\phi(R(a,b)) + \phi(R(b,a)) < 1$, then $I(a,b) = 0$.
(ii) If $\phi(R(a,b)) + \phi(R(b,a)) \geq 1$, then

$$I(a,b) = \phi^{-1}(\phi(R(a,b)) + \phi(R(b,a)) - 1).$$

Therefore, $I(a,b) = \phi^{-1}(\max(\phi(R(a,b)) + \phi(R(b,a)) - 1, 0))$, i.e. $I = R \cap R^{-1}$. By Theorem 28 we obtain the result.

(2) $\Rightarrow$ (1): In the proof of Theorem 28 we have proven that $(P, I, J)$ is a $\phi$-FPS on $A$ associated to $R$. Since $R = P \cup I$, then for all $a, b \in A$ we have

$$\begin{aligned}(P \cup I \cup P^{-1})(a,b) &= \phi^{-1}(\phi(R(a,b)) + \phi(P(b,a))) \\ &= \phi^{-1}(\phi(R(a,b)) + \min(\phi(R(b,a)), 1 - \phi(R(a,b)))) \\ &= \phi^{-1}(\min(\phi(R(a,b)) + \phi(R(b,a)), 1)) \\ &= (R \cup R^{-1})(a,b). \qquad \square\end{aligned}$$

## 5 Conclusion

The reconstruction of a fuzzy set from its subsets by means of Łukasiewicz triplets gives a positive response to a problem which is similar to the formulated by Alsina [1]. Moreover, this result allows to justify the use of Łukasiewicz triplets and it also enables to consider a new definition of the strict preference relation.

On the other hand, in this paper the construction of fuzzy preference structures is accomplished by means of fuzzy partitions. This procedure allows to obtain the minimal formulation for fuzzy preference structures and it is more straightforward than the use of completeness conditions. Furthermore, we show a representation of all fuzzy preference structures. From this result we obtain a interesting geometrical relationship between the relations $R$ and $P$. Finally, we characterize a well-known fuzzy preference structure by means of some properties which are also satisfied in the crisp theory and, consequently, it seems that this fuzzy preference structure is the best-suited.

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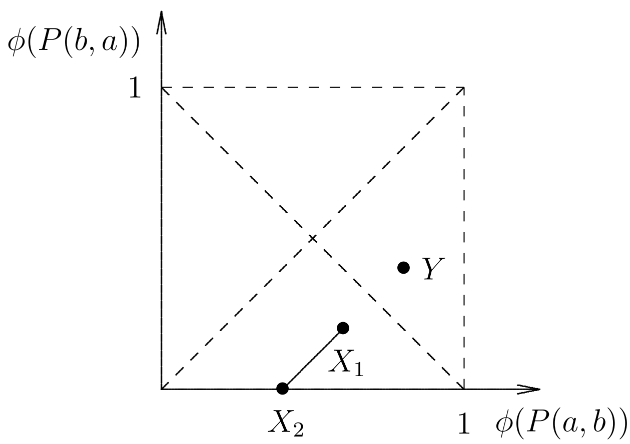


Figure 1

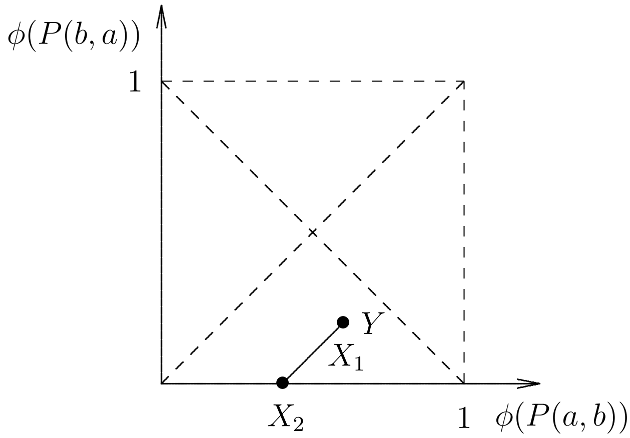


Figure 2