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Research Article

An Analytic Study of the Reversal of Hartmann Flows by Rotating Magnetic Fields

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The effects of a background uniform rotating magnetic field acting in a conducting fluid with a parallel flow are studied analytically. The stationary version with a transversal magnetic field is well known as generating Hartmann boundary layers. The Lorentz force includes now one term depending on the rotation speed and the distance to the boundary wall. As one intuitively expects, the rotation of magnetic field lines pushes backwards or forwards the flow. One consequence is that near the wall the flow will eventually reverse its direction, provided the rate of rotation and/or the magnetic field are large enough. The configuration could also describe a fixed magnetic field and a rotating flow.

1. Introduction

In [1], a generalization of the Hartmann flow was announced. The Hartmann flow is a parallel one influenced by a strong transversal magnetic field [2, 3] and has been studied intensively for its role in many aspects of plasma physics, such as liquid metal pumping [4, 5], plasma convection [6, 7], and geophysics [8]. We now will allow the background magnetic field to rotate while keeping its spatial uniformity. Our aim in this paper is to analyze in depth the action of this field upon the flow through the Lorentz force, emphasizing the possibility of flow reversal at some time near the walls confining the fluid. Given the Galilean invariance of the relevant equations, we could change to a rotating reference frame and interpret the problem as consisting in a fixed magnetic field and a rotating flow; this configuration could perhaps be more easily constructed and have wider applications.

Let us begin by recalling briefly the MHD equations appropriate for our configuration. The flow will be assumed two dimensional, and the background magnetic field will be uniform in space but rotating in time at a rate given by the angle $\theta(t)$:

$$\mathbf{B}(t) = B(\cos\theta(t), \sin\theta(t)). \tag{1.1}$$

Although this field will create a secondary one $\mathbf{b}$, it is assumed that all the terms in $\mathbf{b}$ may be neglected when an analogous one for $\mathbf{B}$ is present; thus, $|\mathbf{u}\times\mathbf{b}| \ll |\mathbf{u}\times\mathbf{B}|$, $|\dot{\mathbf{b}}| \ll |\dot{\mathbf{B}}|$. This does not hold for the current density, as $\mathbf{J} = \nabla\times\mathbf{B} = \mathbf{0}$, $\mathbf{j} = \nabla\times\mathbf{b} \neq \mathbf{0}$. The approximate induction equation becomes

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(-\eta\mathbf{j} + \mathbf{u}\times\mathbf{B}\right), \tag{1.2}$$

where $\mathbf{u}$ is the flow velocity and η the resistivity. By substituting $\mathbf{B}$ by its value in (1.1), uncurling (1.2) to find $\mathbf{j}$ and taking it to the Lorents force $\mathbf{j}\times\mathbf{B}$, we may write the momentum equation as

$$\frac{\partial u}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = \nu\Delta\mathbf{u} + \sigma(\mathbf{u}\times\mathbf{B})\times\mathbf{B} - \sigma B^2\dot{\theta}(y,0) + \nabla f, \tag{1.3}$$

where $\sigma = \eta^{-1}$ is the conductivity, ν the viscosity, and f is the sum of the pressure and the gauge obtained by uncurling (1.2). Detailed calculations may be found in [1]. If in analogy with the classical Hartmann flow we assume a horizontal flow $\mathbf{u}(x,y) = (u(y),0)$, (1.3) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} - \sigma B^2 u\sin^2\theta - \sigma B^2\dot{\theta}y + C, \tag{1.4}$$

where C includes all the conservative forces upon the fluid, such as the pressure and a possible electrostatic field. The interpretation is rather easy; as the magnetic field lines rotate at angular velocity θ, their linear velocity grows linearly with the radius y. This has the undesirable effect of providing infinite energy to the flow, but this should not deter us from studying the model. Other classical flows, such as Karman's, are the object of a vast literature in relation with swirling flows [9, 10] and also possess infinite energy. Their local properties are generalizable to many realistic situations.

We assume that the fluid is bounded either by one no-slip wall or two. In the first case, initial and boundary conditions for (1.4) are

$$\begin{aligned} u(0,t) &= 0, \\ u(y,0) &= \phi(y), \end{aligned} \tag{1.5}$$

whereas for the second one, if the walls are separated by a distance L,

$$u(0,t) = 0,$$
$$u(L,t) = 0, \tag{1.6}$$
$$u(y,0) = \phi(y).$$

Obviously ϕ must satisfy at the walls the same conditions as u.

2. Flow with a Single Wall

We will solve (1.4), (1.5) for $t \geq 0$, $y \geq 0$. To simplify calculations and highlight the contributions of the different terms, we will consider separately the following three problems:

$$\begin{gathered}
\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2} - \sigma B^2 u_1 \sin^2\theta + C, \\
u_1(0,t) = 0, \\
u_1(y,0) = 0, \\
\frac{\partial u_2}{\partial t} = \nu \frac{\partial^2 u_2}{\partial y^2} - \sigma B^2 u_2 \sin^2\theta - \sigma B^2 \dot{\theta} y \\
u_2(0,t) = 0, \\
u_2(y,0) = 0, \\
\frac{\partial u_3}{\partial t} = \nu \frac{\partial^2 u_3}{\partial y^2} - \sigma B^2 u_3 \sin^2\theta, \\
u_3(0,t) = 0, \\
u_3(y,0) = \phi(y).
\end{gathered} \tag{2.1}$$

Then, $u = u_1 + u_2 + u_3$ is the desired solution. From now on, we will denote

$$\begin{gathered}
F(t) = \sigma B^2 \int_0^t \sin^2\theta(s)\,ds, \\
w_j(y,t) = e^{F(t)} u_j(y,t).
\end{gathered} \tag{2.2}$$

Notice that w_j satisfies the same equation as u_j, except that the term in $\sigma B^2 u_j \sin^2\theta$ disappears, and the forcing is multiplied by $e^{F(t)}$. Although the general analysis is valid for any function $\theta(t)$, in order to obtain explicit results we will consider a constant rotation $\theta(t) = \lambda t$; the rotation is counterclockwise if $\lambda > 0$, clockwise otherwise. Then,

$$F(t) = \sigma B^2 \left(\frac{t}{2} - \frac{\sin 2\lambda t}{4} \right), \tag{2.3}$$

so that for large t

$$e^{F(t)} \sim e^{\sigma B^2 t/2},$$
$$e^{-F(t)} \sim e^{-\sigma B^2 t/2}. \tag{2.4}$$

2.1. Effect of the Constant Forcing

Let erf denote the error function,

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{2.5}$$

Then the solution of

$$\begin{gathered} \frac{\partial w_1}{\partial t} = \nu \frac{\partial^2 w_1}{\partial y^2} + Ce^{F(t)} \\ w_1(0,t) = 0 \\ w_1(y,0) = 0 \end{gathered} \tag{2.6}$$

is

$$w_1(y,t) = \int_0^t \operatorname{erf}\left(\frac{y}{\sqrt{4\nu(t-s)}}\right) Ce^{F(s)} ds, \tag{2.7}$$

so that

$$u_1(y,t) = \int_0^t \operatorname{erf}\left(\frac{y}{\sqrt{4\nu(t-s)}}\right) Ce^{-F(t)+F(s)} ds. \tag{2.8}$$

This implies

$$\frac{\partial u_1}{\partial y} = \frac{2C}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{4\nu(t-s)}} \exp\left(-\frac{y^2}{4\nu(t-s)} - F(t) + F(s)\right) ds. \tag{2.9}$$

Let us study the limit of this function when $t \to \infty$ in the case of constant rotation, $\theta(t) = \lambda t$, $\lambda > 0$. It is an easy consequence of Lebesgue's theorem that, for any fixed $t_0 > 0$,

$$\lim_{t\to\infty} \int_0^{t_0} \frac{1}{\sqrt{4\nu(t-s)}} \exp\left(-\frac{y^2}{4\nu(t-s)} - F(t) + F(s)\right) ds = 0. \tag{2.10}$$

Take t_0 large enough that we may substitute $F(t)$ by $\sigma B^2 t/2$. It is enough to consider

$$\begin{aligned}
&\frac{2C}{\sqrt{\pi}}\int_{t_0}^{t}\frac{1}{\sqrt{4\nu(t-s)}}\exp\left(-\frac{y^2}{4\nu(t-s)}-\frac{\sigma B^2(t-s)}{2}\right)ds\\
&\quad=\frac{2C}{\sqrt{\pi}}\int_{0}^{t-t_0}\frac{1}{\sqrt{4\nu s}}\exp\left(-\frac{y^2}{4\nu s}-\frac{\sigma B^2 s}{2}\right)ds,
\end{aligned}\tag{2.11}$$

whose limit when $t \to \infty$ is

$$\frac{2C}{\sqrt{\pi}}\int_{0}^{\infty}\frac{1}{\sqrt{4\nu s}}\exp\left(-\frac{y^2}{4\nu s}-\frac{\sigma B^2 s}{2}\right)ds,\tag{2.12}$$

which is obviously a finite integral. It reaches its maximum at $y = 0$, where its value is

$$\begin{aligned}
\frac{C}{\sqrt{\pi\nu}}\int_0^{\infty}s^{-1/2}e^{-\sigma B^2 s/2}\,ds&=\frac{C}{\sqrt{\pi\nu}}\sqrt{\frac{2}{\sigma B^2}}\int_0^{\infty}r^{-1/2}e^{-r}\,dr\\
&=\frac{C}{\sqrt{\pi\nu}}\sqrt{\frac{2}{\sigma B^2}}\Gamma\left(\frac{1}{2}\right)=\frac{C}{\sqrt{\pi\nu}}\sqrt{\frac{2}{\sigma B^2}}\sqrt{\pi}=C\sqrt{\frac{2}{\nu\sigma B^2}}=\frac{C\sqrt{2}}{B\sqrt{P_m}},
\end{aligned}\tag{2.13}$$

where P_m is the magnetic Prandtl number. Therefore, u_1 also has a limit when $t \to \infty$ for every $y > 0$, given by the integral of (2.12) with respect to y.

2.2. Effect of the Lorentz Force

w_2 satisfies

$$\begin{aligned}
\frac{\partial w_2}{\partial t}&=\nu\frac{\partial^2 w_2}{\partial y^2}-\sigma B^2\dot{\theta}y,\\
w_2(0,t)&=0,\\
w_2(y,0)&=0.
\end{aligned}\tag{2.14}$$

Although there exists an appropriate Green function for this problem, in this particular case it is simpler to assume that the solution is linear in y (which is correct). Setting first

$$w_2(y,t)=h(t)y,\quad h(0)=0,\tag{2.15}$$

and recovering u_2 from it, one finds

$$u_2(y,t)=-y\sigma B^2\int_0^t\exp(-F(t)+F(s))\dot{\theta}(s)ds.\tag{2.16}$$

Therefore,

$$\frac{\partial u_2}{\partial y}(y,t) = -\sigma B^2 \int_0^t \exp(-F(t) + F(s))\dot{\theta}(s)ds. \tag{2.17}$$

This function also has a limit when $t \to \infty$. Its value may be found by an argument similar to the previous one, although simpler; in this case

$$\lim_{t\to\infty} - \lambda\sigma B^2 \int_{t_0}^t \exp\left(-\frac{\sigma B^2(t-s)}{2}\right)ds = -2\lambda. \tag{2.18}$$

Hence, $u_2(y,t)$ tends to $-2\lambda y$ when $t \to \infty$. Notice how the Lorentz force pushes the fluid backwards, the more rapidly the higher the distance from the wall.

2.3. *Effect of the Initial Condition*

w_3 satisfies

$$\begin{gathered} \frac{\partial w_3}{\partial t} = \frac{\partial^2 w_3}{\partial y^2} \\ w_3(0,t) = 0 \\ w_3(y,0) = \phi(y). \end{gathered} \tag{2.19}$$

ϕ must satisfy $\phi(0) = 0$. The solution is found by the method of images: ϕ is extended to $(-\infty,\infty)$ as an odd function. Then,

$$w_3(y,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-z)^2}{4\nu t}\right)\phi(z)\ dz, \tag{2.20}$$

so that

$$u_3(y,t) = \frac{e^{-F(t)}}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-z)^2}{4\nu t}\right)\phi(z)\ dz. \tag{2.21}$$

Therefore, provided ϕ is differentiable,

$$\frac{\partial u_3}{\partial y}(y,t) = \frac{e^{-F(t)}}{\sqrt{4\pi\nu t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-z)^2}{4\nu t}\right)\phi'(z)\ dz. \tag{2.22}$$

All this supposes that ϕ and ϕ' satisfy certain reasonable conditions: for this example. if both are bounded. In that case, given that

$$\frac{1}{\sqrt{4\pi\nu t}}\int_{-\infty}^{\infty}\exp\left(-\frac{(y-z)^2}{4\nu t}\right)dz = 1, \tag{2.23}$$

both u_3 and $\partial u_3/\partial y$ are bounded by a function of the form $M\exp(-F(t))$, which means that they tend to zero as $t \to \infty$.

Thus, the limit when $t \to \infty$ of $u(0,t)$ is

$$k = \frac{C\sqrt{2}}{B\sqrt{P_m}} - 2\lambda. \tag{2.24}$$

If $k < 0$, we may be certain that the fluid inverts its direction and flows backwards from some point on. The factors contributing to this are a small (or negative) potential force C, which includes pressure and electric fields; a large background magnetic field; a large magnetic Prandtl number, which means that the viscosity dominates the resistivity; finally and most obviously, a large rate of rotation of the magnetic field.

3. Flow between Two Walls

When the fluid is stationary along two walls, the Lorentz force does not have so clear an effect as in the previous case, as it is stopped by the upper no-slip wall. We may again decompose the solution in the three components given by (2.1) but with the boundary and initial conditions of (1.6). The solution to these problems may be found again by the method of images, but it is equivalent and somewhat simpler to postulate a Fourier series expression of the form

$$\sum_{k=1}^{\infty} c_k(t)\sin\frac{\pi k}{L}y, \tag{3.1}$$

and find the coefficients c_k.

3.1. Effect of the Constant Forcing

We have

$$\begin{gathered}\frac{\partial w_1}{\partial t} = \nu\frac{\partial^2 w_1}{\partial y^2} + Ce^{F(t)},\\ w_1(0,t) = w_1(L,t) = 0,\\ w_1(y,0) = 0.\end{gathered} \tag{3.2}$$

First we extend the forcing $Ce^{F(t)}$ as an odd function of y in the interval $[-L, L]$. Its Fourier series is

$$Ce^{F(t)} \sum_{k=0}^{\infty} \frac{4}{\pi(2k+1)} \sin \frac{\pi(2k+1)}{L} y, \tag{3.3}$$

so that if we set

$$w_1(t) = \sum_{n=1}^{\infty} \widehat{w}_1(n,t) \sin \frac{\pi n}{L} y, \tag{3.4}$$

we have

$$\begin{gathered}
\frac{\partial \widehat{w}_1}{\partial t}(2k,t) = -\nu \frac{\pi^2 (2k)^2}{L^2} \widehat{w}_1(2k,t), \\
\widehat{w}_1(2k,0) = 0, \\
\frac{\partial \widehat{w}_1}{\partial t}(2k+1,t) = -\nu \frac{\pi^2 (2k+1)^2}{L^2} \widehat{w}_1(2k+1,t) + \frac{4}{\pi(2k+1)} Ce^{F(t)}, \\
\widehat{w}_1(2k+1,0) = 0.
\end{gathered} \tag{3.5}$$

Hence, all the even terms $\widehat{w}_1(2k,t)$ vanish and

$$\begin{aligned}
\widehat{w}_1(2k+1,t) = {} & \frac{4}{\pi(2k+1)} \exp\left(-\frac{\nu\pi(2k+1)^2 t}{L^2} \right) \\
& \times \left(\int_0^t \exp\left(\frac{\nu\pi(2k+1)^2 s}{L^2} \right) C \exp(F(s)) ds \right).
\end{aligned} \tag{3.6}$$

If we define

$$F_m(t) = \sigma B^2 \int_0^t \sin^2\theta(s) ds + \frac{\nu\pi^2 m^2 t}{L^2}, \tag{3.7}$$

we obtain

$$\widehat{u}_1(2k+1,t) = \frac{4C}{\pi(2k+1)} \exp(-F_{2k+1}(t)) \int_0^t \exp(F_{2k+1}(s)) ds. \tag{3.8}$$

Thus,

$$\begin{aligned} u_1(y,t) &= \sum_{k=0}^{\infty}\frac{4C}{\pi(2k+1)}\left(\exp(-F_{2k+1}(t))\int_0^t \exp(F_{2k+1}(s))ds\right) \\ &\quad\times \sin\frac{\pi(2k+1)}{L}y, \\ \frac{\partial u_1}{\partial t}(y,t) &= \sum_{k=0}^{\infty}\frac{4C}{L}\left(\exp(-F_{2k+1}(t))\int_0^t \exp(F_{2k+1}(s))\ ds\right) \\ &\quad\times \cos\frac{\pi(2k+1)}{L}y. \end{aligned} \tag{3.9}$$

In particular

$$\frac{\partial u_1}{\partial t}(0,t) = \sum_{k=0}^{\infty}\frac{4C}{L}\left(\exp(-F_{2k+1}(t))\int_0^t \exp(F_{2k+1}(s))ds\right), \tag{3.10}$$

$$\frac{\partial u_1}{\partial t}(L,t) = -\sum_{k=0}^{\infty}\frac{4C}{L}\left(\exp(-F_{2k+1}(t))\int_0^t \exp(F_{2k+1}(s))ds\right). \tag{3.11}$$

To evaluate the limit of these expressions when $t \to \infty$, let us assume as before that $\theta(t) = \lambda t$, $\lambda > 0$. Analogously to (2.4), we have now

$$\begin{aligned} F_k(t) &\sim \left(\frac{\sigma B^2}{2} + \frac{\nu\pi^2 k^2}{L^2}\right)t, \\ -F_k(t) &\sim -\left(\frac{\sigma B^2}{2} + \frac{\nu\pi^2 k^2}{L^2}\right)t, \end{aligned} \tag{3.12}$$

for t large. Thus, for t_0 fixed,

$$\lim_{t\to\infty}\exp(-F_k(t))\int_0^{t_0}\exp(F_k(s))ds = 0. \tag{3.13}$$

By using an argument similar to the one in (2.10) and glossing over the technicalities concerning the infinite terms in the sums of (3.10) and (3.11) (which hold anyway), we obtain that if we denote

$$\alpha_k = \lim_{t\to\infty}\exp(-F_k(t))\int_0^t \exp(F_k(s))ds, \tag{3.14}$$

then

$$\alpha_k = \left(\frac{\sigma B^2}{2} + \frac{\pi^2 \nu k^2}{L^2}\right)^{-1}; \tag{3.15}$$

moreover

$$\frac{\partial u_1}{\partial t}(0, t) = \frac{4C}{L}\sum_{k=0}^{\infty}\alpha_{2k+1}, \tag{3.16}$$

$$\frac{\partial u_1}{\partial t}(L, t) = -\frac{4C}{L}\sum_{k=0}^{\infty}\alpha_{2k+1}. \tag{3.17}$$

To find these sums, we denote

$$\gamma = \frac{BL}{\pi}\sqrt{\frac{\sigma}{2\nu}}. \tag{3.18}$$

Then,

$$\alpha_k = \frac{2}{\sigma B^2}\frac{\gamma^2}{\gamma^2 + k^2}. \tag{3.19}$$

It may be found by elementary means (e.g, the residue theorem) that

$$\sum_{k=1}^{\infty}\frac{\gamma^2}{\gamma^2 + k^2} = -\frac{1}{2} + \frac{\pi\gamma}{2}\operatorname{cotanh}\pi\gamma, \tag{3.20}$$

Therefore

$$\begin{aligned}\sum_{k=1}^{\infty}\alpha_{2k} &= \frac{2}{\sigma B^2}\left(\frac{\gamma}{2}\right)^2\sum_{k=1}^{\infty}\frac{1}{(\gamma/2)^2 + k^2}\\ &= \frac{1}{\sigma B^2}\left(-1 + \frac{\pi\gamma}{2}\operatorname{cotanh}\frac{\pi\gamma}{2}\right),\\ \sum_{k=0}^{\infty}\alpha_{2k+1} &= \frac{1}{\sigma B^2}\left(\pi\gamma\operatorname{cotanh}\pi\gamma - \frac{\pi\gamma}{2}\operatorname{cotanh}\frac{\pi\gamma}{2}\right).\end{aligned} \tag{3.21}$$

This expression may be written in another form, which is more convenient if we take L and ν as fixed and consider that the variation of γ comes from σB^2. This is

$$\sum_{k=0}^{\infty}\alpha_{2k+1} = \frac{L^2}{2\pi\nu\gamma}\left(\operatorname{cotanh}\pi\gamma - \frac{1}{2}\operatorname{cotanh}\frac{\pi\gamma}{2}\right). \tag{3.22}$$

Given the behavior of the hyperbolic cotangent, it is easy to see that this sum tends to ∞ when $\gamma \to 0$ and to 0 when $\gamma \to \infty$. Combining this with (3.16) and (3.17), we obtain

$$\lim_{t\to\infty} \frac{\partial u_1}{\partial t}(0,t) = \frac{2CL}{\pi\nu\gamma}\left(\operatorname{cotanh} \pi\gamma - \frac{1}{2}\operatorname{cotanh} \frac{\pi\gamma}{2}\right), \tag{3.23}$$

$$\lim_{t\to\infty} \frac{\partial u_1}{\partial t}(L,t) = -\frac{2CL}{\pi\nu\gamma}\left(\operatorname{cotanh} \pi\gamma - \frac{1}{2}\operatorname{cotanh} \frac{\pi\gamma}{2}\right). \tag{3.24}$$

3.2. *Effect of the Lorentz Force*

The Fourier series of the function $y \to -\sigma B^2\dot{\theta}y$ is

$$\sigma B^2\dot{\theta}\sum_{k=1}^{\infty} 2L\frac{(-1)^k}{\pi k}\sin\frac{\pi k}{L}y. \tag{3.25}$$

Hence, $\widehat{w}_2$ satisfies

$$\begin{gathered}\frac{\partial \widehat{w}_2}{\partial t}(k,t) = -\frac{\pi^2\nu k^2}{L^2}\widehat{w}_2(k,t) + \frac{2L(-1)^k}{\pi k}\sigma B^2\dot{\theta}(t)\exp(F(t)) \\ \widehat{w}_2(k,0) = 0,\end{gathered} \tag{3.26}$$

whose solution is

$$\widehat{w}_2(k,t) = \frac{2L(-1)^k}{\pi k}\sigma B^2\int_0^t \exp(F_k(s))\dot{\theta}(s)ds. \tag{3.27}$$

Therefore,

$$\begin{gathered}\widehat{u}_2(k,t) = \frac{2L(-1)^k}{\pi k}\sigma B^2\int_0^t \exp(-F_k(t)+F_k(s))\dot{\theta}(s)ds, \\ u_2(y,t) = \sum_{k=1}^{\infty}\frac{2L(-1)^k}{\pi k}\sigma B^2\left(\int_0^t \exp(-F_k(t)+F_k(s))\dot{\theta}(s)\ ds\right)\sin\frac{\pi k}{L}y,\end{gathered} \tag{3.28}$$

which implies

$$\frac{\partial u_2}{\partial t}(y,t) = 2\sum_{k=1}^{\infty}(-1)^k\sigma B^2\left(\int_0^t \exp(-F_k(t)+F_k(s))\dot{\theta}(s)\ ds\right)\cos\frac{\pi k}{L}y. \tag{3.29}$$

In particular

$$\frac{\partial u_2}{\partial t}(0,t) = 2\sum_{k=1}^{\infty}(-1)^k \sigma B^2 \int_0^t \exp(-F_k(t) + F_k(s))\dot{\theta}(s)ds,$$
$$\frac{\partial u_2}{\partial t}(L,t) = 2\sum_{k=1}^{\infty}\sigma B^2 \int_0^t \exp(-F_k(t) + F_k(s))\dot{\theta}(s)ds. \tag{3.30}$$

To find the limit of this expression when $t \to \infty$ and $\theta(t) = \lambda t$, we use the same argument as in the previous paragraph. We find

$$\lim_{t\to\infty}\frac{\partial u_2}{\partial t}(0,t) = 2\sigma B^2 \lambda \sum_{k=1}^{\infty}(-1)^k \alpha_k,$$
$$\lim_{t\to\infty}\frac{\partial u_2}{\partial t}(L,t) = 2\sigma B^2 \lambda \sum_{k=1}^{\infty}\alpha_k. \tag{3.31}$$

In terms of γ,

$$\lim_{t\to\infty}\frac{\partial u_2}{\partial t}(0,t) = 4\lambda \sum_{k=1}^{\infty}(-1)^k \frac{\gamma^2}{\gamma^2 + k^2}, \tag{3.32}$$

$$\lim_{t\to\infty}\frac{\partial u_2}{\partial t}(L,t) = 4\lambda \sum_{k=1}^{\infty}\frac{\gamma^2}{\gamma^2 + k^2}. \tag{3.33}$$

We know already (3.20) the value of the sum in (3.33). The one in (3.32) sums

$$\sum_{k=1}^{\infty}(-1)^k \frac{\gamma^2}{\gamma^2 + k^2} = -\frac{1}{2} + \frac{\pi\gamma}{2}\operatorname{cosech}\pi\gamma. \tag{3.34}$$

This sum is a decreasing function of γ, tending to 0 when $\gamma \to 0$ and to $-1/2$ when $\gamma \to \infty$.

3.3. *Effect of the Initial Condition*

To deal with the problem

$$\frac{\partial w_3}{\partial t} = \nu \frac{\partial^2 w_3}{\partial y^2},$$
$$w_3(0,t) = w_3(L,t) = 0,$$
$$w_3(y,0) = \phi(y), \tag{3.35}$$

we extend ϕ to $[-L, L]$ as an odd function, with Fourier series

$$\phi(y) = \sum_{k=1}^{\infty} \widehat{\phi}(k) \sin \frac{\pi k}{L} y. \tag{3.36}$$

Then,

$$\begin{aligned} \frac{\partial \widehat{w}_3}{\partial t}(k,t) &= -\frac{\pi^2 \nu k^2}{L^2} \widehat{w}_3(k,t), \\ \widehat{w}_3(k,0) &= \widehat{\phi}(k), \end{aligned} \tag{3.37}$$

whose solution is

$$\widehat{w}_3(k,t) = \exp\left(-\frac{\pi^2 \nu k^2}{L^2} t\right) \widehat{\phi}(k). \tag{3.38}$$

Therefore,

$$\begin{aligned} u_3(y,t) &= \sum_{k=1}^{\infty} \exp(-F_k(t)) \widehat{\phi}(k) \sin \frac{\pi k}{L} y, \\ \frac{\partial u_3}{\partial y}(y,t) &= \sum_{k=1}^{\infty} \frac{\pi k}{L} \exp(-F_k(t)) \widehat{\phi}(k) \cos \frac{\pi k}{L} y. \end{aligned} \tag{3.39}$$

It is trivial to prove that both tend to zero when $t \to \infty$ for all y.

When

$$\frac{\partial \phi}{\partial y}(0) > 0, \quad \frac{\partial \phi}{\partial y}(L) < 0 \tag{3.40}$$

the fluid flows initially in the positive sense. If we show that under certain conditions

$$\lim_{t \to \infty} \frac{\partial u}{\partial t}(0,t) < 0, \tag{3.41}$$

or

$$\lim_{t \to \infty} \frac{\partial u}{\partial t}(L,t) > 0, \tag{3.42}$$

the fluid reverses flow at least in one of the walls. Since $u = u_1 + u_2 + u_3$ and we have found already these limits, we may combine first (3.23) and (3.34) to get

$$\lim_{t\to\infty}\frac{\partial u}{\partial t}(0,t) = \frac{2CL}{\pi\nu\gamma}\left(\operatorname{cotanh}\pi\gamma - \frac{1}{2}\operatorname{cotanh}\frac{\pi\gamma}{2}\right) + 4\lambda\left(-\frac{1}{2} + \frac{\pi\gamma}{2}\operatorname{cosech}\pi\gamma\right). \tag{3.43}$$

On the other hand, combining now (3.24), (3.20), and (3.33), we find

$$\lim_{t\to\infty}\frac{\partial u}{\partial t}(L,t) = -\frac{2CL}{\pi\nu\gamma}\left(\operatorname{cotanh}\pi\gamma - \frac{1}{2}\operatorname{cotanh}\frac{\pi\gamma}{2}\right) + 2\lambda\left(-1 + \pi\gamma\operatorname{cotanh}\pi\gamma\right). \tag{3.44}$$

The expression in (3.43) tends to $-2\lambda < 0$ when $\gamma \to \infty$. Thus, for any C and L, when λ and/or σB^2 are large enough, there exists a flow reversal at this wall.

The limit of (3.44) when $\gamma \to \infty$ is ∞. Since this becomes positive for σB^2 or λ large enough, the fluid will eventually flow backwards at $y = L$. Notice that given the respective size of the limits, this occurs much earlier than the flow reversal at $y = 0$. This was to be expected, as the Lorentz force pushes backwards the flow the more rapidly the larger y is; magnetic field lines move faster the higher the radius.

4. Conclusions

A generalization of the classical Hartmann flow is obtained when we pose a background uniform magnetic field rotating in time. Since the Lorentz force varies both in time and space, the resulting flow cannot be stationary. If we add one or two walls with the no-slip property, the interaction between the magnetic field and the boundary layers creates an interesting phenomenology. We study in particular the phenomenon of flow reversal near the walls for a magnetic field rotating at a constant rate; it is found that for large enough values of this rate, the conductivity of the fluid, or the size of the magnetic field, there exists always flow reversal at some instant. Forces opposing this include the kinetic pressure and a possible electrostatic field. This reversal makes excellent sense, as the Lorentz force (visualized through the rotation of magnetic field lines) pushes backwards the fluid if the sense of rotation of the field is contrary to the original flow.

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