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Almost symmetric numerical semigroups with high type

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Abstract: We establish a one-to-one correspondence between numerical semigroups of genus g and almost symmetric numerical semigroups with Frobenius number F and type $F-2g$, provided that F is greater than or equal to $4g-1$.

Key words: Numerical semigroup, almost symmetric numerical semigroup, Frobenius number, pseudo-Frobenius number, genus, type.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers. A numerical semigroup is a submonoid of $(\mathbb{N},+)$ with finite complement in $\mathbb{N}$. If S is a numerical semigroup, the set $\mathbb{N}\setminus S$ is known as the set of gaps of S. Its cardinality is the genus of S, denoted here by $\mathrm{g}(S)$. The multiplicity of a numerical semigroup, $\mathrm{m}(S)$, is the smallest positive integer not belonging to its gap set. The largest integer not belonging to S is the Frobenius number of S, denoted by $\mathrm{F}(S)$.

Associated to S we can define the following order relation: for $a,b\in\mathbb{Z}$, $a\leq_S b$ if $b-a\in S$. The set of maximal elements of $\mathbb{Z}\setminus S$ with respect to $\leq_S$ is the set of pseudo-Frobenius numbers of S, $\mathrm{PF}(S)$; its cardinality is the type of S, denoted by $\mathrm{t}(S)$.

A numerical semigroup S is irreducible if it cannot be written as the intersection of two numerical semigroups properly containing S. This is equivalent to say that S is maximal (with respect to set inclusion) in the set of numerical semigroups having Frobenius number equal to $\mathrm{F}(S)$. If $\mathrm{F}(S)$ is odd, this is equivalent to $\mathrm{g}(S)=(\mathrm{F}(S)+1)/2$ (S is symmetric); while if $\mathrm{F}(S)$ is even, this is the same as to impose $\mathrm{g}(S)=(\mathrm{F}(2)+2)/2$ (S is pseudosymmetric, see for instance [14, Chapter 3], or [1] for the history behind the names of the invariants defined above). For symmetric numerical semigroups the type is one (and this precisely characterizes them), and for pseudosymmetric numerical semigroups the type is two (though this does not characterize this property). Thus, for any irreducible numerical semigroup S, the equality $\mathrm{g}(S)=(\mathrm{F}(S)+\mathrm{t}(S))/2$ holds. Then converse is not true, but gives rise to a wider family of numerical semigroups: almost-symmetric numerical semigroups. A numerical semigroup is said to be almost symmetric provided that $\mathrm{g}(S)=(\mathrm{F}(S)+\mathrm{t}(S))/2$. It is well known that for any numerical semigroup $\mathrm{g}(S)\geq(\mathrm{F}(S)+\mathrm{t}(S))/2$ (see [13, Proposition 2.2]), so almost symmetric numerical semigroups are those attaining the equality. Indeed, as shown in [13] almost symmetric numerical semigroups have some symmetry properties.

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Almost symmetric numerical semigroups have attracted the attention of many researchers, not only because they generalize the irreducible property in numerical semigroups, but also because they rise in a natural way as a generalization of the Gorenstein property in one-dimensional rings (see [2]). Many papers deal with the almost symmetric property and how to construct examples of these semigroups (see for instance [3, 4, 11] and the references therein). Some manuscripts like [11, 12] and [15] deal with almost symmetric numerical semigroups with small type and small embedding dimension, which is the cardinality of a minimal generating set of the numerical semigroup. The semigroups considered in this manuscript have large type.

The original aim of this note was to clarify a computational evidence noticed by the second author when using the algorithms given in [4] and implemented in the `GAP` [10] package `NumericalSgps` [7] (see [4, Remark 5.1] for further details). To this end, we present a one-to-one correspondence with numerical semigroups with given genus g and almost symmetric numerical semigroups with Frobenius number F and type $F-2g$, for any $F \geq 4g+1$. This, in particular, provides an easy way to construct examples of almost symmetric numerical semigroups and also presents a new way to restate Bras' conjectures on the number of numerical semigroups with genus g [5]; this number is usually denoted by n_g (see Corollary 3.1 and the comment following it).

An important peculiarity of almost symmetric numerical semigroups with Frobenius number F and type $F-2g$, with $F \geq 4g+1$ (for some nonnegative integer g), is that these semigroups are uniquely determined by its sets of pseudo-Frobenius numbers (Corollary 3.2). This is in general far from being true [8], even for almost symmetric numerical semigroups, and it allows us to develop a new and faster algorithm for determining almost symmetric numerical semigroups with Frobenius number F and type $F-2g$, with $F \geq 4g+1$. Thus, our approach can be potentially used to go further in the calculation of unknown elements of the sequence n_g.

2. The correspondence

The definition of almost symmetric numerical semigroups can be stated as follows (see for instance [13]).

Definition 2.1 *A numerical semigroup S is almost symmetric if for any integer a not in S, then $\mathrm{F}(S)-a \in S \setminus \{0\}$ or $a \in \mathrm{PF}(S)$.*

We will use the fact that the above definition is equivalent to

$$\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{t}(S)}{2}. \tag{2.1}$$

Motivated by the notion of set of gaps of a numerical semigroup, in [9] the concept of gapset is introduced.

Definition 2.2 *A **gapset** is a finite subset $G \subset \mathbb{N} \setminus \{0\}$ such that given a and $b \in \mathbb{N} \setminus \{0\}$ with $a+b \in G$, then either $a \in G$ or $b \in G$.*

Notice that if G is a gapset, then $\mathbb{N} \setminus G$ is a numerical semigroup. We are going to give families of gapsets that "produce" almost symmetric numerical semigroups.

Proposition 2.3 *Let S be a numerical semigroup with genus g and let F be a positive integer greater than $2\,\mathrm{F}(S)$. The set*

$$G = \{1, \dots, F\} \setminus \{F-a \mid a \in \mathbb{N} \setminus S\}$$

is the gapset of an almost symmetric numerical semigroup with Frobenius F, type $F-2g$ and multiplcity $F-\mathrm{F}(S)$. Moreover,

$$\begin{aligned}\mathrm{PF}(\mathbb{N}\setminus G)=&\{a\in S \mid 0<a\le \mathrm{F}(S)\}\\ &\cup\ \{\mathrm{F}(S)+1,\dots,F-\mathrm{F}(S)-1\}\\ &\cup\ \big\{F-a \ \mid\ a\in S\cap\{0,\dots,\mathrm{F}(S)\}\big\}.\end{aligned}$$

Proof Set $f=\mathrm{F}(S)$. First of all, we observe that

$$G=\{1,\dots,F-f-1\}\cup\big\{F-a \ \mid\ a\in S\cap\{0,\dots,f\}\big\}.$$

Let us see that G is actually a gapset. To do that we consider a and $b\in\mathbb{N}\setminus\{0\}$ with $a+b\in G$; in particular, $a+b\le F$. Now, if $a\le F-f-1$ or $b\le F-f-1$, then we are done because this would imply a or b in G. Thus, let us assume that $a\ge F-f$ and $b\ge F-f$; in this case, $a+b\ge 2F-2f=F+(F-2f)$ and, since $(F-2f)>0$, we conclude that $a+b>F$, which is incompatible with the condition $a+b\in G$.

Let S' be the numerical semigroup $\mathbb{N}\setminus G$; notice that $\mathrm{F}(S')=F$. Given $a\in G$, if $F-a\in S'$, then $F=a+(F-a)\notin S'$, that is to say, $a\notin\mathrm{PF}(S')$. Thus, to see that S' is almost symmetric, we need to prove that given $a\in G$ with $F-a\in S$, then $a\in\mathrm{PF}(S')$. Thus, let $a\in G$ such that $F-a\in G$. First, we claim that $a\in S$; indeed, if $F-a\le F-f-1$, then $a\ge f+1$; therefore, $a\in S$. Otherwise, $F-a\in\{F-f,\dots,F\}\cap\{F-a \ \mid\ a\in S\}$ and, clearly, $a\in S$. Now, let $b\in S'\setminus\{0\}$ and let us prove that $a+b\in S'$. If $a+b>F$, there is nothing to prove. Otherwise, $a+b=F-c$, for some $c\in\mathbb{N}$. If $a+b\notin S'$, then $F-c\in G$. Arguing as above, we can prove that $c\in S$. Thus, since $a+c\in S$, we conclude that $b=F-(a+c)\in G$, in contradiction with $b\in S'$.

Thus, we have that $S'=\mathbb{N}\setminus G$ is an almost symmetric numerical semigroup. Moreover, since $\mathrm{g}(S')=\#G=F-g$ and S' is almost symmetric, we have that $\mathrm{t}(S')=2\,\mathrm{g}(S')-F=F-2g$. We also observe that the smallest positive integer not in G is $F-f$, so the multiplicity of S' is $F-f$.

Finally, let us see that $\mathrm{PF}(S')=\{a\in S\mid 0<a\le f\}\cup\{f+1,\dots,F-f-1\}\cup\big\{F-a \ \mid\ a\in S\cap\{0,\dots,f\}\big\}$. First, we notice that the right-hand side has cardinality $(f-g)+(F-2f-1)+(f+1-g)=F-2g$. Thus, it suffices to see that all the elements in the right-hand side are in $\mathrm{PF}(S')$. Notice that for every a in the right-hand side, $a\notin S'$, and also $F-a\notin S'$; thus, $a\in\mathrm{PF}(S')$ by the definition of almost symmetric numerical semigroup. □

Definition 2.4 *Let S be a numerical semigroup. A relative ideal I of S is a subset of $\mathbb{Z}$ such that*

1. *$I+S\subseteq I$;*
2. *$a+I\subseteq S$, for some $a\in S$.*

If S is a numerical semigroup and $s\in\mathbb{Z}$, then

$$K_S(s):=\{\mathrm{F}(S)+s-z\mid z\in\mathbb{Z}\setminus S\}$$

is a relative ideal of S. This ideal is the $s-$shifted canonical ideal of S.

Lemma 2.5 *If S is a numerical semigroup and F is a positive integer greater than* $\mathrm{F}(S)$*, then* $\mathbb{N} \setminus K_S(F - \mathrm{F}(S)) = \{0, 1, \ldots, F\} \setminus \{F - a \mid a \in \mathbb{N} \setminus S\}$.

Proof Since $K_S(F - \mathrm{F}(S)) = \{F - z \mid z \in \mathbb{Z} \setminus S\} = \{F - a \mid a \in \mathbb{N} \setminus S\} \cup \{F + 1, \ldots\}$, then $\mathbb{N} \setminus K_S(F - \mathrm{F}(S)) = \{0, 1, \ldots, F\} \setminus \{F - a \mid a \in \mathbb{N} \setminus S\}$. □

Theorem 2.6 *Let F and g be positive integers such that $F \geq 4g - 1$. The correspondence*

$$S \mapsto \{0\} \cup K_S(F - \mathrm{F}(S))$$

is a bijection between the set of numerical semigroups with genus g and the set of almost symmetric numerical semigroups with Frobenius number F and type $F - 2g$.

Proof Let S be a numerical semigroup with genus g. By Lemma 2.5, $\mathbb{N} \setminus K_S(F - \mathrm{F}(S)) = \{0, 1, \ldots, F\} \setminus \{F - a \mid a \in \mathbb{N} \setminus S\}$. Moreover, since $\mathrm{F}(S) \leq 2g - 1$ (see, for instance [14, Lemma 2.14]), then $F > 4g - 1 > 2\,\mathrm{F}(S)$. Thus, by Proposition 2.3, our correspondence is a well-defined application between the set of numerical semigroups with genus g and the set of almost symmetric numerical semigroups with Frobenius number F and type $F - 2g$. Moreover, since $K_{S_1}(F - \mathrm{F}(S_1)) = K_{S_2}(F - \mathrm{F}(S_2))$ if and only if $S_1 = S_2$, we have that our application is clearly injective.

In order to see that it is surjective, let us consider an almost symmetric numerical semigroup S' with Frobenius number F and type $F - 2g$, and set $G' := \{x \in \mathbb{N} \setminus S' \mid F - x \in S'\}$. We prove that G' is a gapset. Take a and $b \in \mathbb{N} \setminus \{0\}$ such that $a + b \in G'$, that is, $a + b \notin S'$ and $F - (a + b) \in S'$, and let us prove that $a \in G'$ or $b \in G'$. To this end, we distinguish two cases:

- If $a \in S'$ or $b \in S'$. If $a \in S'$, then $F - b = (F - (a + b)) + a \in S'$ and $b \notin S'$; otherwise $F = (F - b) + b \in S'$. Thus, $b \in G'$. By arguing analogously, if $b \in S'$, we obtain $a \in G'$.
- If a and $b \notin S'$; in particular, we have that either $F - b \in S'$ or $b \in \mathrm{PF}(S')$, because S' is almost symmetric. In the first case, $b \in G'$ and, in the second case, $F - a = b + (F - (a + b)) \in S$, implies that $a \in G'$.

Thus, we have that $T = \mathbb{N} \setminus G'$ is a numerical semigroup. Since S' is almost symmetric, by (2.1), $\mathrm{g}(S') = (F + F - 2g)/2 = F - g$, and from the definition of almost symmetric numerical semigroup, we have that $\mathrm{g}(T) = \#G' = \mathrm{g}(S') - \mathrm{t}(S') = (F - g) - (F - 2g) = g$. Moreover,

$$\{0\} \cup K_T(F - \mathrm{F}(T)) = \{F - a \mid a \in G'\} \cup \{F + 1, \ldots\}.$$

From the definition of G', $K_T(F - \mathrm{F}(T)) \subseteq S'$. For the other inclusion, take $s \in S'$, with $s < F$. Then $F - s \in G'$, and $s = F - (F - s) \in K_T(F - \mathrm{F}(T))$. □

Observe that we can use Theorem 2.6 to produce almost symmetric numerical semigroups with high type.

Example 2.7 *The numerical semigroup S generated by $21, 24, 25$, and 31 has genus $\mathrm{g}(S) = 55$ and Frobenius number* $\mathrm{F(S)} = 89$*. Let $F = 4\,\mathrm{g}(S) - 1 = 219$. In this case, $K_S(F - \mathrm{F}(S))$ is equal to*

$$\left\{\begin{array}{l} 130, 134, 137, 141, 151, 154, 155, 158, 159, 160, 161, 162, 165, 166, \\ 168, 172, 175, 176, 178, 179, 180, 181, 182, 183, 184, 185, 186, 187, \\ 189, 190, 191, 192, 193, 196, 197, 199, 200, 201, 202, 203, 204, 205, \\ 206, 207, 208, 209, 210, 211, 212, 213, 214, 215, 216, 217, 218, 220, \\ \ldots \end{array}\right\}.$$

Then, by Theorem 2.6, $\{0\}\cup K_S(F-\mathrm{F}(S))$ *is an almost symmetric numerical semigroup with Frobenius number* 219 *and type* $219-2\cdot 55=109$*.*

The inverse map in Theorem 2.6 can be also explicitly described as we will see next. If S is a numerical semigroup, we will write $S^*=S\cup\mathrm{PF}(S)$. It is easy to see that S^* is a relative ideal of S. The ideal S^* is called the dual of $S\setminus\{0\}$ with respect to S. This term is justified by the fact that S^* is equal to $\{z\in\mathbb{Z}\mid z+S\setminus\{0\}\subseteq S\}$.

Let us see that this star operation is the inverse of our application in Theorem 2.6.

Proposition 2.8 *Let* S *be a numerical semigroup with genus* g*. If* F *is a positive integer greater than* $2\,\mathrm{F}(S)$*, then*

$$\Big(\{0\}\cup K_S(F-\mathrm{F}(S))\Big)^*=S.$$

Proof By definition and the condition $F>\mathrm{F}(S)$, $K_S(F-\mathrm{F}(S))\subseteq S$. By Proposition 2.3 and the proof of Theorem 2.6, and using once more that $F>\mathrm{F}(S)$, we have that $\mathrm{PF}\big(\{0\}\cup K_S(F-\mathrm{F}(S))\big)\subseteq S$. Thus, $K_S(F-\mathrm{F}(S))^*\subseteq S$. Now, $\#\Big(\mathbb{N}\setminus\big(\{0\}\cup K_S(F-\mathrm{F}(S))\big)^*\Big)=\mathrm{g}(\{0\}\cup K_S(F-\mathrm{F}(S)))-\mathrm{t}(\{0\}\cup K_S(F-\mathrm{F}(S)))$, and according to the proof of Theorem 2.6, this amount equals $F-\mathrm{g}(S)-(F-2\,\mathrm{g}(S))=\mathrm{g}(S)$. We conclude that $\big(\{0\}\cup K_S(F-\mathrm{F}(S))\big)^*=S$. □

Example 2.9 *Consider, again, the numerical semigroup* S *generated by* $21,24,25$*, and* 31*, and let* $T=\{0\}\cup K_S(F-\mathrm{F}(S))$ *with* $F=219$ *(see Example 2.7). The set of pseudo-Frobenius numbers of* T *is equal to*

$$\begin{gathered}\{21,24,25,31,42,45,46,48,49,50,52,55,56,62,63,66,67,69,70,\\ 71,72,73,74,75,76,77,79,80,81,83,84,86,87,88,90,91,92,93,\\ 94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,\\ 110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,\\ 125,126,127,128,129,131,132,133,135,136,138,139,140,142,143,\\ 144,145,146,147,148,149,150,152,153,156,157,163,164,167,169,\\ 170,171,173,174,177,188,194,195,198,219\}.\end{gathered}$$

Now, it is easy to check that $T^*=S$*, as expected by Proposition 2.3.*

Our correspondence provides a new characterization of almost symmetric numerical semigroups with high type.

Corollary 2.10 *Let* T *be a numerical semigroup with Frobenius number* F *and type* t*, with* $t\geq(F-1)/2$ *and* $F-t$ *even. Then* T *is almost symmetric if and only if* T^* *is a numerical semigroup with genus* $(F-t)/2$*.*

Proof Necessity. If T is almost symmetric, as $F-t$ is even, then $F-t=2g$ for some nonnegative integer g, and $t\geq(F-1)/2$ yields $F\geq 4g-1$. Thus, $T=\{0\}\cup K_S(F-\mathrm{F}(S))$ for some numerical semigroup S of genus g (Theorem 2.6). Notice that $2g-1\geq\mathrm{F}(S)$, whence $2\,\mathrm{F}(S)\leq 4g-2<F$. By Proposition 2.8, we conclude that $T^*=S$, and we are done.

Sufficiency. Let $S=T^*$, and set $g=(F-t)/2$, the genus of S. As $S=T^*=T\cup\mathrm{PF}(T)$, we have $g=\mathrm{g}(T)-t$, and thus

$$\mathrm{g}(T)=\frac{F-t}{2}+t=\frac{\mathrm{F}(T)+\mathrm{t}(T)}{2},$$

proving that T is almost symmetric. $\square$

The depth of a numerical semigroup has shown to play a special role in the study of Wilf's conjecture (see for instance [6, 9]). Let S be a numerical semigroup with Frobenius number F and multiplicity m, and write $F+1 = qm - r$ for some integers q and r with $0 \leq r < m$. Then its depth is $\operatorname{depth}(S) = q$.

Corollary 2.11 *Let S be a numerical semigroup with genus g and let F be a positive integer greater than $2\,\mathrm{F}(S)$. The semigroup $K_S(F - \mathrm{F}(S))$ has depth equal to two.*

Proof By Lemma 2.5, the multiplicity of $K_S(F-\mathrm{F}(S))$ is equal to $F - \mathrm{F}(S)$. Write $F+1 = 2(F-\mathrm{F}(S)) - (F - \mathrm{F}(S) - 1)$. Then $\operatorname{depth}(S) = 2$. $\square$

Depth equal to two has a particular relevance, since Bras' conjecture holds in the restricted class of numerical semigroups having this depth [9].

3. The algorithm

Write $\mathscr{A}(F,t)$ for the set of almost symmetric numerical semigroups with Frobenius number F and type t, and let n_g be the number of numerical semigroups with genus g. As an immediate consequence of Theorem 2.6 we obtain the following result.

Corollary 3.1 *Let $g \in \mathbb{N}$, and let F be a an integer greater than or equal to $4g-1$. Then number of numerical semigroups with genus g is equal to the the number of almost symmetric numerical semigroups with Frobenius number F and type $F-2g$. That is,*

$$F \geq 4g-1 \textit{ implies } \#\mathscr{A}(F, F-2g) = n_g.$$

With this corollary we can restate the weaker version of the conjecture appearing in [5], that is, that the sequence n_g is increasing. Notice that in order to prove that $n_{g+1} > n_g$, one needs to show that

$$\#\mathscr{A}(F, F-2(g+1)) > \#\mathscr{A}(F, F-2g),$$

for F large enough. This opens a new perspective to attack this conjecture.

We recall that the largest n_g known so far appears in `https://github.com/hivert/NumericMonoid`.

As we mentioned in the introduction, almost symmetric numerical semigroups with high type are uniquely determined by their sets of pseudo-Frobenius numbers.

Corollary 3.2 *Let $F \in \mathbb{N}$ and let $t \geq (F-1)/2$. Every almost symmetric numerical semigroup with Frobenius number F and type t is uniquely determined by its pseudo-Frobenius numbers.*

Proof Let S_1' and S_2' be two almost symmetric numerical semigroups with Frobenius number F and type t such that $\mathrm{PF}(S_1') = \mathrm{PF}(S_2')$. By (2.1), the genus of S_1' and S_2', equals $(F+t)/2$; whence $F-t = F+t-2t$ is even. Set $2g = F-t$. Then $2F \geq 4g+2t \geq 4g+F-1$, which implies $F \geq 4g-1$. Thus, by Theorem 2.6, there exist unique numerical semigroups S_1 and S_2 of genus g such that $S_i' = K_{S_i}(F - \mathrm{F}(S_i))$, $i \in \{1,2\}$. Observe that $F \geq 4g-1 \geq \mathrm{F}(S_1) + \mathrm{F}(S_2) + 1$, because, as we already mentioned above, $F(S_i) \leq 2g-1$, $i \in \{1,2\}$ (see, for instance [14, Lemma 2.14]). Moreover, without loss of generality we may suppose that $\mathrm{F}(S_1) \leq \mathrm{F}(S_2)$. Thus, by the last part of Proposition 2.3, we have that $\{a \in S_1 \mid 0 < a \leq \mathrm{F}(S_1)\} \cup \{\mathrm{F}(S_1)+1, \ldots, \mathrm{F}(S_2)\} \subseteq$

$\{a \in S_2 \mid 0 < a \leq \mathrm{F}(S_2)\}$, which implies $S_1 \subseteq S_2$ or equivalently, $\mathbb{N} \setminus S_2 \subseteq \mathbb{N} \setminus S_1$. Now, since both $\mathbb{N} \setminus S_2$ and $\mathbb{N} \setminus S_1$ have cardinality g, we conclude that $S_1 = S_2$, and consequently $S_1' = S_2'$. □

Notice that in general a potential set of pseudo-Frobenius numbers does not uniquely determine a numerical semigroup, [8], even under the almost symmetric condition. It may happen that several numerical semigroups share the same set of pseudo-Frobenius numbers. Thus, the above result opens a new strategy to determine almost symmetric numerical semigroups with large type with respect to the Frobenius number.

Example 3.3 *There are* 103 *almost symmetric numerical semigroups with Frobenius number* 20*, while when computing their sets of psuedo-Frobenius numbers, we only get* 62 *different possible sets.*

```
gap> l:=AlmostSymmetricNumericalSemigroupsWith
     FrobeniusNumber(20);;
gap> Length(l);
103
gap> Length(Set(l,PseudoFrobenius));
62
```

For instance, there are 11 almost symmetric numerical semigroups with Frobenius number 20 and set of pseudo-Frobenius numbers equal to $\{10, 20\}$*. Also, the set of almost symmetric numerical semigroups with set of pseudo-Frobenius numbers* $\{8, 10, 12, 20\}$ *is*

$$\{\langle 7, 9, 15, 17, 19\rangle, \langle 7, 11, 15, 16, 17, 19\rangle, \langle 9, 13, 14, 15, 16, 17, 19, 21\rangle,\\ \langle 11, 13, 14, 15, 16, 17, 18, 19, 21, 23\rangle\}.$$

In [4, Theorem 4.1], it is shown that each almost symmetric numerical semigroup of Frobenius F and type t is obtained from an almost symmetric numerical semigroup of Frobenius F and type $t+2$. More precisely, the following is proved.

Theorem 3.4 *Let* $F \geq 5$ *and* $t + 2 \leq F$ *be a positive integer greater than* 2 *such that* $F + t$ *is even. Then* S' *is an almost symmetric numerical semigroup with Frobenius number* F *and type* t *if and only if there exist an almost symmetric numerical semigroup* S *with Frobenius number* F *and type* $t+2$*, and* $i \in \{t+1, \ldots, \mathrm{m}(S)-1\}$ *such that*

(a) $S' = \{i\} \cup S$*,*

(b) $-i + \mathbb{N} \setminus S' \subseteq \mathbb{Z} \setminus S'$*, and*

(c) $\big(i + (\mathrm{PF}(S) \setminus \{i, F - i\})\big) \subseteq S'$*.*

In this case, $\mathrm{PF}(S) = \mathrm{PF}(S') \cup \{i, F - i\}$.

Let us see that if $t \geq (F - 1)/2$, conditions (b) and (c) in Theorem 3.4 can be replaced by a simpler test.

Proposition 3.5 *Let* $F \geq 5$ *and let* $t \in [(F-1)/2, F-2]$ *be an integer such that* $F + t$ *is even. Then* S' *is an almost symmetric numerical semigroup with Frobenius number* F *and type* t *if and only if there exist an almost symmetric numerical semigroup* S *with Frobenius number* F *and type* $t+2$*, and* $i \in \{t+1, \ldots, \mathrm{m}(S)-1\}$ *such that* *(a)* $S' = \{i\} \cup S$, *and*

(b) $(i + (\mathrm{PF}(S) \setminus \{i, F - i\})) \cap (\mathrm{PF}(S) \setminus \{i, F - i\}) = \varnothing$.

In this case, $\mathrm{PF}(S') = \mathrm{PF}(S) \setminus \{i, F - i\}$ *and* S' *is the unique almost symmetric numerical semigroup with set of pseudo-Frobenius numbers equal to* $\mathrm{PF}(S) \setminus \{i, F - i\}$.

Proof Necessity follows from Theorem 3.4.

For the other implication, first observe that as S is almost symmetric, $F - (t + 2)$ is even. Thus, $g = (F - (t + 2))/2$ is a nonnegative integer. Moreover, $F \geq 4g - 1$, because $t + 2 \geq t \geq (F - 1)/2$. Thus, by Theorem 2.6, there exists a numerical semigroup T of genus g such that $K_T(F - \mathrm{F}(T)) = S$. Now, on the one hand, by Proposition 2.3, we have that $\mathrm{m}(S) = F - \mathrm{F}(T)$ and, on the other hand, we have that, by [14, Lemma 2.14], $\mathrm{F}(T) \leq 2g - 1 \leq (F - 1)/2 \leq t$. Therefore, $\{t + 1, \ldots, \mathrm{m}(S) - 1\} \subseteq \{\mathrm{F}(T) + 1, \ldots, F - \mathrm{F}(T) - 1\}$, in particular, $\{t + 1, \ldots, \mathrm{m}(S) - 1\} \subseteq \mathrm{PF}(S)$ in light of Proposition 2.3. Thus, both i and $F - i$ are pseudo-Frobenius numbers of S and $\mathrm{PF}(S) \setminus \{i, F - i\}$ has cardinality t.

Consider now $S' = S \cup \{i\}$ and set $PF' := \mathrm{PF}(S) \setminus \{i, F - i\}$. Since $i \in \mathrm{PF}(S)$ and $2i \in S$ because $2i \geq 2t + 2 \geq F$. We have that S' is a numerical semigroup and $\mathrm{F}(S') = F$, because $i < F$. Moreover, $PF' + a \in S \subset S'$, for every $a \in S \setminus \{0\}$, because $PF' \subset \mathrm{PF}(S)$. Therefore, if $PF' + i \subset S'$, then we have $PF' \subseteq \mathrm{PF}(S')$.

Suppose that $i + PF' \not\subset S'$. Then there exists $a \in PF' \subset \mathrm{PF}(S)$ such that $i + a \notin S'$. Thus, $a + i \in \mathrm{PF}(S)$. Finally, as $i + a \neq i$, because $a \neq 0$, and $i + a \neq F - i$, because $2i > F$, we conclude that $i + a \in PF'$; in contradiction with condition (b).

Now we have $PF' \subseteq \mathrm{PF}(S')$, and we know that $\mathrm{g}(S') \geq (\mathrm{F}(S') + \mathrm{t}(S'))/2$ [13, Proposition 2.2]. Consequently, $t \leq \mathrm{t}(S') \leq 2(\mathrm{g}(S) - 1) - \mathrm{F}(S) = \mathrm{t}(S) - 2 = t$. Whence $t = \mathrm{t}(S')$ and by (2.1), we deduce that S' is almost symmetric.

The uniqueness of S' follows from Corollary 3.2. □

Example 3.6 *Figure 1 illustrates how Proposition 3.5 can be used to compute the set of almost symmetric numerical semigroups with high type for* $F = 15$.

If $F = 15$, *the only almost symmetric numerical semigroup with type* F *is* $\{0, 16, \ldots\}$. *Then, by Proposition 3.5, the only almost symmetric numerical semigroup with type* $F - 2 = 13$ *is* $S := \{0, 14, 16, \ldots\}$. *This is a general fact; with independence of the value of* F, *the almost symmetric numerical semigroup with Frobenius number* F *and type* F *or* $F - 2$ *are* $\{0, F + 1, \ldots\}$ *or* $\{0, F - 1, F + 1, \ldots\}$, *respectively.*

Now, since S *has multiplicity* $\mathrm{m}(S) = 14$ *and type* $\mathrm{t}(S) = 13$, *we can adjoin to it either* $(\mathrm{t}(S) - 2) + 1 = 12$ *or* $\mathrm{m}(S) - 1 = 13$ *in order to produce possible almost symmetric numerical semigroups. In this case, both operations yield almost symmetric numerical semigroups with Frobenius number* F *and type* $\mathrm{t}(S) - 2$. *Now, we can repeat the same procedure with the new semigroups and so on.*

Note that the possible candidates that can be added not always produce almost symmetric numerical semigroups. For instance, the candidates that can be adjoined to the numerical semigroup $S' := \{0, 12, 14, 16, \ldots\}$ *in the second column of Figure 1 are* $9, 10$, *or* 11. *However, only the second one produces an almost symmetric numerical semigroup which clearly has Frobenius number* $F(S') = 15$ *and type* $9 = \mathrm{t}(S') - 2$.

It is convenient to note that the tree depicted in Figure 1 is exactly the same as the tree of the numerical semigroups with genus from 0 *to* 4*, as we already know by Theorem 2.6.*

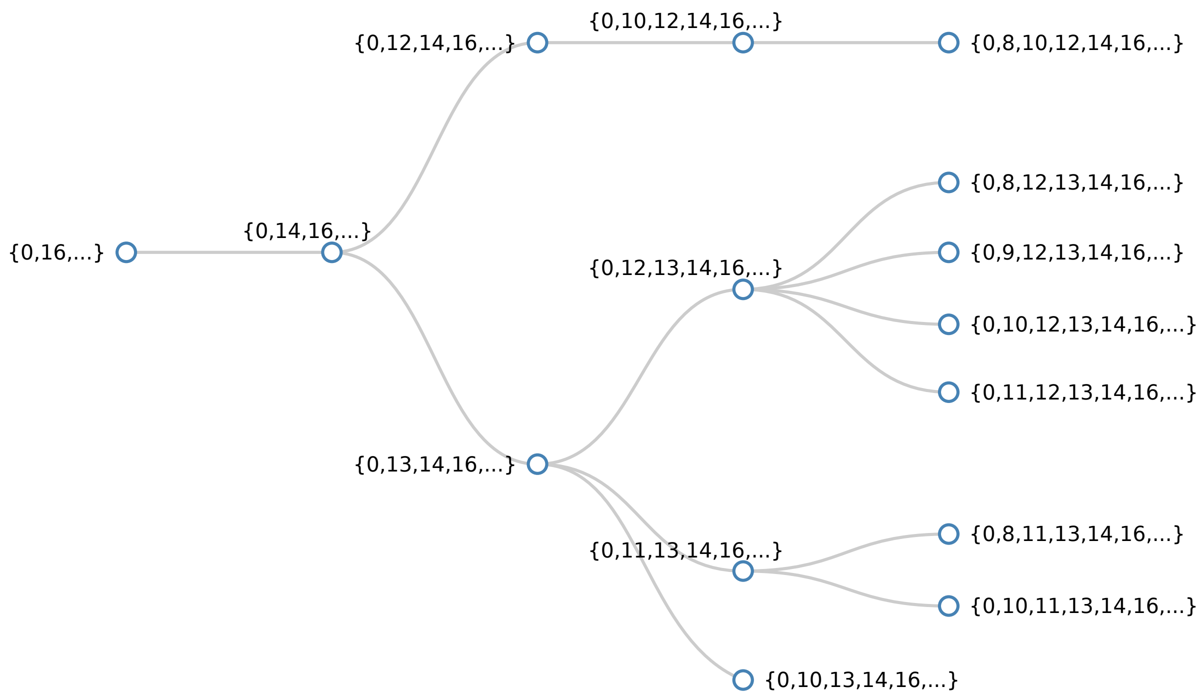

Figure 1. The tree of almost symmetric numerical semigroups with Frobenius number 15 and type $t \in \{15, 13, 11, 7, 9\}$.

We finally observe that we can continue the described procedure beyond type 7*. In fact, this is the descending algorithm in [4]. Nevertheless, after type 9, almost symmetric numerical semigroups are not determined by its pseudo-Frobenius elements. For instance, there are twelve almost symmetric numerical semigroups with Frobenius number* 15 *and type* 5*, but their set of families pseudo-Frobenius elements is*

$$\Big\{\{2,4,11,13,15\},\{3,6,9,12,15\},\{4,5,10,11,15\},\{4,6,9,11,15\},\\ \{4,7,8,11,15\},\{5,6,9,10,15\},\{5,7,8,10,15\},\{6,7,8,9,15\}\Big\}$$

which has cardinality eight.

Let us see that the above results offer an alternative to compute n_g.

Algorithm 3.7 *The following GAP [10] code counts the number of almost symmetric numerical semigroups of Frobenius number* $F = 4g - 1$ *and type* $F - 2j$ *for each* $j \in \{1, 2, \ldots, \lceil (F-1)/4 \rceil\}$*; equivalently, by Corollary 3.1, the number of numerical semigroups with genus j, for each* $j \in \{1, \ldots, g\}$*. The main function is nothing but a recursive step that calls to the* `auxiliar` *function whose correctness relies in Proposition 3.5. We observe that since the pseudo-Frobenius numbers uniquely determine a numerical semigroup by Corolary 3.2, we only need to deal with pseudo-Frobenius sets. Moreover, by Corolary 3.2 again, in order to avoid unnecessary repetitions we* *can restrict the upper range of i in Proposition 3.5, the mentioned restriction is forced with the second argument of the* `auxiliar` *function.*

It is important to emphasize that our GAP code does not require to make calls to other libraries or GAP packages. This makes our method more versatile and suitable to be implemented in other programming languages.

```
auxiliar := function(PF,m,t,s)
 local L,F,PF1,i,k,j;
  L := []; F:=PF[t+2];
  for i in [t+1 .. m-1] do
   PF1:=Difference(PF,[i,F-i]);
   k:=0;
   for j in [1 .. s] do
    if ((PF1[j]+i) in PF1) then
     k:=1;
     break;
    fi;
   od;
   if k=0 then
    Append(L,[[PF1,i]]);
   fi;
  od;
  return L;
end;

counting_function := function(g)
 local F,L,j,M,t,s,N;
 F:=4*g-1;
 L:=[[[1 .. F] ,F]];
 for j in [1 .. g] do
  M:=[];
  t:=Length(L[1][1])-2;
  s:=Int(t/2);
  for N in L do
   Append(M,auxiliar(N[1],N[2],t,s));
  od;
  L:=M;
  Print("n", j, " = ", Length(L), "\n");
  Unbind(M);GASMAN("collect"); #Cleaning Memory
 od;
 return Length(L);
end;
```

A quick comparison with the following GAP command (included in the GAP package `numericalsgps` [7])

`Length(NumericalSemigroupsWithGenus(g))` evidences that our code is slightly faster for $g \geq 23$. For instance, if $g = 26$ our function, `counting_function(26)`, computes $[n_1, \dots, n_{26}]$ in 17.335 seconds, while the above command takes 19.972 seconds to compute n_{26}. Both computations have been performed running GAP in a Intel(R) Core(TM) i7-4770S CPU 3.10 GHz. This simple evidence opens a door to more efficient and faster implementations.

Finally, we observe that, by Proposition 2.3 and Corollary 3.2, we can take advantage of the function `NumericalSemigroupByGaps` included in the GAP package `numericalsgps` to recover the whole set semigroups of genus g from our code. This can be done by just replacing `return Length(L);` with

```
return List(L, j->NumericalSemigroupByGaps(
                 Difference([1 .. (2*g-1)],j[1])));
```

With this modification, the computation of the whole set of numerical semigroups of genus g took 30.351 seconds.

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