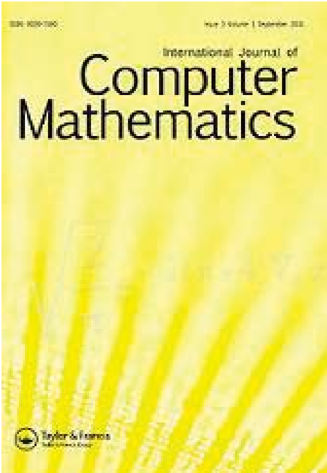

Fourth order method to compute the volume of archaeological vessels using radial sections: Pintia pottery (Spain) as a case study

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ARTICLE

Fourth order method to compute the volume of archaeological vessels using radial sections: Pintia pottery (Spain) as a case study

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ABSTRACT
In the archaeological community there is an interest in knowing the volume of the vessels rescued in the excavations, to study, among other things, if the capacity measures were standardized. The problem is that sometimes it is not possible to check the volume physically because the piece is too delicate or too large or simply incomplete. There is a fairly widespread idea that it is sufficient to know a radial section of the vessel to reconstruct it in 3D. Then, the volume is approximated by dividing the height in small sub-intervals sufficiently small and in each sub-interval the volume is approximated by that of a cylinder. This is equivalent to using the composite rectangular rule for the squared radius. This method works well if the piece made around is regular, more specifically if the horizontal sections are circumferences. The point is that most archaeological pieces have deformations, which makes the previous method very inaccurate. In this work a method that, instead of using a single section, uses several radial sections equally spaced between 0 and 2π angles and then take the average, is proposed. It is shown that the method gives a volume approach of fourth order with respect to the angle. Numerical experiments are presented on an academic example whose horizontal sections are ellipses, another academic example with less symmetries and an example of a Pintia vessel with an evident deformation in which the proposed method is tested.

KEYWORDS
Archaeological vessels volume; tumor volume; volume of hydrocarbon reservoirs; numerical integration; radial section

1. Introduction

The main objective of this article is to calculate accurately the volume of three-dimensional sets for which, in cylindrical coordinates, the radius depends on the height and the rotation angle. A particular case of this type of sets are the bodies of revolution, in which the sections at fixed height are circles, that is, the radius only depends on the height. In this case, the volume is reduced to an one-dimensional integral that can be approximated with some quadrature rule, by measuring the radii of one profile of the set at different heights and, calculating afterwards, a linear combination of these values. For example, it can be used either the composite trapezoidal rule or the

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composite Simpsons rule, with equally spaced nodes. However, if the horizontal sections are not circles but have other shapes, that are not only variable with the height but also with the rotation angle, using a single sets profile does not give an accurate approximation. In this work, we propose the average of revolution volume of several radial sections corresponding to equally spaced angles as an approximation of fourth order (with respect to the angle of rotation) to the volume.

There are several examples where this technique can be used, for instance to approximate the volume of geological structures in exploration of hydrocarbon reservoir [9, 18]. Also it can be employed to estimate the volume of some tumors, for example in mammary tumors which seemed to take on an oblate spheroid geometry [7]. However, in this work we will focus on its application to the computation of volumes of archeological vessels, specifically those corresponding to the Pintia site.

Vaccean are the pre-Roman ethnic group that inhabited the central part of the Duero Basin between the 4th century BC. and the change of the Era. Its location in the sedimentary lands of the Duero and Pisuerga Rivers, has an area of about $45,000\,km^2$. This vast territory was organized in true city-states, arising thanks to a series of factors as the generalization of iron metallurgy and its application to weapons and tools which must have had a decisive impact on a increase in productivity. As a result, there was a greater availability of food and therefore a demographic increase that, with due social, economic and political changes, explains the appearance of the urban phenomenon for the first time in the history of these lands.

Pintia is one of those cities in which the University of Valladolid has been working for the last forty years (1979–2019), which makes it the best known enclave in the vaccea region. Its incineration necropolis of Las Ruedas presents rich tombs that are accompanied by dozens, and exceptionally hundreds, of trousseaus and viatic offerings [14]. The artisan neighborhood of Carralaceña offers the largest and best preserved ovens of its kind in Europe. The type of oven number 2 stands out, in an extraordinary state of preservation, even retaining the two combustion chambers [6]. In the laboratory located above these vaults, a large production would be stacked for joint baking, which gives us an idea of the level of development of these vaccea potteries that we can describe as pseudo-industrial production.

Craft specialization and standardization of products are two concepts of great organizational and social importance, which we can now understand perfectly settled in these vaccea cities. The segregation of the pottery neighborhood in Pintia on the right bank of the Duero River, in Carralaceña (Pesquera de Duero), that is, on the opposite side where the city of Las Quintanas (Padilla de Duero) is located, is very expressive of this situation, counting also with residential neighborhood and own necropolis. A sector of the population, therefore, specialized in non-subsistence production activities and probably with almost exclusive (at least seasonal) dedication to these needs to be able to supply the demand of a city of several thousand inhabitants that, let's not forget, amortizes a good part of its production in the funeral rituals. The incorporation of the potter's wheel from at least the fourth century B.C. to the vaccea territory, and mastering the art of raising the vessels through this movement of revolution, would determine an agile and standardized production that, however, also leaves room for improvisation that always surprises us. Precisely, ceramics is one of the most abundant material culture findings recovered in archaeological excavations and Pintia is no exception in this regard. Fortunately, a good part of them are obtained under optimal conservation conditions that allow an integral reconstruction of them despite their occasional fragmentation. It is important to understand that these ceramic vessels respond at this time, as we have already pointed out, to a standardization of concrete

(a) Handmade scaled drawing of a section.

(b) Picture of the vessel.

Figure 1. Pintia 153V archaeological vessel.

forms, volumes and uses (in relation to the contents obtained through residue analysis [15]). Therefore, beyond the values of their diameter in mouth, maximum diameter or height traditionally recorded, their volume constitutes an increasingly demanded variable in current studies. Thinking about their capacity determines whether they are dynamic, semi-static or static containers [13], and also that the different capacities can respond to certain established measurement patterns [4].

The usual way to calculate the volume of vessels found in deposits is to fill the container with water or, if the piece is porous or fragile, with seeds, and then transfer the contents to a graduated cylinder. However, calculating in this way the capacity of the vessels recovered from the excavations has several difficulties. For example, it is not always possible to complete the puzzle from the rescued pieces, or the vessels are too large, or delicate. For all these reasons, it would be desirable to have a method to estimate the capacity of those vessels for which it is not possible to check their volume in a material way. It is advisable to read the interesting article [21], in which a detailed review of the methods used so far to approximate these volumes is done.

In recent times, 3D modeling is being used based on the technical drawings made of the pieces, which we believe can lead to errors of certain depth in the results obtained. Indeed, methodologically, the basic risk in these measurements may be to use archaeological drawings as a starting point to convert them into solids of revolution, without direct handling of the original ceramic pieces. It is not that the artist is more or less solvent, but that the containers really respond to a exact model of revolution. Frequently, even in the case of famous potters such as vacceos, with standardized productions made in a pseudo-industrialized manner as we have said, ceramic vessels with manifest deformities or variable wall thicknesses are observed along its profile. 2D drawing must necessarily opt for a cut in a specific plane, which can obviate these deformities.

The paper is organized as follows. The common way to approach a turned vessel using a single profile is revised in Section 1. The main target of this work, to improve the precedent approach of the volume of archaeological pottery using several radial sections, with a fourth order method is introduced in Section 2. Section 3 is devoted to the numerical experiments. On one hand, two academic examples are proposed, one of them is used to study the influence of non circumference horizontal sections when only a single profile is used and when a succession of equally spaced radial section are considered, and other example is chosen to check the second order of the method in the coordinate z and the fourth order in the angle. On the other hand, the method proposed in Section 2 is tested with an archaeological example.

2. Classic way to approximate the volume of a turned vessel using a single profile

In the case of pieces done with lathe, that is, made around, or in mathematical language, figures of revolution that are generated by turning a flat curve around an axis, knowing the section of the piece is sufficient to calculate its volume, or at least approach it quite accurately. Taking advantage of the fact that, as part of the study of the pieces found in the excavations, a scaled drawing of a section of the vessel is made systematically, that profile is normally used to calculate an approximation of the volume of the piece.

The general way to compute the volume of a bounded set Ω of three dimensions is to calculate the triple integral of the function one over said set Ω. Applying Fubini's Theorem, this triple integral is reduced to an integral of one dimension when the area of the sections of the set on one of the axes is known. In the bodies of revolution the sections are circles and, therefore, have a known area depending on the radius, ($\pi r(z)^2$), which, in the case of vessels, in general is variable with the height. Thus, the volume V of a set of revolution around the z axis Ω, with radius $r(z)$ between heights a and b is given by

$$V = \iiint_\Omega 1 dxdydz = \pi \int_a^b r(z)^2 dz. \tag{1}$$

Then, known the radius $r(z)$ in terms of the height z, the integral in (1) can be evaluated calculating a primitive of the radius squared and applying the Barrow's rule. The problem is that, from the drawing of the section of the piece, we do not have the expression of the function of the radius and, therefore, we cannot calculate the corresponding primitive. In this case, we can appeal to a numerical quadrature formula [2, 3, 5, 8, 17] using a table of values of the function that we will obtain by measuring the radii of the vessel at different heights and, calculating afterwards, a linear combination of these values. For example, we can use the composite rectangle rule, the composite trapezoidal rule and the composite Simpson's rule with equally spaced nodes, whose convergence for sufficiently regular functions is linear, quadratic, and fourth order respectively.

First, we divide the interval $[a, b]$ into N equal parts with step size $h = (b - a)/N$. The ends of each subinterval are the nodes $z_i = a + ih$, with i from 0 to N and the nodes $z_{i-1/2}$ represent the midpoints of the subintervals. The rectangle, trapezoidal and Simpson's composite rules have the following expressions respectively

$$\begin{aligned}
I_R^C(r^2(z),[a,b],h) &= h\sum_{i=0}^{N-1} r^2(z_i),\\
I_T^C(r^2(z),[a,b],h) &= h((r^2(z_0)+r^2(z_N))/2+\sum_{i=1}^{N-1} r^2(z_i)),\\
I_S^C(r^2(z),[a,b],h) &= (h/6)(r^2(z_0)+r^2(z_N)+2\sum_{i=1}^{N-1} r^2(z_i)+4\sum_{i=1}^{N} r^2(z_{i-1/2})).
\end{aligned}$$

When the step size h tends to zero, the quadrature errors also tend to zero, but also the greater the order of convergence the more quickly the errors converge to zero. Henceforth, we will denote by V_{RS} the approximation to the volume using the composite trapezoidal rule, for the radial section selected

$$V_{RS} = \pi h((r^2(z_0)+r^2(z_N))/2+\sum_{i=1}^{N-1} r^2(z_i)). \tag{2}$$

Most studies devoted to the computation of the volume of archaeological turned vessels start from the hypothesis that they are rotatory figures made around a flat curve. So that the volume is computed from a single radial section, usually drawn by hand, approximating the volume by measuring the radius at different heights equally spaced. For example, in [1, 4], the volume of vessels are calculated from the drawing of their profile as a sum of cylinders, a method that is comparable to the composite rectangle rule, which as we have indicated is first order. The method known as truncated cones [19] is a geometric method equivalent to the trapezoidal composite rule, which converges quadratically. The method proposed in [10] is also among those who requires only a two-dimensional archaeological drawing of each vessel. This profile is digitized by means of the AutoCad vector drawing software. For the reconstruction of each vessel, a typified curve, is obtained by averaging the profiles of the largest possible number of complete vessels from the same typological group. The curve of the inner profile is then vectorized using the image as a base or template. The coordinates obtained in AutoCad are transferred to the computational software Wolfram Mathematica to obtain a vector spline (interpolation of a smooth curve from the defined points) and then a numerical integration method is used to approximate (1), where r(z) is replaced for the spline curve.

Several authors consider three-dimensional reconstruction of ceramic vessels based on its manual drawing imported by a certain software 3D model of the vessel is obtained by revolving the profile around the vertical symmetry axis. For example, in [22], volumes of handmade cylindrical vessels found in excavations of Iron Age IIA in the Negev Highlands in Israel are studied in this way. The same idea extended to other types of revolving pottery is considered in [12, 20]. More recently, in [21] AutoCAD is used to produce a vector drawing of the vessel profile, which later is used to generate a three-dimensional model. Undoubtedly, with these programs, three-dimensional visualization of incomplete pieces is gained, however archaeological ceramics are not perfect rotatory bodies but can have deformations. Thus, the revolving body obtained with this technique depends on the angle chosen for the radial section, and therefore,

Figure 2. Horizontal section of a Pintia's archaeological vessel with remains of bones, courtesy of J.F. Pastor.

the approximation of the volume of the archaeological vessel may have a considerable error [11].

Summarizing, this method works well when the horizontal sections of the vessel are effectively circles. However unfortunately, in archaeological vessels the horizontal sections in general are not circles but have other shapes that are not only variable with the height but also with the rotation angle. For instance in Figure 2 is displayed a horizontal section of an archaeological vessel where it can be clearly seen that the inner curve is not a circumference.

3. Method to compute the volume of archaeological vessels using several radial sections

In general, computing the volume of an archaeological vessel from a single radial section does not give an accurate approximation. Archaeological pottery usually has deformations and consequently the radial sections corresponding to different angles can be significantly different. Here, instead of using a single profile, we consider a method that uses the information of several radial sections corresponding to equally spaced angles; for each section, the formula (2) is computed, and then the average of the sections is proposed as an approximation to the volume.

Using cylindrical coordinates (r, θ, z) instead of Cartesian coordinates (x, y, z) to calculate the volume and applying change of variables theorem we have

$$V = \iiint_{\Omega} 1 dxdydz = \iiint_{\Omega^*} r dr d\theta dz,$$

where Ω^* is the set described in cylindrical coordinates : $0 < \theta < 2\pi$, $a < z < b$, $0 < r < r(\theta, z)$. It can be seen that in general the radius depends on both the height z and the angle θ. Using Fubini's theorem

$$V = \int_0^{2\pi} (\int_a^b (\int_0^{r(\theta,z)} r dr) dz) d\theta = \frac{1}{2} \int_0^{2\pi} (\int_a^b r^2(\theta, z) dz) d\theta. \quad (3)$$

Theorem 3.1. *Let $h = (b-a)/N$ be the step size in the variable z and $N+1$ equally spaced nodes $z_i = a + ih$, $h = 0, \ldots, N$. Let $k = 2\pi/M$ be the step size in the variable θ and $M+1$ equally spaced angles $\theta_j = jk$, $j = 0, \ldots, M$. Be*

$$V_{RS}(\theta) = \pi h \left((r^2(\theta, z_0) + r^2(\theta, z_N))/2 + \sum_{i=1}^{N-1} r^2(\theta, z_i) \right) \tag{4}$$

the volume approach using the composite trapezoidal rule for the radial section of angle θ. Then, the approximation of the volume V with the average V_A of the values $V_{RS}(\theta_j)$, $j = 0, \ldots, M-1$ has second order convergence in h and fourth order in k, if $r(\theta, z) \in C^4([0, 2\pi] \times [a, b])$. More concretely,

$$V = \frac{1}{M} \sum_{j=0}^{M-1} V_{RS}(\theta_j) + O(h^2) + O(k^4) = V_A + O(h^2) + O(k^4). \tag{5}$$

Proof. It is known that the composite trapezoidal rule satisfies

$$\begin{aligned} \int_a^b f(z)dz &= I_T^C(f(z), [a, b], h) - f''(\xi)\frac{(b-a)}{12}h^2 \\ &= h\left((f(z_0) + f(z_N))/2 + \sum_{i=1}^{N-1} f(z_i) \right) - f''(\xi)\frac{(b-a)}{12}h^2, \end{aligned}$$

for $\xi \in (a, b)$ and $f \in C^2([a, b])$. Using the composite trapezoidal rule to approximate the integral (3) with respect to z it has

$$\begin{aligned} V &= \frac{1}{2}\int_0^{2\pi} (I_T^C(r^2(\theta, z), [a, b], h))d\theta - \frac{1}{2}\frac{(b-a)}{12}h^2 \int_0^{2\pi} \frac{\partial^2}{\partial z^2}(r^2(\theta, z))|_{z=\xi_\theta}\, d\theta \\ &= \frac{h}{2}\int_0^{2\pi} ((r^2(\theta, z_0) + r^2(\theta, z_N))/2 + \sum_{i=0}^{N-1} r^2(\theta, z_i))d\theta + O(h^2). \end{aligned} \tag{6}$$

We consider now the auxiliary function

$$g(\theta) = \frac{1}{2}((r^2(\theta, z_0) + r^2(\theta, z_N))/2 + \sum_{i=0}^{N-1} r^2(\theta, z_i)),$$

which satisfies $g(\theta) \in C^4([0, 2\pi])$. It is also known that, there is $\nu_j \in (\theta_j, \theta_{j+1})$ such that the corrected trapezoidal rule

$$I_{CT}(g, [\theta_j, \theta_{j+1}]) = \frac{k}{2}(g(\theta_j) + g(\theta_{j+1})) + \frac{k^2}{12}(g'(\theta_j) - g'(\theta_{j+1})),$$

based on cubic Hermite interpolation of $g(\theta)$ and $g'(\theta)$ at θ_j and θ_{j+1}, satisfies

$$\int_{\theta_j}^{\theta_{j+1}} g(\theta)d\theta = I_{CT}(g, [\theta_j, \theta_{j+1}]) + \frac{k^5}{720}g^{(4)}(\nu_j).$$

The composite corrected trapezoidal rule in the interval $[0, 2\pi]$ fullfills

$$\begin{aligned} \int_0^{2\pi} g(\theta)d\theta &= I_{CT}^C(g, [0, 2\pi], k) + 2\pi \frac{k^4}{720} g^{(4)}(\nu) \\ &= k\left((g(\theta_0) + g(\theta_M))/2 + \sum_{j=1}^{M-1} g(\theta_i)\right) + \frac{k^2}{12}(g'(\theta_0) - g'(\theta_M)) + 2\pi \frac{k^4}{720} g^{(4)}(\nu), \end{aligned} \tag{7}$$

$\nu \in [0, 2\pi]$. In the case of periodic functions, where $g(0) = g(2\pi)$ and $g'(0) = g'(2\pi)$, the composite corrected trapezoidal rule (7) reads

$$\int_0^{2\pi} g(\theta)d\theta = k \sum_{j=0}^{M-1} g(\theta_j) + 2\pi \frac{k^4}{720} g^{(4)}(\nu), \ \nu \in (0, 2\pi). \tag{8}$$

Applying the composite corrected trapezoidal rule to the function $g(\theta)$ in the interval $[0, 2\pi]$ in (6), with size step k,

$$\begin{aligned} V &= h \int_0^{2\pi} g(\theta)\, d\theta + O(h^2) = hk \sum_{j=0}^{M-1} g(\theta_j) + O(h^2) + O(k^4) \\ &= h \frac{2\pi}{M} \frac{1}{2} \sum_{j=0}^{M-1} \left((r^2(\theta_j, z_0) + r^2(\theta_j, z_N))/2 + \sum_{i=0}^{N-1} r^2(\theta_j, z_i) \right) + O(h^2) + O(k^4) \\ &= \frac{1}{M} \sum_{j=0}^{M-1} \pi h \left((r^2(\theta_j, z_0) + r^2(\theta_j, z_N))/2 + \sum_{i=0}^{N-1} r^2(\theta_j, z_i) \right) + O(h^2) + O(k^4) \\ &= \frac{1}{M} \sum_{j=0}^{M-1} V_{RS}(\theta_j) + O(h^2) + O(k^4). \end{aligned}$$

□

Then, it can be concluded that applying the composite trapezoidal rule in the variable z and the composite corrected trapezoidal rule in the variable θ it is equivalent to calculating the average of the approximations with the composite trapezoidal rule for the M radial sections for equally spaced angles. Moreover, this average is an approximation to the volume V of second order in the variable z and fourth order in the variable θ.

Remark 1. If the composite Simpsom's rule is used in (4) instead of the composite trapezoidal rule, the average will be an approximation to the volume V of fourth order in the variable z too.

4. Numerical experiments

4.1. An academic example

In order to validate the method introduced in the previous section, an academic test problem is considered. It is chosen a problem where the volume of the set Ω can

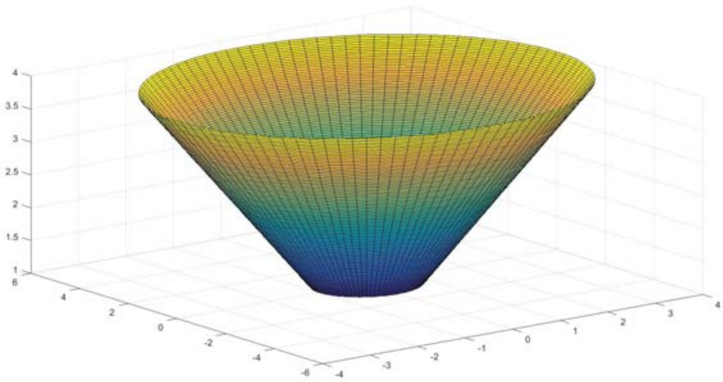

Figure 3. Elliptical paraboloid between the planes $z = 1$ and $z = 4$.

be computed exactly. Moreover, a parameter c is introduced so that the horizontal sections are not circles, but ellipses (unless $c = 1$). In this way we can study how the approaches evolve for several values of c. In this test problem the values $V_{RS}(\theta)$ can be computed exactly, that is, there is no error in h, so that we can focus on the study of the errors in the variable θ.

We consider the set bounded by an elliptical paraboloid and two planes

$$\Omega = \{(x, y, z) \in \mathbb{R}^3 : x^2 + \frac{y^2}{c^2} < z,\ 0 < a < z < b\}.$$

Paraboloid sections with constant z

$$\frac{x^2}{z} + \frac{y^2}{c^2 z} = 1,$$

are ellipses with semiaxis $\sqrt{z}$ and $c\sqrt{z}$ respectively. So the exact volume of Ω is

$$V = \pi \int_a^b \sqrt{z} c \sqrt{z} dz = \pi c \int_a^b z dz = \frac{\pi}{2}(b^2 - a^2)c.$$

Otherwise, $x^2 + \frac{y^2}{c^2} = z$ can be rewritten as $x^2 + y^2 + (\frac{1}{c^2} - 1)y^2 = z$ which, in cylindrical coordinates is

$$r^2 + (\frac{1}{c^2} - 1)r^2 \sin^2(\theta) = z,$$

and then

$$r = \frac{\sqrt{z}}{\sqrt{1 + (\frac{1}{c^2} - 1)\sin^2(\theta)}}.$$

In this particular case, fixing the angle θ and using (1), the volume of the revolution figure corresponding to the profile with radial section of angle θ does not have any error because the trapezoidal rule is exact for the polynomials of degree one and reads

$$V_{RS}(\theta) = \pi \int_a^b r^2(z,\theta)dz = \frac{\pi}{2}(b^2 - a^2)\frac{1}{1 + (\frac{1}{c^2} - 1)\sin^2(\theta)}.$$

In the following experiments we will compare V, the exact value of the volume, with $V_{RS}(\theta_j)$ the revolution volume for the radial section with fixed angle θ_j and V_A, the average of the volumes for radial sections equally spaced.

Although the computation is made to double precision, in the Tables the results are presented with rounding to four decimal figures. The values $a = 1$ y $b = 4$ are considered, so that the ellipses semiaxes are 1 and c for $z = 1$, $\sqrt{z}$ and $\sqrt{z}c$ for $1 < z < 4$ and 2 and $2c$ for $z = 4$. We notice that for $c = 1$ the horizontal sections are circumferences. In Table 1, for $M = 8$ and the values of c 1, 1.1, 1.2 and 1.3, the exact volume, the volume of revolution for angles $j = 2\pi/8$, $j = 0, \ldots, 7$ and the average volume are displayed. The right figures of V and V_A are shown in bold. For a specific c value, for example $c = 1.2$, it can be seen that there is a big difference between the volume of revolution V_{RS} for different angles (by the symmetry of the set half are repeated), however the average approximation is better. Fixed $M = 8$, it can be seen

$M = 8$	$c = 1$	$c = 1.1$	$c = 1.2$	$c = 1.3$
V	**23.5619**	**25.9181**	**28.2743**	**30.6305**
$V_{RS}(0)$	23.5619	23.5619	23.5619	23.5619
$V_{RS}(\frac{2\pi}{8})$	23.5619	25.8009	27.8108	29.6057
$V_{RS}(\frac{2\cdot2\pi}{8})$	23.5619	28.5100	33.9292	39.8197
$V_{RS}(\frac{3\cdot2\pi}{8})$	23.5619	25.8009	27.8108	29.6057
$V_{RS}(\frac{4\cdot2\pi}{8})$	23.5619	23.5619	23.5619	23.5619
$V_{RS}(\frac{5\cdot2\pi}{8})$	23.5619	25.8009	27.8108	29.6057
$V_{RS}(\frac{6\cdot2\pi}{8})$	23.5619	28.5100	33.9292	39.8197
$V_{RS}(\frac{7\cdot2\pi}{8})$	23.5619	25.8009	27.8108	29.6057
$V_A(8)$	**23.5619**	**25.918**4	**28.27**82	**30.6**483

Table 1. V, V_{RS} for angles $j2\pi/8$, $j = 0, \ldots, 7$ and $V_A(8)$, for several values of the parameter c.

that the higher the value of the c parameter, the approximation with the average of M sections is getting worse.

Table 2 shows the absolute error of the volume calculated with V_A, the average of the volumes of revolution for equally spaced angles, for several values of M, the values of c equal to 1.1, 1.2 and 1.3 and the values of b equal to 1.2, 2 and 4. It is observed that according to the number of angles used to make the average increases the approximation is improving. In this specific example even more than expected.

The convergence is observed in all cases, when M increases the error decreases. For M fixed, the error is greater the larger the parameter c is, that is, the more different from a circle are the sections. In archeology an error around 0.01 can be considered acceptable. If we set for example the liter as a unit, it can be seen that for pieces of less than 1 liter (in Table 2, $b = 1.2$) with $M = 4$ is achieved. The same is practically the same for vessels between 1 and 10 liters (in Table 2, $b = 2$). For larger pieces, between 10 and 100 liters another significant figure is needed more to get the same error, so it may be necessary to use $M = 8$ (in Table 2, $b = 4$).

$b=4$ M	$c=1.1$ $V=\mathbf{25.9181}$	$c=1.2$ $V=\mathbf{28.2743}$	$c=1.3$ $V=\mathbf{30.6305}$
2	2.4	4.7	7.1
4	1.2e-1	4.7e-1	1.1
8	2.7e-4	3.9e-3	1.8e-2
16	1.4e-9	2.6e-7	5.1e-6
$b=2$ M	$c=1.1$ $V=\mathbf{5.1836}$	$c=1.2$ $V=\mathbf{5.6549}$	$c=1.3$ $V=\mathbf{6.1261}$
2	4.7e-1	9.4e-1	1.4
4	2.4e-2	9.4e-2	2.1e-1
8	5.3e-5	7.7e-4	3.5e-3
16	2.7e-10	5.3e-8	1.0e-6
$b=1.2$ M	$c=1.1$ $V=\mathbf{0.7603}$	$c=1.2$ $V=\mathbf{0.8294}$	$c=1.3$ $V=\mathbf{0.8985}$
2	6.9e-2	1.4e-1	2.1e-1
4	3.5e-3	1.4e-2	3.1e-2
8	7.8e-6	1.1e-4	5.2e-4
16	4.0e-11	7.7e-9	1.5e-7

Table 2. $|V - V_A|$, absolute error of the volume calculated with the average for several values of b, c and M.

4.2. *Another academic example*

In the previous section it is observed that the volume of revolution calculated with sections corresponding to different angles can give very different values. On the other hand the more equally spaced angles are considered to compute the average the approximation is improving, even faster than expected. However, in the former academic example, neither the order 2 in h, because the trapezoidal rule integrates exactly the function z, nor the order 4 in k, because of the symmetries, are observed. For the purpose of confirming the convergence orders of the Theorem 3.1, another academic example, in which the composite trapezoidal rule is not exact in the variable z and there are no symmetries with respect to the angle, is proposed.

We consider the following bounded set described in cylindrical coordinates as

$$\Omega^* = \{(r,\theta,z) \in \mathbb{R}^3 : 0 < \theta < 2\pi,\ 0 < 1 < z < 2,\ 0 < r < z(10^{-3}\theta^4(\theta-2\pi)^2+1)\}.$$

So the exact volume of Ω^* is

$$\begin{aligned} V^* &= \frac{1}{2}(\int_0^{2\pi} (10^{-3}\theta^4(\theta-2\pi)^2+1)^2 d\theta)(\int_1^2 z^2 dz) \\ &= \frac{7}{6}\left(10^{-6}\frac{4\cdot 3\cdot 2}{9\cdot 10\cdot 11\cdot 12\cdot 13}(2\pi)^{13} + 2\pi + 2\cdot 10^{-3}\frac{2}{5\cdot 6\cdot 7}(2\pi)^7\right) \\ &\sim 20.234012820909374. \end{aligned}$$

In order to check the convergence order 2 with respect h in (5), the value $M = 128$ is fixed and the values $0.5, 0.25, 0.125, 0.0625, 0.03125$ for h are considered, in such a way the error due to k is negligible in comparison to the error due to h. Denoting $error(h) = |V^* - V_A(h)|$, the evolution of $\log 2(error(h)/error(h/2))$ is displayed in Table 4.2, where second order is evident.

h	0.5	0.25	0.125	0.0625	0.03125
$\log 2(error(h)/error(h/2))$	2.0000	2.0000	2.0000	2.0001	2.0003

Table 3. Evolution of $\log 2(error(h)/error(h/2))$ for $h = 0.5, 0.25, 0.125, 0.0625, 0.03125$ and $M = 128$.

With aim of checking now the convergence order 4 with respect k in (5), the value $h = 0.0001$ is fixed and the values $M = 4, 8, 16, 32, 64$ for the corresponding values of k are considered. So the error due to h is negligible in comparison to the error due to k. Denoting $error(k) = |V^* - V_A(k)|$, the evolution of $\log 2(error(k)/error(k/2))$ is displayed in Table 4.2, where fourth order is clearly seen.

M	4	8	16	32	64
$\log 2(error(k)/error(k/2))$	1.8758	3.3938	3.8687	3.9790	4.1800

Table 4. Evolution of $\log 2(error(k)/error(k/2))$ for the corresponding $M = 4, 8, 16, 32, 64$ and $h = 0.0001$.

4.3. An archaeological example

To test our measurement method, we have selected a ceramic denoted by 153V, made around, of some complexity, derived from an exceptional tomb corresponding to a female youth individual (between 13 and 20 years old) of the vacceo aristocracy, from the end of the second century B.C. This set has more than a hundred ceramic and metal pieces that are specifically the trousseau and viatic offerings arranged for this young woman in her burial tomb of the necropolis of Las Ruedas, Pintia [16].

The piece 153V is not a simple cylinder, like some of the kalathos or top hat glasses [1] or others [22] in which this type of volumetric calculations are considered, but it responds to a winding figure, of the bitroncoconic type, with a raised foot and an excavated edge. Piece 152V has also a marked deformation, the result of its manipulation in fresh, creating evident dissymmetries in its development.

We have a fairly accurate tomography of vessel 153V thanks to J.F. Pastor, Professor of Anatomy and Radiology Department of the University of Valladolid. In Figure 4 the tomographic exterior and interior images of vessel 153V are displayed, where the mentioned dissymmetry and variable wall thickness can be observed. The vessel was measured with water to overflow and a volume of 3 liters was obtained.

Considering the handmade drawn of Figure 1(a) and using (2) with the left profile, for $h = 0.3597$ mm, a volume of 2.7724 l is achieved, which implies an absolute error of 0.2276 l.

The tomography we have is made with vertical sections from front to back of the piece. In Figure 5 two perpendicular sections obtained from the tomography data are displayed. We select the central cut, so we have access to two profiles, which we will consider corresponding to angle 0 and angle π. Using measurements in each profile by means of counting pixels in the graphic file, we compute with $h = 0.34668$ mm $V_{RS}(0) = 3.4330$ l and $V_{RS}(\pi) = 2.8030$ l. We notice for this vessel that is quite deformed, the difference between both volume approach, one of them is an excess approach and the other one is a default approach. The average of this two approachs yields $V_A(2) = 3.1180$ l, whose absolute error is 0.1180 l, less than the corresponding to use only one profile. Then, extracting the radial sections perpendicular to these, $V_{RS}(\pi/2) = 2.2481$ l and $V_{RS}(3\pi/2) = 3.4065$ l are obtained. Computing the average

(a) Exterior view. (b) Interior view.

Figure 4. Tomographic images of vessel 153V, courtesy of J.F. Pastor.

(a) Sections corresponding to 0 and π angles. (b) Sections corresponding to $\pi/2$ and $3\pi/2$ angles.

Figure 5. Perpendicular sections of vessel 153V.

of the four profiles $V_A(4) = 2.9726$ l is reached, for which the absolute error 0.0274 l has improved.

5. Conclusions and Future work

From all the above, the following conclusions can be drawn:

(1) The widespread idea that, knowing one radial section an archaeological vessel can be rebuilt, is not entirely correct. 3D reconstruction of ceramic vessels based on manual drawing of one radial section allows an approach to the three-dimensional visualization of incomplete pieces, however the approximation to the volume with a single section can be very little precise.

(2) If the piece is complete it is best to measure the volume directly on the original piece. If, for conservation, large format or fragmentation reasons, direct measurement is not possible, it is recommended to use several radial sections. In the case of incomplete vessels, the larger the conserved part, the more sections we can obtain and the better we will approximate the original volume.

(3) The proposed method to approximate the volume using several equally spaced

radial sections is of order four with respect to the angle.

(4) If the piece is not very large, with four equally spaced radial sections a volume approximation, acceptable in archaeological terms, is obtained. If the piece is larger, more sections may be necessary.

As future work, we plan to apply the method to calculate the volume on a sufficiently large group of Pintia vessels.

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