

ENTANGLEMENT-ASSISTED QUANTUM ERROR CORRECTING CODES FROM RS CODES AND BCH CODES WITH EXTENSION DEGREE 2

CARLOS GALINDO, FERNANDO HERNANDO AND DIEGO RUANO

ABSTRACT. Entanglement-assisted quantum error correcting codes (EAQECCs) constructed from Reed-Solomon codes and BCH codes are considered in this work. It is provided a complete and explicit formula for the parameters of EAQECCs coming from any Reed-Solomon code, for the Hermitian metric, and from any BCH code with extension degree 2 and consecutive cyclotomic cosets, for both the Euclidean and the Hermitian metric. The main task in this work is the computation of a completely general formula for c , the minimum number of required maximally entangled quantum states.

1. INTRODUCTION

Quantum error correcting codes (QECCs) are mostly defined using classical linear codes [4, 13]. They were first introduced over the binary field and then the construction was extended to an arbitrary finite field [15]. Namely, QECCs over \mathbb{F}_q , the finite field with q elements, are usually constructed from self-orthogonal classical codes. We can consider classical codes over \mathbb{F}_q if we decide to use the Euclidean metric, and over \mathbb{F}_{q^2} , when using the Hermitian metric.

Brun, Dvetak and Hsieh in [2] proposed to share entanglement between encoder and decoder to simplify the theory of quantum error-correction and increase the communication capacity, giving rise to entanglement-assisted quantum error correcting codes (EAQECCs). An important advantage of this new construction is that one may consider an arbitrary classical linear code, without the self-orthogonality restriction. EAQECCs were also first defined over the binary field and then the construction was extended to an arbitrary finite field [11].

The main difficulty for determining the parameters of an EAQECC, with respect to QECCs, is the computation of the parameter c , the minimum number of required maximally entangled quantum states in $\mathbb{C}^q \otimes \mathbb{C}^q$. Computationally speaking, given a concrete EAQECC, the computation of c is not intense but it is a difficult task to provide a formula for a given family of codes. This is the main goal of the article.

Several articles have explored the construction of EAQECCs from classical linear codes as binary BCH codes [18, 21, 8], constacyclic codes [17, 27], constacyclic LCD codes [24], cyclic codes [25], generalized Reed-Solomon codes [19, 20, 14], negacyclic BCH codes [6, 7] and algebraic-geometry codes [23]. These papers address particular cases and determine

Key words and phrases. EAQECC; entanglement-assisted quantum codes; Reed-Solomon codes; BCH codes; subfield subcodes; cyclotomic cosets.

Published in Quantum Information Processing 20, 158 (2021). This work was supported in part by the Spanish MICINN/FEDER grants PGC2018-096446-B-C21, PGC2018-096446-B-C22 and RED2018-102583-T, by the Spanish MINECO grant RYC-2016-20208 (AEI/FSE/UE), by the Junta de CyL (Spain) grant VA166G18, by the Generalitat Valenciana (Spain) grant AICO-2019-223 and by Universitat Jaume I grant P1-1B2018-10.

their parameters but no general formula for the parameters of EAQECCs using the previous families of codes is known. A more general procedure is given in [22], which provides parameters for EAQECCs obtained from Reed-Solomon (RS) codes both for the Euclidean and Hermitian metric, however not all RS codes in the Hermitian case are considered. In this regard and in this paper, we give a formula for the parameters of all the EAQECCs that can be obtained from RS codes with respect to the Hermitian metric (see Section 4).

The main aim of this article is to consider BCH codes over an arbitrary field \mathbb{F}_q , that can be understood as cyclic codes but also as subfield subcodes of RS codes. We consider BCH codes defined from consecutive cyclotomic cosets and hence we can bound their minimum distance by the well-known BCH bound. We give a complete and explicit formula for the parameters of the EAQECCs coming from *any* BCH code (defined by consecutive cyclotomic cosets) with extension degree 2, both for the Euclidean and the Hermitian metric. Moreover, we also determine the parameters when we extend the classical codes by evaluating at zero. With our formulae the reader can easily determine the parameters of the EAQECCs obtained as mentioned.

The computation of c is performed following the geometric decomposition of a linear code [26], that was stated in [11] for EAQECCs. The computations are carried out by a careful analysis of the involved cyclotomic cosets and the q -adic decomposition of its elements that allow us to compute the geometric decomposition of a linear code. The geometric decomposition of a linear code endowed with an inner product provides a basis of the code included in a specific basis of the ambient linear space. This last basis $\{v_i\}_{i \in I}$ satisfies that each v_i is either orthogonal to v_j for all $j \neq i$ (symmetric) or orthogonal to v_j for all j except for a unique reciprocal element v_k with $k \neq i$ (asymmetric). This means that one can determine the hull, or radical, of the linear code easily. In this article, we compute such a decomposition by identifying symmetric and asymmetric cosets (see Definition 4) with respect to the inner product considered. We notice that the method is similar to the one in [18] for quaternary BCH codes.

In Section 5, we give a general formula for the parameters of EAQECCs coming from BCH codes with the Euclidean metric and extension degree 2. We show that they have parameters that are worse than the ones obtained with RS codes with Hermitian metric (studied in Section 4). Namely, they have the same length, $q^2 - 1$ or q^2 , and bound for the minimum distance but the dimension is lower for BCH codes with the Euclidean metric. However, BCH codes with the Euclidean metric have still some interest since one may obtain codes whose parameter c is greater than the one obtained with RS codes. In this way, we increase the constellation of known EAQECCs that is limited at this moment. Moreover, the computation of c for BCH codes with the Euclidean metric helps understanding the computation of c for BCH codes with the Hermitian metric treated in Section 6, that is rather technical.

Finally, the main results of this article are given in Section 6, where BCH codes with the Hermitian metric and extension degree 2 are considered. We give a *completely* general formula for their parameters. Furthermore, using that formula we present some EAQECCs with good parameters, by giving tables of them over different finite fields whose parameters exceed the Gilbert-Varshamov (GV) bound [11]. We also compare their parameters with the codes available in the literature, when it is possible. We obtain long codes over \mathbb{F}_q , with length $q^4 - 1$ and q^4 , and with good parameters $[[n, k, d; c]]_q$. Our codes satisfy $k > c$ and thus, they give rise to catalytic quantum codes [3]. Note that one cannot consider the Singleton bound, $n + 2 \geq k + 2d + c$, for EAQECCs defined over a non-binary field since

that bound is only proved for codes defined over a binary field with $d \leq (n + 2)/2$ [16], despite several articles in the literature consider it for arbitrary fields.

The article is organized as follows: we introduce RS and BCH codes in Section 2 and EAQECCs coming from them in Section 3. Section 4 is devoted to the study of EAQECCs from RS codes with respect to the Hermitian metric. To conclude, EAQECCs obtained from BCH codes with extension degree 2 are presented in Section 5 in the Euclidean case and in Section 6 for the Hermitian case.

2. RS AND BCH CODES

We introduce RS codes and BCH codes in this section. We regard BCH codes as subfield subcodes of evaluation codes as in [1, 5] instead of as cyclic codes. This construction has the advantage that it can be extended to evaluation by polynomials in several variables.

Let p be a prime number and consider the finite field \mathbb{F}_{p^ℓ} with p^ℓ elements. Let $n = p^\ell - 1$ and $\mathbb{F}_{p^\ell}[X]$ the ring of polynomials in one variable with coefficients in \mathbb{F}_{p^ℓ} . Consider classes of univariate polynomials in the quotient ring $\mathbb{F}_{p^\ell}[X]/J$, where J is the ideal of $\mathbb{F}_{p^\ell}[X]$ generated by $X^n - 1$. Define

$$\text{ev} : \mathbb{F}_{p^\ell}[X]/J \rightarrow \mathbb{F}_{p^\ell}^n; \quad \text{ev}(f) = (f(P_1), f(P_2), \dots, f(P_n)),$$

where $Z = \{P_1, P_2, \dots, P_n\}$ is the zero locus of J in \mathbb{F}_{p^ℓ} . Let Δ be a subset of $\mathcal{H} := \{0, 1, \dots, n - 1\}$. Then the RS code, $D_\Delta \subseteq \mathbb{F}_{p^\ell}^n$, is the code generated by

$$\{\text{ev}(X^i) \mid i \in \Delta\}.$$

Usually one considers $\Delta = \{0, 1, \dots, k - 1\}$ and the RS code has parameters $[p^\ell - 1, k, p^\ell - k]_q$. Moreover, one can extend the previous code by evaluating at 0 as well and therefore, one obtains a code with parameters $[p^\ell, k, p^\ell + 1 - k]_q$.

Let r be a positive integer such that r divides ℓ . We consider first codes over the field \mathbb{F}_{p^ℓ} and then their subfield subcodes over the field \mathbb{F}_{p^r} . BCH codes can be defined as subfield subcodes of the form $D_\Delta \cap (\mathbb{F}_{p^r})^n$ and extend RS codes in the sense that one can consider that a RS code is a BCH with extension degree one, that is, $r = \ell$. This is why we may consider the same notation for both families of codes in this article.

In the ring \mathbb{Z}_n , we consider minimal cyclotomic cosets with respect to $q = p^r$, minimal means that it contains exactly the elements of the form xq^t , $t \geq 0$, in \mathbb{Z}_n for some $x \in \mathbb{Z}_n$ under the identification $\mathbb{Z}_n = \mathcal{H}$. We denote by I_x the minimal cyclotomic coset $\{xq^t : t \geq 0\}$. For every minimal cyclotomic coset, pick its least element, then let \mathcal{A} be the set of all minimal representatives and $\{I_x\}_{x \in \mathcal{A}}$ is the set of minimal cyclotomic cosets with respect to q . Moreover, let $i_x := \#(I_x)$, with $\#$ denoting the cardinality of a set. For convenience, we write

$$\mathcal{A} = \{m_0 = 0 < m_1 < m_2 < \dots\} = \{m_j\}_{j=0}^z.$$

We will use the following two results which can be found in [10, 9].

Proposition 1. *Set $\Delta = \cup_{j=t'}^t I_{m_j}$, $t' < t$. Then the subfield-subcode of D_Δ over \mathbb{F}_q ,*

$$E_\Delta = D_\Delta|_{\mathbb{F}_q} = E_\Delta \cap (\mathbb{F}_q)^n,$$

has dimension $\sum_{j=t'}^t i_{m_j}$.

The forthcoming Proposition 2 uses duality with respect to the two metrics we consider in this work. These metrics are induced by two inner products: the Euclidean inner product, where $x \cdot y = \sum_{i=1}^n x_i y_i$ and the source code is defined over the finite field with q elements, and the Hermitian inner product, where $x \cdot y = \sum_{i=1}^n x_i y_i^q$ and the source code

is defined over the finite field with $q^2 = p^r$ elements (for a suitable r that divides ℓ); in this last case, we have to replace q with q^2 in the above description. Both cases, will allow us to obtain EAQECCs over the finite field with q elements by [11].

For the sake of clarity, we consider a toy example through the article only for illustrating the main concepts and notation.

Example 1. Let $p = 2$, $\ell = 4$ and $r = 2$. We consider the Hermitian inner product, thus $q^2 = p^r = 4$ and $n = p^\ell - 1 = 15$. The cyclotomic cosets in \mathbb{Z}_{15} with respect to 4 are:

$$\begin{aligned} I_0 &= \{0\}, I_1 = \{1, 4\}, I_2 = \{2, 8\}, I_3 = \{3, 12\}, I_5 = \{5\}, \\ I_6 &= \{6, 9\}, I_7 = \{7, 13\}, I_{10} = \{10\}, I_{11} = \{11, 14\}. \end{aligned}$$

And therefore, the set of all minimal representatives is $\mathcal{A} = \{0, 1, 2, 3, 5, 6, 7, 10, 11\}$. \square

Proposition 2. *The minimum distance of the (Euclidean or Hermitian) dual code of E_Δ , C_Δ , where $\Delta = \cup_{j=0}^t I_{m_j}$, is larger than or equal to $m_{t+1} + 1$ (BCH bound).*

Therefore for $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j}$, the code C_Δ is known as a BCH code and it has parameters $[n, n - \sum_{j=0}^t i_{m_j}, \geq m_{t+1} + 1]_{p^r}$.

For the Euclidean case, set $I_x^\perp := I_{n-x}$ and define

$$\begin{aligned} \Delta(t)^\perp &:= (I_{m_0} \cup I_{m_1} \cup \dots \cup I_{m_t})^\perp \\ &:= \mathcal{H} \setminus (I_{m_0}^\perp \cup I_{m_1}^\perp \cup \dots \cup I_{m_t}^\perp) \\ &= \mathcal{H} \setminus (I_{n-m_0} \cup I_{n-m_1} \cup \dots \cup I_{n-m_t}). \end{aligned}$$

Analogously, for the Hermitian case with base field \mathbb{F}_{q^2} , we set $I_x^\perp := I_{n-qx}$ and $\Delta(t)^\perp := \mathcal{H} \setminus (I_{n-qm_0} \cup I_{n-qm_1} \cup \dots \cup I_{n-qm_t})$. With the above notations, by [12, 9], it holds

$$C_{\Delta(t)} = E_{\Delta(t)}^\perp = E_{\Delta(t)^\perp}.$$

Example 2. This is a continuation of Example 1 with $p = 2$, $\ell = 4$, $r = 2$, $q^2 = 4$ and $n = 15$, we consider the Hermitian inner product. Let $t = 6$, then $m_6 = 7$ and

$$\Delta(6) = I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_5 \cup I_6 \cup I_7.$$

Moreover, we have that

$$\begin{aligned} \Delta(6)^\perp &= \mathcal{H} \setminus (I_{n-0q} \cup I_{n-1q} \cup I_{n-2q} \cup I_{n-3q} \cup I_{n-5q} \cup I_{n-6q} \cup I_{n-7q}) \\ &= \{0, 1, \dots, 14\} \setminus (I_0 \cup I_7 \cup I_{11} \cup I_6 \cup I_5 \cup I_3 \cup I_1) \\ &= I_2 \cup I_{10}. \end{aligned}$$

Therefore, $C_{\Delta(6)} = E_{\Delta(6)^\perp}$ has parameters $[15, 3, 11]_4$. \square

As for RS codes, we can extend BCH codes by evaluating at 0 as well. Then we obtain a code with parameters $[n + 1, n + 1 - \sum_{j=0}^t i_{m_j}, \geq m_{t+1} + 1]_{p^r}$. In this paper, we only consider extension degree 1 or 2 (i.e., $\ell = r$ or $\ell = 2r$) and the length n of our codes will be $p^\ell - 1$ or p^ℓ .

3. EAQECC

We compute in this section parameters of entanglement-assisted quantum error correcting codes coming from RS codes and BCH codes. As we have mentioned in the previous section, for easing the notation, we consider RS codes as BCH codes with extension degree 1. That is, for RS codes one has that $r = \ell$, $n = p^\ell - 1 = p^r - 1$, and all the cyclotomic cosets have size 1.

From Corollary 1 and Theorem 4 in [11], we have

Theorem 3. *Let E be a linear code over \mathbb{F}_q (over \mathbb{F}_{q^2}) with length n and dimension k and let C be its Euclidean (Hermitian) dual code that has minimum distance d . Then there exists an EAQECC with parameters*

$$[[n, n - 2k + c, d; c]]_q,$$

where $c = \dim E - \dim(E \cap C)$.

This implies that, for $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j}$, there exists an EAQECC with parameters

$$[[n, n - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c]]_q.$$

Hence, the only task for completely determining its parameters remains computing c .

It is not feasible to give a formula for c in the general case. The aim of this article is to provide closed formulas for extension degree equal to 1 (RS codes) and 2. Thus, assuming that we do not evaluate at 0, in the first case $\ell = r$, $n = q - 1$ when considering codes over \mathbb{F}_q and Euclidean duality and $n = q^2 - 1$ when we use codes over \mathbb{F}_{q^2} and Hermitian duality, and in the second case $\ell = 2r$, $n = q^2 - 1$ when we consider subfield subcodes over \mathbb{F}_q from codes over \mathbb{F}_{q^2} and Euclidean duality and $n = q^4 - 1$ when we use subfield-subcodes over \mathbb{F}_{q^2} from codes over \mathbb{F}_{q^4} and Hermitian duality.

For RS codes, one can find a formula for c in [22] for the Euclidean case and a partial result for the Hermitian case. We give a general formula for the Hermitian case in the next section. First, we need to introduce some notation.

Definition 4. We say that a minimal cyclotomic coset is symmetric if $I_x^\perp = I_{n-x} = I_x$ in the Euclidean case ($I_x^\perp = I_{n-qx} = I_x$, in the Hermitian case), and asymmetric otherwise. Let I_x be asymmetric and $I_y = I_{n-x}$ ($I_y = I_{n-qx}$, in the Hermitian case). Moreover, assume that x and y are the minimal representatives of I_x and I_y , respectively. Without loss of generality, we may assume that $x < y$, then we say that I_x and I_y are asymmetric reciprocal cosets, that I_x is the first reciprocal asymmetric coset (FR-asymmetric coset) and that I_y is the second reciprocal asymmetric coset (SR-asymmetric coset).

Let I_R consists of the asymmetric cosets in Δ whose reciprocal coset does not belong to Δ and I_L of the symmetric cosets in Δ and the asymmetric cosets whose reciprocal coset belongs to Δ as well. We have that $\Delta = \Delta(t) = I_R \sqcup I_L$ (that is, $\Delta = I_R \cup I_L$ and $I_R \cap I_L = \emptyset$), and in this way it holds that $E_{I_R} = E_\Delta \cap E_\Delta^\perp$ is the hull, or radical, of E_Δ since $E_\Delta^\perp = E_{\Delta^\perp}$ (see the paragraph after Proposition 2). Note that I_R and I_L are a union of minimal cyclotomic cosets. In this way, the value c is given by $\#I_L$ (see Section 2 in [11] for more details) since

$$c = \dim E_\Delta - \dim(E_\Delta \cap C_\Delta) = \#\Delta - \#(\Delta \cap \Delta^\perp) = \#\Delta - \#(I_R) = \#(I_L).$$

For a cyclotomic coset I_x that belongs to I_R , we have that $I_x \subseteq \Delta^\perp$ and, on the other hand, for a cyclotomic coset I_x that belongs to I_L , we have that $I_x \not\subseteq \Delta^\perp$. Therefore, for obtaining regular quantum codes, one considers a set Δ such that $I_L = \emptyset$ and hence $E_\Delta \subseteq E_\Delta^\perp$. On the contrary, for constructing LCD codes, one considers a set Δ such that $I_R = \emptyset$ and hence $E_\Delta \cap E_\Delta^\perp = \{0\}$.

Remark 5. Notice that since we are considering consecutive minimal cyclotomic cosets for constructing our codes, the cardinality of I_L will be given by the cardinality of the symmetric cosets in Δ plus two times the cardinality of the SR-asymmetric cosets in Δ .

Example 3. This is a continuation of Examples 1 and 2, with $p = 2$, $\ell = 4$, $r = 2$, $q^2 = 4$, $n = 15$, $t = 6$ and $m_6 = 7$, where we consider the Hermitian inner product.

The symmetric cosets in \mathbb{Z}_{15} are $I_0 = \{0\}$, $I_5 = \{5\}$, $I_{10} = \{10\}$ and the pairs of asymmetric cosets are: $I_1 = \{1, 4\}$ (FR-asymmetric) and $I_7 = \{7, 13\}$ (SR-asymmetric), $I_2 = \{2, 8\}$ (FR-asymmetric) and $I_{11} = \{11, 14\}$ (SR-asymmetric), and $I_3 = \{3, 12\}$ (FR-asymmetric) and $I_6 = \{6, 9\}$ (SR-asymmetric).

Thus, for $\Delta = \Delta(6) = I_0 \cup I_1 \cup I_2 \cup I_3 \cup I_5 \cup I_6 \cup I_7 = I_R \sqcup I_L$, we have that

$$I_R = I_2, \text{ and } I_L = I_0 \cup I_1 \cup I_3 \cup I_5 \cup I_6 \cup I_7,$$

because $I_L = I_2$ since I_2 is an FR-asymmetric coset whose reciprocal coset, I_{11} , does not belong to Δ and because I_L is a union of symmetric cosets, I_0 and I_5 , and pairs of asymmetric cosets, I_1 and I_7 and I_3 and I_6 included in Δ . Therefore, $c = \#I_L = 10$. Moreover, by Remark 5, one can alternatively compute c by considering the cardinality of the symmetric cosets, I_1 and I_5 , plus two times the cardinality of the SR-asymmetric cosets, I_6 and I_7 in Δ : $1 + 1 + 2(2 + 2) = 10$. This second form of computing c is the method that we follow in the rest of the article to obtain a general formula for c . In particular, for this code, c may be computed following our forthcoming Theorem 26, case (2), since $(q^4 - 1)/(q + 1) = 5 \leq m_t < q^3 + q = 10$.

Finally, one has that considering $\Delta = \Delta(6)$, one obtains an EAQECC with parameters

$$\left[\left[n, n - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c \right] \right]_2 = [[15, 1, \geq 11; 10]]_2.$$

□

Furthermore, we can also consider RS codes and BCH codes where we evaluate at zero to construct EAQECC, that is, the ideal J at the beginning of Section 2 is generated by $X^{n+1} - X$. In this case, we obtain an EAQECC with parameters

$$\left[\left[n + 1, n - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c - 1 \right] \right]_q,$$

because the length is increased by one unit and the parameter c is decreased by one unit since the cyclotomic coset $I_0 = \{0\}$ is symmetric when we do not evaluate at zero and it is FR-asymmetric otherwise by [9] (because the reciprocal coset of $I_0 = \{0\}$ is I_n that is not contained in \mathcal{H}). The dimension of the EAQECC remains therefore the same as before. This allows us to increase the constellation of codes that we may construct.

4. HERMITIAN RS CODES

We give in this section a general formula for c in the case of EAQECCs coming from RS codes with respect to the Hermitian metric.

Theorem 6. *Let $\Delta = \{0, 1, \dots, t\} \subseteq \mathcal{H}$ and consider the RS code $D_\Delta = E_\Delta$ over \mathbb{F}_{q^2} . Let $b_0 + b_1q$ be the q -adic expression of t , then the parameters of the corresponding EAQECC are:*

$$[[q^2 - 1, (q - b_1)^2 - 2b_0 - 2, t + 2; b_1^2 + 1]]_q$$

when $b_0 + b_1 < q - 1$. And

$$[[q^2 - 1, (q - b_1 - 1)^2, t + 2; b_1^2 + 2(b_0 + b_1 - q) + 4]]_q$$

otherwise ($b_0 + b_1 \geq q - 1$).

Proof. We start by noticing that, in this case and with the notation as in Section 2, every considered coset has cardinality one and $m_t = t$. By Theorem 3, it suffices to compute the value c corresponding to the entanglement and by Remark 5 we need to decide which values a in Δ determine a symmetric coset (contributing one to c) and which ones determine SR-asymmetric cosets (contributing two in the computation of the value c).

Let $a \in \Delta$, with q -adic expression $a = a_0 + a_1q$, then a represents an SR-asymmetric (respectively, a symmetric) coset if and only if $a_0 + a_1 < q - 1$ (respectively, $a_0 + a_1 = q - 1$). Indeed, a represents an SR-asymmetric (respectively, a symmetric) coset if and only if $(q^2 - 1) - qa < a$ (respectively, $(q^2 - 1) - qa = a$). Noticing that the q -adic expression of $q^2 - 1$ is $(q - 1)q + (q - 1)$ and that of qa is $qa = a_1 + a_0q$, the result follows straightforwardly.

Then for computing c we only need to compute twice number of values (in Δ) $a = a_0 + a_1q$, $0 \leq a_0, a_1 \leq q - 1$ such that $a_0 + a_1 \geq q - 1$ plus the number of values (in Δ) $a = a_0 + a_1q$, $0 \leq a_0, a_1 \leq q - 1$ such that $a_0 + a_1 = q - 1$.

Δ contains the number 0 whose coset is symmetric, thus we add 1 to our computations.

The number of elements a giving rise to symmetric cosets satisfying $a_1 \leq b_1 - 1$ equals to b_1 because each value a_1 determines the corresponding a_0 .

Next, we compute how many values a satisfy $a_0 + a_1 \geq q - 1$ and $a_1 \leq b_1 - 1$. Since for any fixed a_1 , it holds that a_0 satisfies $q - 1 - a_1 < a_0 \leq q - 1$, the number of solutions is $\sum_{a_1=0}^{b_1-1} a_1$. This sum equals $(1/2)b_1(b_1 - 1)$ giving a contribution to c of $b_1(b_1 - 1)$. We have proved the first part of the result because we have obtained $c = 1 + b_1(b_1 - 1) + b_1 = 1 + b_1^2$. Note that we cannot obtain $a_0 + a_1 \geq q - 1$ when $b_0 + b_1 < q - 1$ and $a_1 = b_1$.

For proving our second statement, where $b_0 + b_1 \geq q - 1$, it suffices to add to the value $1 + b_1^2$ twice the number of integers a with q -adic expression $a_0 + b_1q$, $a_0 \leq b_0$, such that $a_0 + b_1 > q - 1$ plus the number of integers a with q -adic expression $a_0 + b_1q$ such that $a_0 + b_1 = q - 1$, which is exactly one. As a consequence,

$$c = 1 + b_1^2 + 2(b_0 + b_1 - q + 1) + 1 = b_1^2 + 2(b_0 + b_1 - q) + 4.$$

□

Considering the ideal J introduced in Section 2 but now generated by $X^{n+1} - X$ instead of $X^n - 1$, a very similar argument proves the following result.

Theorem 7. *Let $\Delta = \{0, 1, \dots, t\} \subseteq \mathcal{H}$ and consider the RS code $D_\Delta = E_\Delta$ over \mathbb{F}_{q^2} . Set $t = b_0 + b_1q$, then the parameters of the corresponding EAQECC are:*

$$[[q^2, (q - b_1)^2 - 2b_0 - 2, t + 2; b_1^2]]_q$$

when $b_0 + b_1 < q - 1$. And

$$[[q^2, (q - b_1 - 1)^2, t + 2; b_1^2 + 2(b_0 + b_1 - q) + 3]]_q$$

otherwise ($b_0 + b_1 \geq q - 1$).

Remark 8. In [22, Theorem 5], parameters for EAQECCs with length q^2 coming from Reed-Solomon codes were given for some intervals. Hence, our Theorem 7 extends [22, Theorem 5] in the sense that we consider all possible cases.

5. EUCLIDEAN BCH EAQECC WITH EXTENSION DEGREE 2

We consider in this section BCH codes over \mathbb{F}_q with extension degree equal to 2. Hence, the cyclotomic cosets have one or two elements. Namely, $n = q^2 - 1$, $q = p^r$ and $l = 2r$, and the cyclotomic coset whose minimal representative is x is equal to $I_x = \{x, xq\}$. In

addition, its reciprocal cyclotomic coset is $I_{n-xq} = \{n - xq, n - x\}$, where $n - xq$ is the minimal representative of I_{n-xq} since $x \leq xq$. Moreover, I_x has cardinality 1 if and only if $x = xq$.

Lemma 9. *Let I_x be such that x is its minimal representative. One has that*

- I_x is an FR-asymmetric coset if and only if $n - xq > x$.
- I_x is a symmetric coset if and only if $n - xq = x$.
- I_x is an SR-asymmetric coset if and only if $n - xq < x$.

Proof. We have that I_{n-xq} is the reciprocal coset of I_x and that $n - xq$ is the minimal representative of I_{n-xq} . Hence, one has that

$$I_x = I_{n-x} \text{ if and only if } n - xq = x.$$

Let I_x be asymmetric, then $x \neq n - xq$ (otherwise it would be symmetric). Then I_x is an SR-asymmetric cyclotomic coset if $x > n - xq$, otherwise it is an FR-asymmetric cyclotomic coset. □

The following proposition characterizes symmetric and SR-asymmetric cosets by the q -adic representation of its minimal representative.

Proposition 10. *Let I_x be such that x is its minimal representative. Let $x = a_0 + a_1q$ with $0 \leq a_0, a_1 < q$, the q -adic representation of x . Then*

- I_x is a symmetric coset if and only if $a_0 + a_1 = q - 1$.
- I_x is an SR-asymmetric coset if and only if $a_0 + a_1 > q - 1$.

Proof. By Lemma 9, we have that I_x is symmetric if and only if $n - xq = x$. That is, if $n - (a_0 + a_1q)q = a_0 + a_1q$, that is equivalent to $n = (a_0 + a_1)(q + 1)$ since q^2 is equivalent to 1 modulo $q^2 - 1$. Moreover, since $n = q^2 - 1$, we have that I_x is a symmetric coset if $a_0 + a_1 = q - 1$.

By Lemma 9, we have that I_x is an SR-asymmetric coset if and only if $n - xq < x$. That is, if $n - (a_0 + a_1q)q < a_0 + a_1q$, that is equivalent to $n < (a_0 + a_1)(q + 1)$. Thus, we have that I_x is an SR-symmetric coset if and only if $a_0 + a_1 > q - 1$. □

For computing c , first we compute how many minimal cyclotomic cosets are symmetric or SR-asymmetric. Afterwards, we will determine the number of symmetric and FR-asymmetric cosets and their cardinality. The coset I_0 is always in I_L since it is symmetric and hence, it will not be considered in the following computations. We have a characterization of the cyclotomic cosets in I_L by Proposition 10, namely $a_0 + a_1 \geq q - 1$ (i). Moreover, given a cyclotomic coset I_x we should check whether the cyclotomic coset I_x is in $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j}$, that is, we should check that $x \leq m_t$ (ii). Finally we should consider that x is a minimal representative of I_x , which is equivalent to $a_0 \geq a_1$ (iii), since $x = a_0 + a_1q \leq a_1 + a_0q = xq$ if and only if $a_0 \geq a_1$.

Summarizing, setting $b_0 + b_1q$ the q -adic expression of m_t , we have to count the number of elements $a_0 + a_1q$, with $0 \leq a_0, a_1 \leq q - 1$ (and $(a_0, a_1) \neq (0, 0)$), such that

- (i) $a_0 + a_1 \geq q - 1$,
- (ii) $a_0 + a_1q \leq b_0 + b_1q$, and
- (iii) $a_0 \geq a_1$.

We first consider the case $a_1 < b_1$. Note that in this case, condition (ii) always holds. Our strategy consists of computing the number of pairs that satisfy (i) and then subtracting the number of pairs that do not satisfy (iii). Let $a_1 = i$, with $i \in \{0, 1, \dots, b_1 - 1\}$. We

have that (i) is equivalent to $q - 1 - i \leq a_0 \leq q - 1$ and hence the number of possible values for a_0 is $i + 1$ and the total number of possible values for a_0 and a_1 is

$$(1) \quad \sum_{i=0}^{b_1-1} i + 1 = \frac{b_1(b_1 + 1)}{2}.$$

Now we consider pairs that do not satisfy equation (iii), that is, $a_0 < a_1$. Again, let $a_1 = i$, with $i \in \{0, 1, \dots, b_1 - 1\}$, thus we have that $q - 1 - i \leq a_0 \leq i - 1$. Note that the previous inequalities are satisfied only when $(i - 1) - (q - 1 - i) + 1 = 2i - q + 1 \geq 1$, that is, when $i \geq q/2$. This is equivalent to $i \geq q/2$ when the characteristic is 2 and $i \geq (q + 1)/2$ when the characteristic is odd. Summarizing, both inequalities are equivalent to $i \geq \lceil q/2 \rceil$ in arbitrary characteristic. Thus, the total number of pairs that satisfy (i) and do not satisfy (iii) is equal to

$$\sum_{i=\lceil q/2 \rceil}^{b_1-1} 2i - q + 1 = \begin{cases} 0, & \text{if } b_1 \leq \lceil q/2 \rceil, \\ (b_1 - q/2)^2, & \text{if } q \text{ is even and } b_1 > \lceil q/2 \rceil, \\ (b_1 - (q + 1)/2)(b_1 - (q + 1)/2 + 1), & \text{if } q \text{ is odd and } b_1 > \lceil q/2 \rceil. \end{cases}$$

Which, in arbitrary characteristic, equals

$$(2) \quad \left(\max \left\{ 0, b_1 - \left\lceil \frac{q}{2} \right\rceil \right\} \right) \left(b_1 - \left\lfloor \frac{q}{2} \right\rfloor \right).$$

Let us now consider the case $a_1 = b_1$. In this case, condition (i) is equivalent to $a_0 \geq q - 1 - b_1$, condition (ii) is equivalent to $a_0 \leq b_0$, and condition (iii) is equivalent to $a_0 \geq b_1$. Summarizing we have that

$$\max\{q - 1 - b_1, b_1\} \leq a_0 \leq b_0.$$

Thus, for $a_1 = b_1$ the number of pairs that satisfy (i)-(iii) is

$$(3) \quad \max\{0, b_0 - \max\{q - 1 - b_1, b_1\} + 1\}.$$

As mentioned, we have counted the number of cyclotomic cosets that are symmetric or SR-asymmetric, but we do not know yet their cardinality and how many of them are symmetric and SR-asymmetric. With respect to the number of symmetric cosets of cardinality one, we have that $x = xq$ and $n - x = x$, hence $(q^2 - 1) - x = x$ that implies that $x = (q^2 - 1)/2$. Therefore, there is a unique symmetric coset with cardinality one, I_x with $x = (q^2 - 1)/2$, if the characteristic of \mathbb{F}_q is different from 2. Otherwise, there is no symmetric coset with cardinality one. Therefore, the number of symmetric cosets of cardinality one in Δ is

$$(4) \quad \begin{cases} 0, & \text{if } q \text{ is even} \\ 1, & \text{if } q \text{ is odd and } m_t \geq (q^2 - 1)/2. \end{cases}$$

We compute now the number of SR-asymmetric cosets that have cardinality one. For an asymmetric coset (FR or SR) with cardinality one, we have that $x = xq$, hence $x(q - 1)$ is zero modulo $q^2 - 1$, and this implies that x is a multiple of $(q + 1)$. Therefore, the number of asymmetric cosets with cardinality one is $\lfloor (m_t)/(q + 1) \rfloor$. However, we are interested in the number of SR-asymmetric cosets with cardinality one. To compute it, we should subtract the number of FR-asymmetric cosets with cardinality one. Actually, among the $q - 2$ multiples of $q + 1$ (we recall that we do not consider 0) the first half are minimal representatives of FR-asymmetric cosets and the second half are the minimal representatives of SR-asymmetric cosets. Thus, if the characteristic of \mathbb{F}_q is even, there are $(q - 2)/2$ FR-symmetric cosets with cardinality one. If the characteristic of \mathbb{F}_q is odd, we should take into account that the coset $\{(q^2 - 1)/2\}$ is the only symmetric coset with

cardinality one. Hence, there are $(q-3)/2+1$ FR-asymmetric cosets with cardinality one. Therefore, for arbitrary characteristic, the number of SR-asymmetric cosets with cardinality one in Δ is

$$(5) \quad \max \left\{ 0, \left\lfloor \frac{m_t}{q+1} \right\rfloor - \left\lfloor \frac{q-1}{2} \right\rfloor \right\}.$$

Finally we compute the number of symmetric cosets with cardinality 2. In this case, $n-xq=x$, hence $n=x(q+1)$ and therefore x is a multiple of $q-1$. Since we are not considering 0, there are q possible multiples. However, not all of them are minimal representatives, and only a half of them will be minimal representatives of a coset. Hence, the number of symmetric cosets with cardinality 2 in Δ is

$$(6) \quad \min \left\{ \left\lfloor \frac{m_t}{q-1} \right\rfloor, \left\lfloor \frac{q}{2} \right\rfloor \right\}.$$

Thus, we are ready to state and prove the main result in this section.

Theorem 11. *Consider the BCH code E_Δ over the field \mathbb{F}_q with extension degree 2 and length q^2-1 given by $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j} \subseteq \mathcal{H}$. Set $b_0 + b_1 t$ the q -adic expression of m_t . Then the parameters of the corresponding EAQECC are*

$$\left[\left[q^2-1, q^2-1-2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c \right] \right]_q,$$

where c is equal to

$$\begin{aligned} c = & 1 + 4 \left(\frac{b_1(b_1+1)}{2} - \left(\max \left\{ 0, b_1 - \left\lfloor \frac{q}{2} \right\rfloor \right\} \right) \left(b_1 - \left\lfloor \frac{q}{2} \right\rfloor \right) \right. \\ & \left. + \max \{ 0, b_0 - \max \{ q-1-b_1, b_1 \} + 1 \} \right) \\ & - 3\delta - 2 \left(\max \left\{ 0, \left\lfloor \frac{m_t}{q+1} \right\rfloor - \left\lfloor \frac{q-1}{2} \right\rfloor \right\} \right) - 2 \left(\min \left\{ \left\lfloor \frac{m_t}{q-1} \right\rfloor, \left\lfloor \frac{q}{2} \right\rfloor \right\} \right), \end{aligned}$$

and where δ is equal to 1 if q is odd and $m_t \geq (q^2-1)/2$, and it is equal to 0 otherwise.

Proof. By Remark 5, we have that c is equal to the sum of the cardinality of the symmetric cosets in Δ plus two times the cardinality of the SR-asymmetric cosets in Δ .

The coset $I_0 = \{0\}$ is symmetric and it always contributes with 1 to the value of c . Consider the above referenced values from (1) to (6). Then (1)–(2)+(3) are the number of symmetric and SR-asymmetric cosets in Δ . If they were all SR-asymmetric cosets with cardinality 2, they would contribute with 4 times (1)–(2)+(3) to the value c . However, this may not be the case and we should adjust the previous computation.

Note that a coset with cardinality one contributes 1 to c if it is symmetric and it contributes 2 to c if it is SR-asymmetric. Finally, a coset with cardinality 2 that is symmetric contributes 2 to c .

Therefore,

$$c = 1 + 4 \times ((1) - (2) + (3)) - 3 \times (4) - 2 \times (5) - 2 \times (6),$$

and the result holds. \square

If one constructs the above codes as described in Section 2 but with the ideal J generated by $X^{n+1} - X$ instead of $X^n - 1$, a very similar argument proves the following result.

Theorem 12. Consider the BCH code E_Δ over the field \mathbb{F}_q with extension degree 2 and length q^2 given by $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j} \subseteq \mathcal{H}$. Set $b_0 + b_1 t$ the q -adic expression of m_t . Then the parameters of the corresponding EAQECC are

$$\left[\left[q^2, q^2 - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c \right] \right]_q,$$

where c is equal to

$$\begin{aligned} c = & 4 \left(\frac{b_1(b_1 + 1)}{2} - \left(\max \left\{ 0, b_1 - \left\lceil \frac{q}{2} \right\rceil \right\} \right) \left(b_1 - \left\lfloor \frac{q}{2} \right\rfloor \right) \right. \\ & \left. + \max \{ 0, b_0 - \max \{ q - 1 - b_1, b_1 \} + 1 \} \right) \\ & - 3\delta - 2 \left(\max \left\{ 0, \left\lfloor \frac{m_t}{q+1} \right\rfloor - \left\lfloor \frac{q-1}{2} \right\rfloor \right\} \right) - 2 \left(\min \left\{ \left\lfloor \frac{m_t}{q-1} \right\rfloor, \left\lfloor \frac{q}{2} \right\rfloor \right\} \right), \end{aligned}$$

and where δ is equal to 1 if q is odd and $m_t \geq (q^2 - 1)/2$, and it is equal to 0 otherwise.

Note that considering an RS code over the finite field with q^2 elements and the Hermitian metric produces an EAQECC with better parameters than the corresponding one obtained with a BCH over the field with q elements and the Euclidean metric, because both have the same length and bound for the minimum distance and the dimension is larger for the RS code with the Hermitian metric. However, we notice that it is possible to obtain a BCH code with a higher value for c , which increases the constellation of known EAQECCs.

6. HERMITIAN BCH EAQECC WITH EXTENSION DEGREE 2

We consider now BCH codes over \mathbb{F}_{q^2} and the Hermitian inner product to construct EAQECCs. Set $\ell = 2r$ and $n = q^4 - 1$ in this section. Thus, the cyclotomic coset with minimal representative x is equal to $I_x = \{x, q^2x\}$, and the reciprocal cyclotomic coset of I_x is equal to I_{n-qx} .

We can again characterize symmetric cosets and SR-asymmetric cosets in $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j}$, but such a characterization is not as precise as for the Euclidean case for cosets with cardinality 2. Let x be the minimal representative of I_x with cardinality 2. In the Euclidean case, we have that $n - qx$ is the minimal representative of its reciprocal coset I_{n-x} . However, in the Hermitian case the reciprocal coset of I_x is $I_{n-qx} = \{n - qx, n - q^3x\}$ and we do not know a priori who is its minimal representative because any one of them can be the smallest element modulo $q^4 - 1$.

Let (a_0, a_1, a_2, a_3) be the q -adic expansion of x , i.e., $x = a_0 + a_1q + a_2q^2 + a_3q^3$, with $0 \leq a_i < q$ for $0 \leq i \leq 3$. We denote in this section by x the 4-tuple (a_0, a_1, a_2, a_3) as well. Note that $y = q^2x$ has q -adic expansion (a_2, a_3, a_0, a_1) . We begin by studying the cosets with cardinality one since they can be easily characterized.

Lemma 13. $I_x = \{x\}$ holds if and only if $(q^2 + 1) \mid x$. Moreover, I_x has cardinality 1 if and only if the q -adic expansion of x is of the form (a, b, a, b) , with $0 \leq a, b < q$.

Proof. $I_x = \{x\}$ if and only if $x = q^2x$. That is, $x(q^2 - 1) = 0$ modulo $n = q^4 - 1$. Let $x = (a_0, a_1, a_2, a_3)$, since $I_x = \{x\}$ we have that $x = q^2x$ and $q^2(a_0, a_1, a_2, a_3) = (a_2, a_3, a_0, a_1)$. Therefore $(a_0, a_1, a_2, a_3) = (a_2, a_3, a_0, a_1)$ and the result holds. \square

Next, we characterize symmetric and SR-asymmetric cosets with cardinality one.

Lemma 14. Let $x = (a_0, a_1, a_2, a_3)$ with $I_x = \{x\}$, then

- I_x is a symmetric coset if and only if $a_2 + a_3 = q - 1$.
- I_x is an SR-asymmetric coset if and only if $a_2 + a_3 > q - 1$.

Proof. Since $I_x = \{x\}$, one has that I_x is symmetric if $n - qx = x$, that is, if

$$(q - 1, q - 1, q - 1, q - 1) - (a_3, a_2, a_3, a_2) = (a_2, a_3, a_2, a_3),$$

which is equivalent to $a_2 + a_3 = q - 1$. Analogously, I_x is an SR-asymmetric coset if $n - qx < x$, that is, if

$$(q - 1, q - 1, q - 1, q - 1) - (a_3, a_2, a_3, a_2) < (a_2, a_3, a_2, a_3),$$

which is equivalent to $a_2 + a_3 > q - 1$. \square

Thus, we have completely characterized the cosets with cardinality one. We consider now cosets with cardinality two. As we have mentioned before, its study will be more elaborated and complicated, although we have some positive news described in the following result.

Lemma 15. *There is no symmetric coset with cardinality 2 in the Hermitian case.*

Proof. Let I_x be with cardinality 2 and symmetric, then $n - qx = q^2x$ or $n - qx = x$. Let us show, by contradiction, that these equalities do not hold. If $n - qx = q^2x$, then $q^4 - 1 = q(q - 1)x$ which implies that q divides $q^4 - 1$, contradiction.

If $n - qx = x$, then $n = (q + 1)x$. This implies that $x = (q - 1)(q^2 + 1) = -1 + q - q^2 + q^3 = (q - 1, 0, q - 1, 0)$ and, by Lemma 13, I_x has cardinality 1, contradiction. \square

Hence, among the cosets with cardinality two, we should only count SR-asymmetric cosets.

Lemma 16. *Let I_x be with cardinality 2 and such that x is its minimal representative. Then I_x is an SR-asymmetric coset if and only if there exists $z \in I_x$ such that $n - qz < x$.*

Proof. Let $I_x = \{x, y\}$. Note that the minimal representative of the reciprocal cyclotomic coset of I_x is $n - qx$ or $n - qy$ and the result follows as in Lemma 9. \square

The following result is used to characterize which element is the minimal representative of the reciprocal coset of a coset with two elements.

Lemma 17. *Let $I_x = \{x, y\}$ be with cardinality 2 and such that $x = (a_0, a_1, a_2, a_3)$ is its minimal representative. One has that:*

- $qx > qy$ if and only if $a_2 > a_0$ or $a_2 = a_0$ and $a_1 > a_3$.
- $qx < qy$ if and only if $a_2 < a_0$ or $a_2 = a_0$ and $a_1 < a_3$.

Proof. We have that $qx = (a_3, a_0, a_1, a_2)$ and $qy = (a_1, a_2, a_3, a_0)$. We prove the case $qx > qy$, because the case $qx < qy$ is analogous. The inequality $qx > qy$ holds if and only if (a_3, a_0, a_1, a_2) is greater than (a_1, a_2, a_3, a_0) with respect to the lexicographical ordering. That is, if $a_2 > a_0$ or if $a_2 = a_0$ and $a_1 > a_3$. Note that since $qx \neq qy$, we cannot have that $a_2 = a_0$ and $a_1 = a_3$ and the result holds. \square

Next we characterize the cyclotomic cosets with two elements that are SR-asymmetric cosets.

Lemma 18. *Let $I_x = \{x, y\}$ with cardinality 2, where $x = (a_0, a_1, a_2, a_3)$ is its minimal representative. Then, I_x is an SR-asymmetric coset if and only if*

- either $a_2 + a_3 > q - 1$, or $a_2 + a_3 = q - 1$ and $a_1 + a_2 > q - 1$, if $qx > qy$,
- either $a_0 + a_3 > q - 1$, or $a_0 + a_3 = q - 1$ and $a_3 + a_2 > q - 1$, if $qx < qy$.

Proof. Let us assume that $qx > qy$, the case $qx < qy$ follows analogously. In this case, the minimal representative of I_{n-qx} is $n - qx$. We should compute when $n - qx \leq x$, that is, when $(q - 1, q - 1, q - 1, q - 1) - (a_3, a_0, a_1, a_2) \leq (a_0, a_1, a_2, a_3)$, since $n = (q - 1, q - 1, q - 1, q - 1)$. That is,

$$(7) \quad (q - 1, q - 1, q - 1, q - 1) \leq (a_3 + a_0) + (a_0 + a_1)q + (a_1 + a_2)q^2 + (a_2 + a_3)q^3.$$

We claim that (7) holds if $a_2 + a_3 > q - 1$, or if $a_2 + a_3 = q - 1$ and $a_1 + a_2 > q - 1$. Equivalently, we are claiming that if $a_2 + a_3 = q - 1$ and $a_1 + a_2 = q - 1$, then Inequality (7) does not hold. Indeed, in that case we would have that $a_1 = a_3$ and, by Lemma 17, $a_2 > a_0$ (since $qx > qy$). As a consequence, (7) is not true since $a_0 + a_1 < a_2 + a_1 = q - 1$. \square

The following definition is a key concept to study cosets with cardinality 2.

Definition 19. We define an *interlude* as the set of natural numbers that are between two consecutive cosets of cardinality one and that are minimal representatives of a coset with cardinality 2. By Lemma 13, the cyclotomic cosets of cardinality 1 are of the form $\{(a, b, a, b)\}$, with $0 \leq a, b < q$. Hence, an interlude is formed by integers x of the form

$$(a, b, a, b) < x < (a + 1, b, a + 1, b), \text{ with } 0 \leq a < q - 1 \text{ and } 0 \leq b < q,$$

or

$$(q - 1, b, q - 1, b) < x < (0, b + 1, 0, b + 1) \text{ with } 0 \leq b \leq q - 2, \text{ and } x < q^2x,$$

to ensure that x is the minimal representative. We denote the interlude bounded by (a, b, a, b) and $(a + 1, b, a + 1, b)$ by $[(a, b, a, b), (a + 1, b, a + 1, b)]_M$.

We now characterize the elements in an interlude. We will see that they are all consecutive which explains the name.

Lemma 20. *Let $0 \leq a < q - 1$ and $0 \leq b < q$, then $[(a, b, a, b), (a + 1, b, a + 1, b)]_M =$*

$$\{(i, j, a, b) : 0 \leq i < q \text{ and } b < j < q, \text{ or } a < i < q \text{ and } j = b\}.$$

Let $0 \leq b \leq q - 2$, then

$$[(q - 1, b, q - 1, b), (0, b + 1, 0, b + 1)]_M = \{(i, j, q - 1, b) : 0 \leq i < q \text{ and } b < j < q\}.$$

Proof. Let us consider $(a, b, a, b) < x < (a + 1, b, a + 1, b)$, then x can be written in two different ways:

- (i) $x = (i, j, a, b)$, with $0 \leq i < q$ and $b < j < q$, or $a < i < q$ and $j = b$.
- (ii) $x = (i, j, a + 1, b)$, with $0 \leq i < q$ and $0 \leq j < b$, or $0 \leq i < a + 1$ and $j = b$.

We claim that all elements of the form described in (i) are minimal representatives, and thus they are in an interlude, and that all the elements in (ii) are not minimal representatives, and hence they are not in an interlude. Then $x = (i, j, a, b)$ is the minimal representative of its cyclotomic coset if $x < q^2x$ and then $q^2(i, j, a, b) = (a, b, i, j) > (i, j, a, b)$ if and only if $b < j$ or $j = b$ and $a < i$. That is, we get all the elements in (i). Consider now $x = (i, j, a + 1, b)$, again x is the minimal representative of its coset if $q^2(i, j, a + 1, b) = (a + 1, b, i, j) > (i, j, a + 1, b)$, which implies that $b < j$, or that $a + 1 < i$ and $b = j$. Note that these conditions are not satisfied by any element in (ii).

We consider now $(q - 1, b, q - 1, b) < x < (0, b + 1, 0, b + 1)$, then x can be written in two different ways:

- (iii) $x = (i, j, q - 1, b)$, with $0 \leq i < q$ and $b < j < q$.
- (iv) $x = (i, j, 0, b + 1)$, with $0 \leq i < q$ and $0 \leq j < b + 1$.

Again, we claim that the elements in (iii) are in an interlude and all the elements in (iv) are not. One has that $x = (i, j, q-1, b)$ is the minimal representative of its cyclotomic coset if $x < q^2x$ and then $q^2(i, j, q-1, b) = (q-1, b, i, j) > (i, j, a, b)$ if and only if $b < j$. That is, we obtain all the elements in (iii). Consider now $x = (i, j, 0, b+1)$, again x is the minimal representative of its coset if $q^2(i, j, 0, b+1) = (0, b+1, i, j) > (i, j, 0, b+1)$, which implies that $b+1 < j$. This condition is not satisfied by any element in (iv), which concludes the proof. \square

Remark 21. Let I_x be with cardinality 2 and $x = (a_0, a_1, a_2, a_3)$ its minimal representative, then x is in the interlude $[(a_2, a_3, a_2, a_3), (a_2+1, a_3, a_2+1, a_3)]_M$ if $a_2 < q-1$, and in $[(q-1, a_3, q-1, a_3), (0, a_3+1, 0, a_3+1)]_M$ if $a_2 = q-1$.

FR-asymmetric cosets were described in [9, Theorem 3.11], where they were used to construct regular quantum codes:

Lemma 22. *Let $0 < x < (q^4 - 1)/(q + 1)$, then I_x is an FR-asymmetric coset and x is its minimal representative.*

Next we characterize the SR-asymmetric cyclotomic cosets whose minimal representative is in the interlude $[(q^4 - 1)/(q + 1), q^3 + q]_M$.

Lemma 23. *Let x be the minimal representative of I_x , a coset with cardinality 2, with $(q^4 - 1)/(q + 1) < x < q^3 + q$. Then I_x is an SR-asymmetric coset. Furthermore, there are $q^2 - q$ minimal representatives of SR-asymmetric cosets in the interlude $[(q^4 - 1)/(q + 1), q^3 + q]_M$*

Proof. Note that $(q^4 - 1)/(q + 1) = (q - 1, 0, q - 1, 0)$ and $q^3 + q = (0, 1, 0, 1)$. By Lemma 20, an element x in $[(q^4 - 1)/(q + 1), q^3 + q]_M$ is given by $x = (a_0, a_1, a_2, a_3) = (i, j, q-1, 0)$, with $0 \leq i < q$ and $0 < j < q$. Hence, there are $q(q-1)$ elements in the interlude that are minimal representatives of cosets, we claim that all these cosets are SR-asymmetric.

In fact, $qx = (0, i, j, q-1)$ and $qy = (j, q-1, 0, i)$, where $I_x = \{x, y\}$. Thus, $qx > qy$ and by Lemma 17, it holds that $a_2 > a_0$ or $a_2 = a_0$ and $a_1 > a_3$. By Lemma 18, the result follows because $a_2 + a_3 = q-1$ and $a_1 + a_2 > q-1$ since $(q-1) + 0 = (q-1)$ and $j + (q-1) > q-1$ (because $j > 0$). \square

Our aim in the following lemmas is to count the quantity of SR-asymmetric cosets in an arbitrary interlude. We divide our study in two cases, for $a_2 + a_3 < q-1$ and $a_2 + a_3 \geq q-1$. These results will be used when proving the main theorem.

Lemma 24. *There are $a_3(q - a_3)$ minimal representatives of SR-asymmetric cosets with cardinality 2 in the interlude $[(a_2, a_3, a_2, a_3), (a_2+1, a_2, a_2+1, a_3)]_M$, with $a_2 + a_3 < q - 1$.*

Proof. Let $x \in [(a_2, a_3, a_2, a_3), (a_2+1, a_2, a_2+1, a_3)]_M$. By Lemma 20, $x = (i, j, a_2, a_3)$, with $0 \leq i < q$ and $a_3 < j < q$, or $a_2 < i < q$ and $j = a_3$. Since $a_2 + a_3 < q - 1$, by Lemma 18 and Lemma 15, we have that I_x is an SR-asymmetric coset with cardinality two if and only if $qx < qy$. Therefore, we have that $qx < qy$. By Lemma 18, I_x is an SR-asymmetric coset if $i + a_3 > q - 1$ or if $i + a_3 = q - 1$ and $a_2 + a_3 > q - 1$. Therefore, I_x is an SR-asymmetric coset if $i + a_3 > q - 1$ because we are assuming that $a_2 + a_3 < q - 1$.

Let $a_3 < j < q$, then there are a_3 possible values for those i such that $i + a_3 > q - 1$, since $i < q$, namely $q - 1 - a_3 + 1, q - 1 - a_3 + 2, \dots, q - 1$. We consider now the case $j = a_3$, then there are also a_3 values i satisfying $i + a_3 > q - 1$, but we have a stronger restriction: $a_2 < i < q$ in this case, however, the assumption $a_2 + a_3 < q - 1$ implies that all possible values of i satisfy the restriction $a_2 < i < q$ as well. Summarizing, j can take

$q - a_3$ possible values and, for all of them, i may take a_3 values. Hence, there are $(q - a_3)a_3$ SR-asymmetric cosets with cardinality 2 in the given interlude. \square

Lemma 25. *There are $q^2 - qa_3 - a_2 - 1$ minimal representatives of SR-asymmetric cosets with cardinality 2 in the interlude $[(a_2, a_3, a_2, a_3), (a_2 + 1, a_2, a_2 + 1, a_3)]_M$, with $a_2 + a_3 \geq q - 1$, and in the interlude $[(q - 1, a_3, q - 1, a_3), (0, a_3 + 1, 0, a_3 + 1)]_M$ (considering $a_2 = q - 1$). Actually, every cyclotomic coset generated by an element in these interludes is an SR-asymmetric coset.*

Proof. We start by proving our first statement. Let $x \in [(a_2, a_3, a_2, a_3), (a_2 + 1, a_2, a_2 + 1, a_3)]_M$. By Lemma 20, $x = (i, j, a_2, a_3)$, with $0 \leq i < q$ and $a_3 < j < q$, or $a_2 < i < q$ and $j = a_3$. We consider first the case $qx > qy$. If $a_2 + a_3 > q - 1$ then I_x is an SR-asymmetric coset by Lemma 18. If $a_2 + a_3 = q - 1$, I_x is an SR-asymmetric coset if $j + a_2 > q - 1$ by Lemma 18. Note that $j + a_2 > q - 1$ if and only if $j > a_3$, by Lemma 20, since we are assuming that $a_2 + a_3 = q - 1$ in this case. We consider now the case $qx < qy$. By Lemma 17, $i > a_2$ since $j \geq a_3$. Moreover, since we assume that $a_2 + a_3 \geq q - 1$ and $i > a_2$, one has that $i + a_3 > q - 1$ and by Lemma 18, I_x is an SR-asymmetric coset. Finally, we determine how many minimal representatives of SR-asymmetric cosets are. For every j such that $a_3 < j < q$ we have q possible values for i (since $0 \leq i < q$), hence we have $(q - 1 - a_3)q$ minimal representatives. For $a_3 = j$, we have $q - a_2 - 1$ minimal representatives and then the first statement of this result holds.

We prove now the second statement. Assume that x is in $[(q - 1, a_3, q - 1, a_3), (0, a_3 + 1, 0, a_3 + 1)]_M$. By Lemma 20, $x = (a_0, a_1, a_2, a_3) = (i, j, q - 1, a_3)$, with $0 \leq i < q$ and $a_3 < j < q$, and hence there are $q(q - 1 - a_3)$ possible values for i and j . It remains to prove that I_x is an SR-asymmetric coset for all x . The case $a_3 = 0$ follows from Lemma 23, hence we can assume that $a_3 > 0$. Moreover, we have that $qx > qy$ by Lemma 17, since $a_2 = q - 1 \leq a_0$ and $a_1 = j > a_3$. Therefore, I_x is an SR-asymmetric coset by Lemma 18, because $a_2 + a_3 = q - 1 + a_3 > q - 1$ (since $a_3 > 0$). \square

We can now prove the main result that gives the parameters of the EAQECCs we construct in the Hermitian case.

Theorem 26. *Consider the BCH code E_Δ over the field \mathbb{F}_{q^2} with extension degree 2 and length $q^4 - 1$, where $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j} \subseteq \mathcal{H}$. Set (b_0, b_1, b_2, b_3) the q -adic expression of m_t . Then, considering the Hermitian inner product, the parameters of the corresponding EAQECC are*

$$\left[\left[q^4 - 1, q^4 - 1 - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c \right] \right]_q,$$

where

- (1) If $m_t < \frac{q^4 - 1}{q + 1}$, then $c = 1$.
- (2) If $\frac{q^4 - 1}{q + 1} \leq m_t < q^3 + q$, then $c = 2 + 4 \left(m_t - \frac{q^4 - 1}{q + 1} \right)$.
- (3) If $q^3 + q \leq m_t$, then

$$\begin{aligned}
c &= 1 + 4q(q-1) + b_3^2 + \max\{0, 2(b_2 + b_3 - q + 2) - 1\} \\
&+ 4 \left(\sum_{j=1}^{b_3-1} \left[(q-1-j)(q-j)j + (j+1)(q^2 - qj - 1) - \frac{(2(q-1)-j)(j+1)}{2} \right] \right. \\
&+ \delta (b_2 b_3 (q - b_3) + b_3 (b_1 - b_3) + \max\{0, b_0 + b_3 - q + 1\}) \\
&+ (1 - \delta) [(q-1-b_3)b_3(q-b_3) + (b_2 + b_3 - q + 1)((q^2 - qb_3 - 1) \\
&\quad \left. - (b_2 + q - b_3 - 2)/2) + (m_t - (b_2 + b_3 q + b_2 q^3 + b_3 q^3))] \right),
\end{aligned}$$

δ being 1 if $b_2 + b_3 < q - 1$ and $\delta = 0$ otherwise.

Proof. By Remark 5, we have that c is equal to the cardinality of the symmetric cosets in Δ plus two times the cardinality of the SR-asymmetric cosets in Δ . Note that the symmetric cosets of cardinality 1 contribute 1 to c and the SR-asymmetric cosets of cardinality 1 contribute 2 to c . The SR-asymmetric cosets of cardinality 2 contribute 4 to c and that there are no symmetric cosets with cardinality 2 by Lemma 15.

We have that the coset $I_0 = \{0\}$ is symmetric and it always contributes 1 to the value of c .

(1) Let $m_t < \frac{q^4-1}{q+1}$, then the results holds by Lemma 22.

(2) Let $(q^4 - 1)/(q + 1) \leq m_t < q^3 + q = (q^4 - 1)/(q + 1) + q^2 + 1$. By Lemma 13, the cyclotomic coset generated by $(q^4 - 1)/(q + 1)$ has cardinality 1 and it is symmetric by Lemma 14.

By Lemma 23, an element greater than $(q^4 - 1)/(q + 1)$ and smaller than or equal to m_t , that is a minimal representative of a cyclotomic coset, generates an SR-asymmetric coset. Hence there are $m_t - (q^4 - 1)/(q + 1)$ SR-asymmetric cosets.

(3) Let $q^3 + q \leq m_t$, by Lemma 23 we have $q(q - 1)$ minimal representatives of SR-asymmetric cosets in the interlude $[(q^4 - 1)/(q + 1), q^3 + q]_M$, that together with the symmetric coset with one element of $(q^4 - 1)/(q + 1)$ contribute

$$4q(q - 1) + 1$$

to c .

We count first the symmetric and SR-asymmetric cosets with cardinality 1, that is $I_x = \{x\}$ with $(i, j, i, j) = x \leq m_t = (b_0, b_1, b_2, b_3)$. Note that $(q - 1, 0, q - 1, 0)$ has been already considered in part (2). By Lemma 14, we have that I_x is symmetric if $i + j = q - 1$ and that I_x is an SR-asymmetric coset if $i + j > q - 1$.

- Let $1 \leq j < b_3$, we have that I_x is symmetric for $i = q - 1 - j$ and that is an SR-asymmetric coset for $q - j \leq i < q$. Hence, for $1 \leq j < b_3$, we have $b_3 - 1$ symmetric cosets and $\sum_{j=1}^{b_3-1} j = b_3(b_3 - 1)/2$ SR-asymmetric cosets.
- For the case $j = b_3$, we should consider $(i, b_3, i, b_3) \leq (b_0, b_1, b_2, b_3)$ that is equivalent to $i \leq b_2$ by Lemma 20. Moreover, one has that $i + b_3 \geq q - 1$. Hence, there are $\max\{0, b_2 + b_3 - q + 2\}$ possible values for i . Only one of them generates a symmetric coset and the others are SR-asymmetric cosets.

Hence, the cosets with cardinality one, excepting $(q^4 - 1)/(q + 1)$, contribute

$$\begin{aligned}
&b_3 - 1 + b_3(b_3 - 1) + \max\{0, 2(b_2 + b_3 - q + 2) - 1\} = \\
&-1 + b_3^2 + \max\{0, 2(b_2 + b_3 - q + 2) - 1\}
\end{aligned}$$

to c .

Now, we focus on the rest of the SR-asymmetric cosets with cardinality 2. We first count the number of different interludes that are smaller than the interlude that contains m_t and then count the number of SR-asymmetric cosets in each interlude. By Remark 21, $m_t = (b_0, b_1, b_2, b_3)$ is in the interlude $[(b_2, b_3, b_2, b_3), (b_2+1, b_3, b_2+1, b_3)]_M$ if $b_2 < q-1$ and in $[(q-1, b_3, q-1, b_3), (0, b_3+1, 0, b_3+1)]_M$ if $b_2 = q-1$. Thus, the interludes that we should count are of the form $[(i, j, i, j), (i+1, j, i+1, j)]_M$ or $[(q-1, j, q-1, j), (0, j+1, 0, j+1)]_M$ with $0 \leq i < q$ and $0 < j < b_3$, or $0 \leq i < b_2$ and $j = b_3$. We divide the study in two cases $j < b_3$ and $j = b_3$.

Let $j < b_3$. We divide again the study in two cases, the first one where we can use Lemma 24 and the second one where we can use Lemma 25.

- There are $q-1-j$ interludes of type $[(i, j, i, j), (i+1, j, i+1, j)]_M$, with $i+j < q-1$. By Lemma 24, they contain $(q-j)j$ SR-asymmetric cosets. Hence the case $j < b_3$ and $i+j < q-1$ contributes

$$4 \sum_{j=1}^{b_3-1} (q-1-j)(q-j)j$$

to c .

- We consider now the opposite case, $i+j \geq q-1$, where we know that every minimal representative in the interlude is an SR-asymmetric coset by Lemma 25. Namely, there are $q^2 - qj - i - 1$ elements in the interlude with first element (i, j, i, j) with $0 < j < b_3$ and $q-1-j \leq i < q$. Therefore, we have in total

$$\sum_{i=q-1-j}^{q-1} (q^2 - qj - i - 1) = (j+1)(q^2 - qj - 1) - \sum_{i=q-1-j}^{q-1} i,$$

which means that the case $j < b_3$ and $i+j \geq q-1$ contributes

$$4 \left((j+1)(q^2 - qj - 1) - \frac{(2(q-1) - j)(j+1)}{2} \right)$$

to c .

Let $j = b_3$. As above, we divide the study in two cases.

- We consider first the case $b_2 + b_3 < q-1$. Note that there are b_2 interludes of the form $[(i, j, i, j), (i+1, j, i+1, j)]_M$ with $0 \leq i < b_2$ and $j = b_3$ that are smaller than (b_2, b_3, b_2, b_3) . Moreover, by Lemma 24, they contain $b_3(q - b_3)$ minimal representatives.

Furthermore, we should count how many elements in the interlude containing m_t , $[(b_2, b_3, b_2, b_3), (b_2+1, b_3, b_2+1, b_3)]_M$ (recall Remark 21), are smaller than or equal to m_t . In this interlude the elements are (i, k, b_2, b_3) with $0 \leq i < q$ and $b_3 < k < q$, or $b_2 < i < q$ and $k = b_3$.

Those elements that are minimal representatives of an SR-asymmetric coset satisfy $i + b_3 > q-1$ by Lemma 18, because $b_2 + b_3 < q-1$ (we have that $qx > qy$ does not hold if $b_2 + b_3 < q-1$). Hence, combining both restrictions we have either $q - b_3 \leq i < q$ and $b_3 < k < q$ or $\max\{b_2 + 1, q - b_3\} \leq i < q$. Note that under the hypothesis that $b_2 + b_3 < q-1$, we have that $\max\{b_2 + 1, q - b_3\} = q - b_3$ holds. Hence, the restrictions for belonging to the interlude and generating an SR-asymmetric coset are either

$$q - b_3 \leq i < q \text{ and } b_3 < k < q$$

or

$$q - b_3 \leq i < q.$$

Moreover, these elements are smaller than or equal to m_t , that is, $k < b_1$ or $k = b_1$ and $i \leq b_0$. From these restrictions we get (note that $b_3 \leq b_1$ by Lemma 20):

$$q - b_3 \leq i < q \text{ and } b_3 < k < b_1,$$

or

$$q - b_3 \leq i \leq b_0 \text{ and } k = b_1,$$

or

$$q - b_3 \leq i < q \text{ and } k = b_3.$$

Thus the number of possible representatives of an SR-asymmetric coset is:

$$\begin{aligned} & b_3(b_1 - b_3 - 1) + \max\{0, b_0 + b_3 - q + 1\} + b_3 = \\ & b_3(b_1 - b_3) + \max\{0, b_0 + b_3 - q + 1\} \end{aligned}$$

Hence, this case contributes

$$4(b_2 b_3(q - b_3) + b_3(b_1 - b_3) + \max\{0, b_0 + b_3 - q + 1\})$$

to c .

- We consider now the case $b_2 + b_3 \geq q - 1$. Note that there are b_2 interludes of the form $[(i, j, i, j), (i + 1, j, i + 1, j)]_M$ with $0 \leq i < b_2$ and $j = b_3$ whose elements are smaller than (b_2, b_3, b_2, b_3) . For $0 \leq i < q - 1 - b_3$, we have that $i + b_3 < q - 1$, and by Lemma 24, they contain $b_3(q - b_3)$ minimal representatives, which means that the first $q - 1 - b_3$ interludes have a total of $(q - 1 - b_3)b_3(q - b_3)$ minimal representatives of SR-asymmetric cosets.

For the remaining cosets, $q - 1 - b_3 \leq i < b_2$, we have that $i + b_3 \geq q - 1$ holds and thus we should consider Lemma 25. Every minimal representative of these interludes generates an SR-asymmetric coset and there are $q^2 - qb_3 - i - 1$. Therefore, the number of minimal representatives in this case is

$$\begin{aligned} & \sum_{i=q-1-b_3}^{b_2-1} (q^2 - qb_3 - i - 1) = \\ & (b_2 + b_3 - q + 1)(q^2 - qb_3 - 1) - \sum_{i=q-1-b_3}^{b_2-1} i = \\ & (b_2 + b_3 - q + 1)(q^2 - qb_3 - 1) - (b_2 + q - b_3 - 2)(b_2 + b_3 - q + 1)/2. \end{aligned}$$

Furthermore, we should count how many elements in the interlude that contains m_t , $[(b_2, b_3, b_2, b_3), (b_2 + 1, b_3, b_2 + 1, b_3)]_M$, are smaller than or equal to m_t . Since, by Lemma 25, all elements in the interlude of m_t are minimal representatives of an SR-asymmetric coset, we conclude that there are $m_t - (b_2 + b_3q + b_2q^3 + b_3q^3)$ elements.

Hence, this case contributes

$$\begin{aligned} & (q - 1 - b_3)b_3(q - b_3) + (b_2 + b_3 - q + 1)((q^2 - qb_3 - 1) \\ & - (b_2 + q - b_3 - 2)/2) + (m_t - (b_2 + b_3q + b_2q^3 + b_3q^3)) \end{aligned}$$

to c .

n	k	$d \geq$	c	n	k	$d \geq$	c
80	75	3	1	80	71	4	1
80	67	5	1	80	63	6	1
80	47	11	1	80	45	12	1
80	41	13	1	80	37	14	1
80	21	18	1	80	17	21	1
80	16	22	2	80	16	23	6
80	16	24	10	80	16	25	14

TABLE 1. Parameters of EAQECCs over \mathbb{F}_3

Summing all contributions to c we get the formula given in the statement for this part (3). \square

As above, if the ideal J introduced in Section 2 is generated by $X^{n+1} - X$, with a very similar proof to that of the previous theorem, the following result holds.

Theorem 27. *Consider the BCH code E_Δ over the field \mathbb{F}_{q^2} with extension degree 2 and length q^4 , where $\Delta = \Delta(t) = \cup_{j=0}^t I_{m_j} \subseteq \mathcal{H}$. Set (b_0, b_1, b_2, b_3) the q -adic expression of m_t . Then the parameters of the corresponding EAQECC are*

$$\left[\left[q^4, q^4 - 2 \sum_{j=0}^t i_{m_j} + c, \geq m_{t+1} + 1; c \right] \right]_q,$$

where

- (1) If $m_t < \frac{q^4-1}{q+1}$, then $c = 0$.
- (2) If $\frac{q^4-1}{q+1} \leq m_t < q^3 + q$, then $c = 1 + 4 \left(m_t - \frac{q^4-1}{q+1} \right)$.
- (3) If $q^3 + q \leq m_t$, then

$$\begin{aligned} c = & 4q(q-1) + b_3^2 + \max\{0, 2(b_2 + b_3 - q + 2) - 1\} \\ & + 4 \left(\sum_{j=1}^{b_3-1} \left[(q-1-j)(q-j)j + (j+1)(q^2 - qj - 1) - \frac{(2(q-1)-j)(j+1)}{2} \right] \right. \\ & + \delta (b_2 b_3 (q - b_3) + b_3 (b_1 - b_3) + \max\{0, b_0 + b_3 - q + 1\}) \\ & + (1 - \delta) \left[(q-1-b_3)b_3(q-b_3) + (b_2 + b_3 - q + 1)((q^2 - qb_3 - 1) \right. \\ & \left. \left. - (b_2 + q - b_3 - 2)/2) + (m_t - (b_2 + b_3q + b_2q^3 + b_3q^3)) \right] \right), \end{aligned}$$

where $\delta = 1$ if $b_2 + b_3 < q - 1$ and $\delta = 0$ otherwise.

This family of codes contains codes with good parameters. As a sample, to finish this section and the paper, we present several tables, Tables 1, 2, 3 and 4, containing parameters of codes obtained with our formulae, over different finite fields, that exceed the Gilbert-Varshamov (GV) bound for EAQECCs [11].

Most of the EAQECCs in the literature (see Section 1) are binary or q -ary with length smaller than or equal to q^2 . Hence, we cannot compare our codes with them. Other articles about BCH codes consider just concrete subfamilies. To the best of our knowledge, the only article with EAQECCs having the same length as ours is [25]. There the authors

n	k	$d \geq$	c	n	k	$d \geq$	c
255	250	3	1	255	246	4	1
255	242	5	1	255	238	6	1
255	194	18	1	255	192	19	1
255	176	23	1	255	172	24	1
255	148	30	1	255	144	31	1
255	136	35	1	255	134	36	1
255	118	40	1	255	114	41	1
255	81	54	6	255	81	55	10
255	81	56	14	255	81	57	18
255	81	64	46	255	81	69	50
255	79	70	50	255	75	71	50

TABLE 2. Parameters of EAQECCs over \mathbb{F}_4

n	k	$d \geq$	c	n	k	$d \geq$	c
624	619	3	1	624	615	4	1
624	611	5	1	624	607	6	1
624	527	27	1	624	525	28	1
624	521	29	1	624	517	30	1
624	513	31	1	624	509	32	1
624	489	37	1	624	485	38	1
624	395	63	1	624	391	64	1
624	273	97	1	624	269	98	1
624	261	100	1	624	257	105	1
624	256	107	6	624	256	108	10
624	256	113	30	624	256	114	34
624	256	115	38	624	256	116	42

TABLE 3. Parameters of EAQECCs over \mathbb{F}_5

n	k	$d \geq$	c	n	k	$d \geq$	c
2400	2399	2	1	2400	2395	3	1
2400	2391	4	1	2400	2387	5	1
2400	2383	6	1	2400	2379	7	1
2400	2207	51	1	2400	2205	52	1
2400	2201	53	1	2400	2197	54	1
2400	2141	68	1	2400	2137	69	1
2400	1919	126	1	2400	1915	127	1
2400	1911	128	1	2400	1907	129	1
2400	1907	130	1	2400	1899	31	1
2400	1829	152	1	2400	1825	153	1
2400	1781	164	1	2400	1777	165	1
2400	1737	175	1	2400	1733	176	1

TABLE 4. Parameters of EAQECCs over \mathbb{F}_7

provide a few codes with length $q^4 - 1$. Indeed, for a given finite field \mathbb{F}_q , they give two codes, with parameters $[[q^4 - 1, q^4 - 1 - 5, 3; 1]]_q$ and $[[q^4 - 1, q^4 - 1 - 7, 4; 1]]_q$. These codes are constructed from almost MDS constacyclic codes using the Hartmann-Tzeng bound. We notice that those with minimum distance 3 are contained in the set of codes presented in this section but the ones with minimum distance 4 are not contained in the codes presented in this section, and have better parameters. Notice also that the cyclotomic cosets presented in this section are consecutive to bound the minimum distance with the BCH bound but the codes with minimum distance 4 in [25] are constructed from non-consecutive cyclotomic cosets.

REFERENCES

- [1] Jürgen Bierbrauer. The theory of cyclic codes and a generalization to additive codes. *Des. Codes Cryptogr.*, 25(2):189–206, 2002.
- [2] Todd Brun, Igor Devetak, and Min-Hsiu Hsieh. Correcting quantum errors with entanglement. *Science*, 314(5798):436–439, 2006.
- [3] Todd A. Brun, Igor Devetak, and Min-Hsiu Hsieh. Catalytic quantum error correction. *IEEE Trans. Inform. Theory*, 60(6):3073–3089, 2014.
- [4] A. Robert Calderbank, Eric M. Rains, P. W. Shor, and Neil J. A. Sloane. Quantum error correction via codes over $\text{GF}(4)$. *IEEE Trans. Inform. Theory*, 44(4):1369–1387, 1998.
- [5] Ignacio Cascudo. On squares of cyclic codes. *IEEE Trans. Inform. Theory*, 65(2):1034–1047, 2019.
- [6] Jianzhang Chen, Yuanyuan Huang, Chunhui Feng, and Riqing Chen. Entanglement-assisted quantum MDS codes constructed from negacyclic codes. *Quantum Inf. Process.*, 16(12):Paper No. 303, 22, 2017.
- [7] Xiaojing Chen, Shixin Zhu, and Xiaoshan Kai. Entanglement-assisted quantum negacyclic BCH codes. *Internat. J. Theoret. Phys.*, 58(5):1509–1523, 2019.
- [8] Zongrun Du, Chengju Li, and Sihem Mesnager. Constructions of self-orthogonal codes from hulls of BCH codes and their parameters. *IEEE Trans. Inform. Theory*, pages 1–1, 2020.
- [9] Carlos Galindo, Olav Geil, Fernando Hernando, and Diego Ruano. On the distance of stabilizer quantum codes from J -affine variety codes. *Quantum Inf. Process.*, 16(4):Art. 111, 32, 2017.
- [10] Carlos Galindo and Fernando Hernando. Quantum codes from affine variety codes and their subfield subcodes. *Des. Codes Cryptogr.*, 76:89–100, 2015.
- [11] Carlos Galindo, Fernando Hernando, Ryutaroh Matsumoto, and Diego Ruano. Entanglement-assisted quantum error-correcting codes over arbitrary finite fields. *Quantum Inf. Process.*, 18(4):Art. 116, 18, 2019.
- [12] Carlos Galindo, Fernando Hernando, and Diego Ruano. Stabilizer quantum codes from J -affine variety codes and a new Steane-like enlargement. *Quantum Inf. Process.*, 14(9):3211–3231, 2015.
- [13] Daniel Gottesman. Class of quantum error-correcting codes saturating the quantum Hamming bound. *Phys. Rev. A (3)*, 54(3):1862–1868, 1996.
- [14] Guanmin Guo and Ruihu Li. New entanglement-assisted quantum MDS codes derived from generalized Reed-Solomon codes. *Internat. J. Theoret. Phys.*, 59(4):1241–1254, 2020.
- [15] Avanti Ketkar, Andreas Klappenecker, Santosh Kumar, and Pradeep Kiran Sarvepalli. Nonbinary stabilizer codes over finite fields. *IEEE Trans. Inform. Theory*, 52(11):4892–4914, 2006.
- [16] Ching-Yi Lai and Alexei Ashikhmin. Linear programming bounds for entanglement-assisted quantum error-correcting codes by split weight enumerators. *IEEE Trans. Inform. Theory*, 64(1):622–639, 2018.
- [17] Yang Liu, Ruihu Li, Liangdong Lv, and Yuena Ma. Application of constacyclic codes to entanglement-assisted quantum maximum distance separable codes. *Quantum Inf. Process.*, 17(8):Paper No. 210, 19, 2018.
- [18] Liang-Dong Lü and Ruihu Li. Entanglement-assisted quantum codes constructed from primitive quaternary BCH codes. *Int. J. Quantum Inf.*, 12(3):1450015, 14, 2014.
- [19] Gaojun Luo and Xiwang Cao. Two new families of entanglement-assisted quantum MDS codes from generalized Reed-Solomon codes. *Quantum Inf. Process.*, 18(3):Paper No. 89, 12, 2019.
- [20] Gaojun Luo, Xiwang Cao, and Xiaojing Chen. MDS codes with hulls of arbitrary dimensions and their quantum error correction. *IEEE Trans. Inform. Theory*, 65(5):2944–2952, 2019.
- [21] Liangdong Lv, Ruihu Li, Qiang Fu, and Xueliang Li. Maximal entanglement entanglement-assisted quantum codes from quaternary BCH codes. In *2015 IEEE Advanced Information Technology, Electronic and Automation Control Conference (IAEAC)*, pages 709–713, 2015.

- [22] Francisco R. F. Pereira. Entanglement-assisted quantum codes from cyclic codes. *arXiv cs.IT*, 1911.06384, 2019.
- [23] Francisco R. F. Pereira, Ruud Pellikaan, Giuliano G. La Guardia, and Francisco M. de Assis. Application of complementary dual AG codes to entanglement-assisted quantum codes. In *2019 IEEE International Symposium on Information Theory (ISIT)*, pages 2559–2563, 2019.
- [24] Jianfa Qian and Lina Zhang. On MDS linear complementary dual codes and entanglement-assisted quantum codes. *Des. Codes Cryptogr.*, 86(7):1565–1572, 2018.
- [25] Jianfa Qian and Lina Zhang. Constructions of new entanglement-assisted quantum MDS and almost MDS codes. *Quantum Inf. Process.*, 18(3):Paper No. 71, 12, 2019.
- [26] Diego Ruano. The metric structure of linear codes. In *Singularities, algebraic geometry, commutative algebra, and related topics*, pages 537–561. Springer, Cham, 2018.
- [27] Mustafa Sariand Emre Kolotoğlu. An application of constacyclic codes to entanglement-assisted quantum MDS codes. *Comput. Appl. Math.*, 38(2):Paper No. 75, 13, 2019.

Current address: Carlos Galindo and Fernando Hernando: Instituto Universitario de Matemáticas y Aplicaciones de Castellón and Departamento de Matemáticas, Universitat Jaume I, Campus de Riu Sec. 12071 Castelló (Spain), Diego Ruano: IMUVA-Mathematics Research Institute, Universidad de Valladolid, 47011 Valladolid (Spain).

Email address: Galindo: galindo@uji.es; Hernando: carrillf@uji.es; Ruano: diego.ruano@uva.es