Universidad de Valladolid

FACULTAD DE CIENCIAS ECONÓMICAS Y EMPRESARIALES

DEPARTAMENTO DE ECONOMÍA APLICADA

TESIS DOCTORAL:

# Majorities based on differences: Consistency analysis and extensions
# Mayorías basadas en diferencias: análisis de la consistencia y extensiones

Presentada por Patrizia Pérez Asurmendi para optar al grado de Doctor con Mención Internacional por la Universidad de Valladolid

Año Académico 2013/2014

Dirigida por:
José Luis García Lapresta y Bonifacio Llamazares Rodríguez

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*A Pablo*

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# Contenido/Contents

# Introducción

La Teoría de la Elección Social centra su atención en el análisis del proceso de toma de decisiones colectivas. Teniendo en cuenta las opiniones o los gustos de los miembros de un grupo social sobre distintas alternativas, se deriva un juicio colectivo por medio de una regla de agregación que asigna una preferencia colectiva a las preferencias individuales.

Entre las reglas utilizadas comúnmente para la agregación de las preferencias individuales, destacan las *reglas de puntuación* y las *reglas mayoritarias*.

En las primeras, las alternativas reciben puntuaciones que dependen de la posición que tales alternativas ocupan en las preferencias individuales. A la hora de agregar, se suman las puntuaciones de las distintas alternativas y se ordenan en función de dichos resultados. Entre las diferentes reglas de puntuación destacan las reglas de *pluralidad*, *antipluralidad* y *Borda*.

En el caso de las reglas mayoritarias, los individuos expresan sus preferencias entre pares de alternativas. Dichas preferencias individuales se agregan por medio de reglas mayoritarias concretas, obteniéndose una comparación social por pares de las alternativas.

Ambos tipos de reglas proveen a las decisiones sociales de ciertas propiedades que resultan útiles a la hora de justificar el resultado social del proceso de votación. Las reglas mayoritarias evitan que las decisiones entre dos alternativas se vean influenciadas por los juicios sobre otras alternativas; dicho de otra manera, las reglas mayoritarias dan lugar a decisiones sociales que cumplen la propiedad arrowiana de ser independientes de alternativas irrelevantes [3].

Por su parte, en las reglas de puntuación las decisiones sociales son consistentes; es decir, no se producen ciclos ni intransitividades en las decisiones colectivas. Precisamente, las posibles inconsistencias en las decisiones sociales constituyen la gran desventaja de las reglas mayoritarias. Este problema fue apuntado ya en el siglo XVIII por Condorcet [18] en su conocida *Paradoja del Voto*. En el ejemplo clásico se describe la agregación de las preferencias de tres individuos sobre tres alternativas $x, y, z$. El primero de los individuos prefiere la alternativa $x$ a la $y$ y la $y$ a la $z$, el segundo la $y$ a la $z$ y la $z$ a la $x$ y el último la $z$ a la $x$ y la $x$ a la $y$, y se asume que votan entre los pares de alternativas siendo fieles a dichas preferencias. Como resultado de la votación, la alternativa $x$ vence a la $y$, la $y$ a la $z$ y la $z$ vence a la $x$, lo que da lugar a un ciclo en la preferencia colectiva.

A pesar de ello, las reglas mayoritarias poseen algunas características atractivas para recomendar su uso. Por ejemplo, Dasgupta y Maskin [23] argumentan que estas reglas representan mejor las preferencias individuales que otras reglas de agregación. Además, las reglas mayoritarias resultan fácilmente comprensibles desde el punto de vista de los votantes: dadas dos alternativas, únicamente se requiere que los individuos voten a favor de una de ellas o que se declaren indiferentes entre ambas. Para declarar cuál es la alternativa vencedora en cada par, simplemente se cuentan los votos a favor de cada alternativa.

Entre las reglas mayoritarias, la *mayoría simple* ocupa una posición predominante. Dadas dos alternativas, $x$ e $y$, la alternativa $x$ vence por mayoría simple a la alternativa $y$ si en el proceso de votación $x$ obtiene más votos que $y$. El estudio de sus propiedades (véanse entre otros, May [93], Campbell y Kelly [8] y Merlin [95]) y de sus inconvenientes ha dado lugar a una amplia literatura en el campo de la Teoría de la Elección Social. Esta regla resulta ser la más decisiva entre todas las posibles reglas neutrales, es decir, entre todas aquéllas que tratan igual a las alternativas que se comparan, dado que el requisito exigido a la alternativa ganadora es muy débil. Sin embargo esta propiedad, aparentemente deseable, incluye una vertiente perniciosa: la decisión social entre dos alternativas puede ser revertida con facilidad.

Este problema induce la introducción de otras reglas mayoritarias como la *mayoría absoluta*, las *mayorías cualificadas* (también conocidas en la literatura como *supermayorías*), la *unanimidad* o las *mayorías por diferencia de votos* (véanse entre otros, Fishburn [40], Ferejohn y Grether [38], Saari [102] y García-Lapresta y Llamazares [49]).

En el caso de la mayoría absoluta, dadas dos alternativas $x$ e $y$, la alternativa $x$ vence a la alternativa $y$ si el número de votos que obtiene es mayor que la mitad del número de votantes. Por tanto, el apoyo que se exige a la alternativa vencedora es como mínimo la mitad de los votantes más uno. Nótese que la mayoría absoluta equivale a la mayoría simple cuando los votantes no se muestran indiferentes entre las alternativas. Es decir, si todos los posibles electores votan por alguna alternativa, ambas mayorías exigen que la alternativa vencedora alcance más de la mitad de la totalidad del número de votos posibles.

En el caso de las mayorías cualificadas, la alternativa vencedora tiene que alcanzar un número de votos superior o igual a una cuota, fijada antes del proceso de votación, que tiene que ser mayor que la mitad del número de votantes.

Por su parte, la unanimidad exige que la alternativa vencedora obtenga todos los posibles votos. Obviamente, esta condición es tan fuerte que en la práctica resulta complicado obtener una alternativa ganadora, ya que es suficiente con que las preferencias de un solo individuo diverjan de las de los demás para que no pueda alcanzarse un resultado social.

En las mayorías por diferencia de votos, una alternativa se declara vencedora si supera una determinada diferencia de votos, establecida antes del proceso de votación, con respecto a la alternativa perdedora. Estas reglas se sitúan entre la mayoría simple, en la que la diferencia de votos requerida a la alternativa vencedora es nula, y la unanimidad, donde la diferencia de votos exigida a la vencedora es igual al número total de posibles votos menos uno. En la definición de las reglas se admite la posibilidad de que los individuos se declaren indiferentes entre las alternativas que se comparan. Si dicha opción se elimina de las preferencias individuales, las mayorías por diferencia de votos son equivalentes a las mayorías cualificadas. Cabe destacar que en este caso, se establece un compromiso entre la decisividad y la estabilidad del resultado social que se recoge expresamente en la segunda caracterización axiomática para estas mayorías dada por Llamazares en [85].

La consideración de la diferencia de votos en estas mayorías provoca que la indiferencia social entre las alternativas se declare con una mayor frecuencia que bajo la mayoría simple. Este hecho podría influir en la posibilidad de que las decisiones colectivas sean consistentes, es decir, que estén libres de ciclos o intransitividades.

Con independencia de las características concretas de las reglas mayoritarias ya descritas, en todas ellas la representación de las preferencias individuales es, en cierto sentido, incompleta. Los individuos sólo declaran si prefieren una alternativa a otra o si son indiferentes entre dichas alternativas.

Sin embargo, en la vida real los individuos diferencian entre distintos niveles de preferencia a la hora de declarar sus preferencias sobre pares de alternativas. La importancia de tener en cuenta la forma innata en la que los individuos gradúan sus preferencias se apunta en la literatura por autores como Morales [97], Sen [105] y Nurmi [100]. Todo esto motiva la introducción de las *relaciones de preferencia recíprocas* (véanse Bezdek *et al.* [6], Nurmi [99], Tanino [112] y Nakamura [98]) que permiten a los individuos declarar distintos grados de preferencia entre pares de alternativas. Para cada par, los individuos declaran sus intensidades de preferencia por medio de valores numéricos en el intervalo unidad de la siguiente manera: 0 y 1 representan respectivamente la preferencia absoluta por cada una de las alternativas que se comparan; el valor intermedio 0,5 representa la indiferencia entre ambas; finalmente, los valores entre los extremos y el intermedio representan las preferencias no taxativas en favor de cada una de las alternativas. La reciprocidad de estas relaciones de preferencia significa que, dadas dos alternativas $x$ e $y$, la intensidad con la que se prefiere la alternativa $x$ a la $y$ es 1 menos la intensidad con la que la alternativa $y$ se prefiere a la $x$.

En este ámbito, diferentes reglas de votación se extienden al contexto de las relaciones de preferencia recíprocas (véanse por ejemplo, García-Lapresta y Llamazares [48], García-Lapresta y Martínez-Panero [52], Llamazares y García-Lapresta [87] y García-Lapresta y Llamazares [50]).

En el caso de las reglas mayoritarias basadas en relaciones de preferencia recíprocas, la comparación colectiva entre dos alternativas consiste en la confrontación entre la suma de las intensidades de los votantes por una alternativa y la suma de las intensidades de los votantes por la otra alternativa. En este ámbito, las *mayorías por diferencia de apoyo* (García-Lapresta y Llamazares [50]) asignan una relación de preferencia colectiva a las relaciones de preferencia recíprocas exigiendo a la alternativa ganadora alcanzar una diferencia de intensidad colectiva, establecida antes del proceso de votación, con respecto a la alternativa perdedora, es decir, un determinado *umbral de apoyo*.

Hasta ahora se ha considerado que las preferencias individuales entre distintas alternativas se representan por medio de valores numéricos. Implícitamente se ha asumido que los individuos son capaces de graduar sus preferencias en un espectro continuo, de una manera objetiva. No obstante, en la práctica, los juicios de las personas se plantean más bien en términos lingüísticos. Es decir, los individuos manifiestan por medio de palabras sus gustos y sus valoraciones de manera subjetiva e imprecisa. Siguiendo esta idea, las decisiones colectivas se basarían en la agregación de *preferencias lingüísticas* (véanse Zadeh [122, 125, 126, 127]). La dificultad que plantea dicha agregación da lugar a diversos estudios cuyo objetivo es hacer manejable la computación de los términos lingüísticos (véanse entre otros, Delgado *et al.* [29, 30], Herrera *et al.* [71, 70] y Herrera y Martínez [72]).

Una de las cuestiones a las que se pretende dar respuesta en esta tesis es si la consideración de las intensidades de preferencia, así como la introducción de los umbrales de apoyo, inducen decisiones colectivas consistentes. Con objeto de responder a dicha cuestión, en el Capítulo 1 se estudia la transitividad de la relación de preferencia fuerte generada por la agregación de las relaciones de preferencia recíprocas bajo las mayorías por diferencia de apoyo, y en el Capítulo 2 se estudia la triple-aciclidad de dicha relación de preferencia fuerte.

Por una parte, en ambos casos se obtienen umbrales que permiten garantizar la consistencia de la preferencia colectiva bajo unas determinadas condiciones de racionalidad individual. Dichas condiciones extienden la noción de transitividad clásica al contexto de las intensidades de preferencia (véanse por ejemplo, Dubois y Prade [37], De Baets *et al.* [28], De Baets y Van de Walle [27], Dasgupta y Deb [22], Świtalski [109, 110], Herrera-Viedma *et al.* [73], Díaz *et al.* [34], García-Lapresta y Meneses [53], De Baets y De Meyer [24], De Baets *et al.* [26], Díaz *et al.* [35, 32, 33] y Chiclana *et al.* [15]). Por otra parte, los resultados requieren que los umbrales sean muy elevados y que las preferencias individuales sean altamente racionales.

En el Capítulo 3 se completan los resultados anteriores por medio de la estimación de las probabilidades con las que se producen resultados colectivos consistentes bajo las mayorías por diferencia de votos y las mayorías por diferencia de apoyo. Este enfoque, ampliamente utilizado en la literatura, parte del supuesto de que los resultados inconsistentes se producen en condiciones poco habituales (Gehrlein y Fishburn [61]). En este sentido, destacan los estudios anteriores para la mayoría simple (Gehrlein y Fishburn [61], Fishburn y Gerhlein [42] y Gerhlein [56]), las mayorías cualificadas (Balasko y Crès [4], y Tovey [113]) y las reglas de puntuación (Gehrlein y Fishburn [62, 63, 64], Cervone *et al.* [12] y Diss *et al.* [36]).

En dicho capítulo se distingue entre tres tipos de resultados relacionados con la consistencia de las decisiones colectivas tanto para las mayorías por diferencia de votos como para las mayorías por diferencia de apoyo; aquéllos en los que la relación de preferencia fuerte es transitiva, aquéllos en los que la relación de preferencia fuerte es triple-acíclica y aquéllos en los que la relación de preferencia débil es transitiva.

Para las preferencias colectivas que se derivan de la aplicación de las mayorías por diferencia de votos (teniendo en cuenta preferencias individuales lineales, en las que no se permite la indiferencia entre alternativas, y órdenes débiles, en las que los votantes pueden declararse indiferentes entre las alternativas en comparación), se sigue el modelo introducido por Wilson y Pritchard [117] y Lepelley *et al.* [82]. Para su aplicación se asume la hipótesis de *cultura imparcial anónima* o *modelo IAC*; en concreto, se establece que en una votación cualquier combinación de preferencias individuales es equiprobable a priori (para conocer más sobre el modelo IAC, véase Gehrlein y Fishburn [61]).

Para el estudio de las probabilidades de resultados consistentes para las preferencias colectivas que se derivan de la aplicación de las mayorías por diferencia de apoyo, se aplica el método de Montecarlo. Este planteamiento se inspira en los estudios previos de Campbell y Tullock [9], Klahr [80], DeMeyer y Plott [31] y Jones [76]. Las probabilidades simuladas que se obtienen complementan los resultados teóricos que se presentan en los Capítulos 1 y 2.

En el Capítulo 4 se extienden las mayorías por diferencia de votos al contexto de las preferencias lingüísticas. Para su definición formal se consideran las dos vertientes fundamentales en la modelización de las preferencias lingüísticas. En concreto, se introduce la regla a través de la representación cardinal proporcionada por los conjuntos difusos y sus funciones de pertenencia (Zadeh [122] y Hanss [69]) y de la representación ordinal recogida en el modelo de las 2-tuplas (Herrera y Martínez [72]). Adicionalmente, se presenta la equivalencia entre ambas modelizaciones bajo determinadas condiciones de regularidad y se estudian las propiedades que cumplen estas mayorías lingüísticas.

Esta memoria finaliza con un capítulo de conclusiones en el que se resumen los resultados obtenidos y se plantean futuras líneas de trabajo.

Conforme esta tesis se iba desarrollando, los distintos resultados han sido presentados en congresos de carácter científico nacionales, internacionales y en un seminario.

**Congresos nacionales:**

- I Jornadas de Trabajo sobre Sistemas de Votación. Valladolid, 2009.
- VI Encuentro de la Red Española de Elección Social: REES. Reus (Tarragona), 2009.
- XV Congreso Español sobre Tecnologías y Lógica Fuzzy: ESTYLF. Punta Umbría (Huelva), 2010.
- III Simposio sobre Lógica Fuzzy y Soft Computing; Congreso Español de Informática: CEDI. Valencia, 2010.

**Congresos internacionales:**

- 11th International Student Conference on Applied Mathematics and Informatics: ISCAMI. Bratislava (Eslovaquia), 2010.
- 10th International Meeting of the Society for Social Choice and Welfare. Moscú (Rusia), 2010.
- 32nd Linz Seminar on Fuzzy Set Theory. Decision Theory: qualitative and quantitative approaches. Linz (Austria), 2011.
- 7th International Summer School on Aggregation Operators: AGOP. Pamplona, 2013.

**Seminario:**

- GATE Semminar. Saint-Etienne (Francia), 2012.

Por último, los resultados presentados en esta memoria han dado lugar a los siguientes artículos:

- Capítulo 1
  Llamazares, B., Pérez-Asurmendi, P. y García-Lapresta, J. L. Collective transitivity in majorities based on difference in support. *Fuzzy Sets and Systems* 216 (2013), 3–15.

- Capítulo 2
  Llamazares, B. y Pérez-Asurmendi, P. Triple-acyclicity in majorities based on difference in support. En revisión.

- Capítulo 3
  Diss, M. y Pérez-Asurmendi, P. Consistent collective decisions under majorities based on differences. En evaluación.

- Capítulo 4
  Pérez-Asurmendi, P. y Chiclana, F. Linguistic majorities with difference in support. *Applied Soft Computing*, aceptado para su publicación.

# Introduction

The Theory of Social Choice focuses on the analysis of the collective decision making process. Taking into account the opinions and likes of the members of a social group in regard to various alternatives, a social choice is derived by means of an aggregation rule that assigns a collective preference to individual preferences.

Among the rules commonly used to aggregate individual preferences *scoring rules* and *majority rules* stand out.

In the former, alternatives are awarded points depending on the position that they hold in individual preferences. Scores are aggregated by adding up the points of the different alternatives and ordering them as a function of the results. Among the different scoring rules the rules of *plurality*, *antiplurality* and *Borda* stand out.

In the case of majority rules, individuals indicate their preferences between pairs of alternatives. Such individual preferences are aggregated by means of specific majority rules, yielding a social pairwise comparison between the alternatives.

Both types of rule provide social decisions with certain properties that are useful for justifying the social result of the voting process. Majority rules prevent decisions between two alternatives from being influenced by judgments concerning other alternatives. In other words, majority rules provide social decisions that fulfil the Arrowian property of independence of irrelevant alternatives [3].

By contrast, in the case of scoring rules social decisions are consistent, i.e. there are neither cycles nor intransitivities in collective decisions. Possible inconsistencies in social decisions precisely constitute the major drawback of majority rules. This problem was pointed out in the 18th century by Condorcet [18] in his well-known *Voting Paradox*. The classical example describes the aggregation of the preferences of three individuals concerning three alternatives $x, y, z$. The first individual prefers alternative $x$ to $y$ and $y$ to $z$, the second one prefers alternative $y$ to $z$ and $z$ to $x$ and the third one prefers $z$ to $x$ and $x$ to $y$. It is assumed that voters cast their votes between the pairs according to their preferences. As a result of the voting process, alternative $x$ defeats $y$, $y$ defeats $z$ and $z$ defeats $x$, providing a cycle on the collective preference.

In spite of this, majority rules have some interesting characteristics to recommend their use. For instance, Dasgupta and Maskin [23] argue that they represent individual preferences better than other aggregation rules. Moreover, from the voters' point of view, majority rules are easy to understand: given two alternatives, voters are only required to cast a vote in favour of one of them or declare themselves indifferent between the two. Declaring the winning alternative in each pair is merely a matter of counting the votes in favour of each alternative.

Among majority rules, *simple majority* holds a prominent position. Given two alternatives, $x$ and $y$, alternative $x$ defeats alternative $y$ under simple majority if $x$ gets more votes than $y$ in the voting process. The study of its properties (see among others, May [93], Campbell and Kelly [8] and Merlin [95]) and its drawbacks has produced a large body of literature in the field of the Theory of Social Choice. This is the most decisive of all the possible neutral rules, in other words, of all those that treat all the alternatives compared equally. Nevertheless, this apparently desirable property has its downside: a social decision between two alternatives could be easily reversed.

This problem leads to the introduction of other majority rules such as the *absolute majority*, *qualified majorities* (also known in the relevant literature as *supermajorities*), *unanimity* and *majorities based on difference of votes* (see among others, Fishburn [40], Ferejohn and Grether [38], Saari [102] and García-Lapresta and Llamazares [49]).

In the case of the absolute majority, given two alternatives $x$ and $y$ alternative $x$ defeats alternative $y$ if the number of votes obtained by $x$ is greater than half of the number of voters. Thereof, the support required of the winning alternative is at least the half of the voters plus one. Notice that the absolute majority is equivalent to the simple majority when voters are not indifferent between the alternatives. In other words, if all possible voters cast a vote for one of the alternatives, both majority rules require the winning alternative to reach more than half of the total number of possible votes.

In the case of qualified majorities the winning alternative has to reach a number of votes equal to or greater than a quota set before the voting process, which must be greater than the half of the number of voters.

Unanimity requires the winning alternative to obtain all the possible votes. Obviously, this condition is so strong that in practice it is very difficult to reach a winning alternative, given that it suffices for the preferences of one voter to differ from the rest for a social outcome not to be reached.

In majorities based on difference of votes, one alternative is declared the winner if the votes for it exceed the votes cast for the losing alternative by a difference set before the voting process. These rules are located between the simple majority rule, for which the required difference of votes is zero, and unanimity, for which the required difference of votes is the total number of possible votes minus one. In the definition of these rules, individual indifference between the alternatives compared is allowed. If that option is ruled out from individual preferences, majorities based on difference of votes are equivalent to qualified majorities. It is worth remarking that in this case a compromise between the decisiveness of the rule and the stability of the social outcome is established and explicitly included in the second axiomatic characterisation of these majorities given by Llamazares in [85].

Taking into consideration the difference of votes in these majorities means that social indifference is more frequently declared than under the simple majority rule. This could influence the possibility of making consistent collective decisions, i.e. being decision free from cycles or intransitivities.

Regardless of the specific characteristics of the majority rules described above, in all of them the representation of individual preferences is somehow incomplete. Individuals only declare whether they prefer one alternative to another or whether they are indifferent between the two alternatives.

Nevertheless, in real life individuals distinguish between different levels of preference in declaring their preferences regarding pairs of alternatives. The importance of taking into account the natural way in which individuals scale their preferences has been pointed out in the relevant literature by authors such as Morales [97], Sen [105] and Nurmi [100]. All this has led to the introduction of *reciprocal preference relations* (see Bezdek *et al.* [6], Nurmi [99], Tanino [112] and Nakamura [98]) which allow individuals to declare different degrees of preference between pairs of alternatives. For each pair, individuals declare their intensities of preference by means of numerical values in the unit interval as follows: 0 and 1 represent the absolute preferences for each alternative compared, respectively; the intermediate value 0.5 represents indifference between the two alternatives; finally, figures between the extreme and intermediate values represent non emphatic preferences in favour of each alternative. The reciprocity of these preference relations means that given two alternatives $x$ and $y$, the intensity with which alternative $x$ is preferred to alternative $y$ is 1 minus the intensity with which alternative $y$ is preferred to alternative $x$.

In this field, different voting rules are extended to the context of reciprocal preference relations (see for instance, García-Lapresta and Llamazares [48], García-Lapresta and Martínez-Panero [52], Llamazares and García-Lapresta [87] and García-Lapresta and Llamazares [50]).

In the case of majority rules based on reciprocal preference relations, the collective comparison between two alternatives consists of comparing the sum of the individual intensities for one alternative with the sum of the individual intensities for the other one. In this framework, *majorities based on difference in support* (García-Lapresta and Llamazares [50]) assign a collective preference to reciprocal preference relations, requiring the winning alternative to attain a certain difference in collective intensity compared to that attained by the losing alternative, set before the voting process, i.e. a determined *threshold of support*.

So far individual preferences between different alternatives are considered to be represented by numerical values. It is implicitly assumed that individuals are able to graduate their preferences objectively over a continuous spectrum. However, in practice personal judgments are usually proposed in linguistic terms. In other words, individuals declare their likes and valuations subjectively and softly, using words. Following that idea, collective decisions can be considered to be based on the aggregation of linguistic preferences (see Zadeh [122, 125, 126, 127]). The difficulty of such aggregation has given rise to several studies that seek to make the computation of linguistic terms tractable (see among others, Delgado *et al.* [29, 30], Herrera *et al.* [71, 70]

and Herrera and Martínez [72]).

One of the questions that this thesis sets out to solve is whether considering intensities of preference and introducing thresholds of support leads to consistent collective decisions. To answer that question, Chapter 1 examines the transitivity of the strict preference relation generated by the aggregation of reciprocal preference relations under majorities based on difference in support, and Chapter 2 looks at the triple-acyclicity of such strict preference relations.

On the one hand, thresholds to guarantee the consistency of collective preferences under certain individual rationality conditions are found in both cases. Such conditions extend the classical notion of transitivity to the context of intensities of preference (see for instance, Dubois and Prade [37], De Baets *et al.* [28], De Baets and Van de Walle [27], Dasgupta and Deb [22], Świtalski [109, 110], Herrera-Viedma *et al.* [73], Díaz *et al.* [34], García-Lapresta and Meneses [53], De Baets and De Meyer [24], De Baets *et al.* [26], Díaz *et al.* [35, 32, 33] and Chiclana *et al.* [15]). On the other hand, the results require thresholds to be quite high and individual preferences to be highly rational.

Chapter 3 extends the results of the previous chapters by estimating the probabilities of the occurrence of consistent collective outcomes under majorities based on difference of votes and majorities based on difference in support. This approach, which is widely used in the relevant literature, assumes that inconsistent outcomes appear in conditions that are hardly usual (Gehrlein and Fishburn [61]). In that sense, previous studies of simple majorities (Gehrlein and Fishburn [61], Fishburn and Gerhlein [42] and Gerhlein [56]), qualified majorities (Balasko and Crès [4], and Tovey [113]) and scoring rules (Gehrlein and Fishburn [62, 63, 64], Cervone *et al.* [12] and Diss *et al.* [36]) stand out.

In the said chapter, three different types of result related to consistency are considered both for majorities based on difference of votes and for majorities based on difference in support: those in which the strict preference relation is transitive, those in which it is triple-acyclic and those in which the weak preference relation is transitive.

For collective preferences derived from the application of the majorities based on difference of votes (bearing in mind linear individual preferences, in which indifference between alternatives is not allowed, and weak orderings, in which voters can be indifferent between the alternatives compared), the model introduced by Wilson and Pritchard [117] and Lepelley *et al.* [82] is followed. The *anonymous impartial culture* hypothesis or *IAC model* is assumed in its application: specifically, it is established that any combination of individual preferences is a priori equiprobable in a voting process (to learn more about the IAC model, see Gehrlein and Fishburn [61]).

For the study of the probability of there being consistent outcomes for collective preferences derived from the application of majorities based on difference in support, the Montecarlo method is applied. This approach is based on previous studies by Campbell and Tullock [9], Klahr [80], DeMeyer and Plott [31] and Jones [76]. The simulated probabilities obtained complement the theoretical results presented in Chapters 1 and 2.

In Chapter 4, majorities based on difference of votes are extended to the framework of linguistic preferences. For their formal definition, the two main models for managing linguistic preferences are considered. Specifically, the rule is introduced by means of cardinal representation based on fuzzy sets and their membership functions (Zadeh [122] and Hanss [69]) and ordinal representation collected in the 2-tuple model (Herrera and Martínez [72]). Moreover, the equivalence of such modelling under certain regularity conditions is introduced and the properties of such linguistic majorities are studied.

This report ends with some concluding remarks that summarise the results obtained and suggest directions for future research.

During the writing of this thesis, the various results have been presented at national and international congresses and in a seminar.

**National congresses:**

- I Jornadas de Trabajo sobre Sistemas de Votación. Valladolid, 2009.
- VI Encuentro de la Red Española de Elección Social: REES. Reus (Tarragona), 2009.
- XV Congreso Español sobre Tecnologías y Lógica Fuzzy: ESTYLF. Punta Umbría (Huelva), 2010.
- III Simposio sobre Lógica Fuzzy y Soft Computing; Congreso Español de Informática: CEDI. Valencia, 2010.

**International congresses:**

- 11th International Student Conference on Applied Mathematics and Informatics: ISCAMI. Bratislava (Eslovaquia), 2010.
- 10th International Meeting of the Society for Social Choice and Welfare. Moscú (Rusia), 2010.
- 32nd Linz Seminar on Fuzzy Set Theory. Decision Theory: qualitative and quantitative approaches. Linz (Austria), 2011.
- 7th International Summer School on Aggregation Operators: AGOP. Pamplona, 2013.

**Seminar:**

- GATE Seminar. Saint-Etienne (Francia), 2012.

Lastly the findings compiled in this thesis have resulted in the following papers:

- Chapter 1
  Llamazares, B., Pérez-Asurmendi, P. y García-Lapresta, J. L. Collective transitivity in majorities based on difference in support. *Fuzzy Sets and Systems* 216 (2013), 3–15.
- Chapter 2
  Llamazares, B., Pérez-Asurmendi, P. Triple-acyclicity in majorities based on difference in support. Under review.
- Chapter 3
  Diss, M., Pérez-Asurmendi, P. Consistent collective decisions under majorities based on differences. Submitted.
- Chapter 4
  Pérez-Asurmendi, P. y Chiclana, F. Linguistic majorities with difference in support. *Applied Soft Computing*, forthcoming.

# Chapter 1

# Collective transitivity on majorities based on difference in support

[This chapter has been previously published (jointly with Bonifacio Llamazares and José Luis García-Lapresta) in the journal *Fuzzy Sets and Systems* 216, pp. 3–15, 2013.]

A common criticism to simple majority voting rule is the slight support that such rule demands to declare an alternative as a winner. Among the distinct majority rules used for diminishing this handicap, we focus on majorities based on difference in support. With these majorities, voters are allowed to show intensities of preference among alternatives through reciprocal preference relations. These majorities also take into account the difference in support between alternatives in order to select the winner. In this paper we have provided some necessary and sufficient conditions for ensuring transitive collective decisions generated by majorities based on difference in support for all the profiles of individual reciprocal preference relations. These conditions involve both the thresholds of support and some individual rationality assumptions that are related to transitivity in the framework of reciprocal preference relations.

## 1.1 Introduction

Majority voting systems are doubtless the most popular methods, within some organizations, of aggregating individual opinions in order to make collective decisions. Nowadays, in almost all democratic parliaments and institutions, decisions over collective issues are usually taken through majority systems as simple majority, absolute majority or qualified majorities. In fact, some authors argue that majority rules represent voters' views better than other voting systems (see Dasgupta and Maskin [23]).

From a practical point of view, these methods are easy to understand by the voters: Given two alternatives, each individual casts a vote for his/her preferred alternative or he/she abstains when he/she is indifferent between them (if it is allowed). The social decision simply consists of the selection of the most preferred alternative of a predetermined majority of the voters, if such an alternative exists.

These facts have promoted a broad literature about all those kinds of majority systems; their properties, advantages and drawbacks have been deeply studied in almost all of them. Moreover, majority rules are usually set against rank order voting systems. Only majority systems lead to decisions independent of irrelevant alternatives; however, rank order voting systems fulfil collective transitivity whereas majority systems do not. In other words, cycles on collective decision are possible under majority rules. It is commonly known that these cycles represent one of the most troublesome problems on Voting Theory given that their occurrence leads to the impossibility of getting a social outcome. The well-known voting paradox (Condorcet [18]) describes the pointed problem; whenever a social choice involves three or more alternatives and three or more voters, cycles might appear.

A common feature of majority rules and other classic voting systems, is that they require individuals to declare dichotomous preferences. In other words, voters can only declare if an alternative is preferred to another, or if they are indifferent. All kinds of preference modalities are identified and voters' opinions are misrepresented. In this way, some authors have pointed out the necessity of having more information about individuals preferences.

Quoting the Nobel Prize Laureate Amartya K. Sen [106, p. 162]: '…the method of majority decision takes no account of intensities of preference, and it is certainly arguable that what matters is not merely the *number* who prefer $x$ to $y$ and the *number* who prefer $y$ to $x$, but also by how much each prefers one alternative to the other'. This idea had already been considered in the 18th Century by the Spanish mathematician Morales [97] who stated that 'opinion is not something that can be quantified but rather something which has to be weighed' (see English translation in McLean and Urken [94, p. 204], or '. . . majority opinion . . . is something which is independent of any fixed number of votes' (see English translation in McLean and Urken [94, p. 214]).

The importance of considering intensities of preference in the design of appropriate voting systems has also been advocated by Nurmi [100]. Following this approach, García-Lapresta and Llamazares [48] provided some axiomatic characterizations of several decision rules that aggregate reciprocal preferences through different kind of means. Moreover, in García-Lapresta and Llamazares [48, Prop. 2], simple majority has been obtained as a specific case of some means-based decision rules. Likewise, other kinds of majorities can be obtained through operators that aggregate reciprocal preferences (on this, see Llamazares and García-Lapresta [87, 88] and Llamazares [84, 86]).

Majority rules based on difference of votes were introduced by García-Lapresta and Llamazares [49] in order to avoid some of the drawbacks of simple and absolute majorities. Furthermore, in García-Lapresta and Llamazares [50] these majority rules were extended by allowing individuals to show their intensities of preference among alternatives. They introduced *majorities based on difference in support* or $\widetilde{M}_k$ *majorities* and provided a characterization by means of some independent axioms.

As previously mentioned, the voting paradox constitutes a key aspect of voting systems, and it is also crucial in this research. We devote this paper to analyze when majorities based on difference in support ($\widetilde{M}_k$ majorities) provide transitive collective preference relations for every profile of individual reciprocal preferences satisfying some rationality conditions. In other words, we study when the aggregation of individual intensities of preferences through $\widetilde{M}_k$ majorities provides transitive social decisions.

The paper is organized as follows. Section 1.2 is devoted to introducing some classes of transitivity conditions in the field of reciprocal preferences. Moreover, some well-known majority rules are reviewed, and the extension of majorities based on difference of votes to the field of reciprocal preferences, the class of majorities based on difference in support, is introduced. Section 1.3 includes the results and some illustrative examples. Finally, Section 1.4 provides some concluding remarks.

## 1.2 Preliminaries

Consider $m$ voters, $V = \{1, \ldots, m\}$, with $m \geq 2$, showing the intensity of their preferences on $n$ alternatives, $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$, through *reciprocal* preference relations $R^p : X \times X \longrightarrow [0,1]$, for $p = 1, \ldots, m$, i.e., $r^p_{ij} + r^p_{ji} = 1$ for all $i, j \in \{1, \ldots, n\}$, where $r^p_{ij} = R^p(x_i, x_j)$. The information contained in $R^p$ can be represented by a $n \times n$ matrix with coefficients in $[0,1]$

$$R^p = \begin{pmatrix} r^p_{11} & r^p_{12} & \cdots & r^p_{1n} \\ r^p_{21} & r^p_{22} & \cdots & r^p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ r^p_{n1} & r^p_{n2} & \cdots & r^p_{nn} \end{pmatrix},$$

where, by reciprocity, all the diagonal elements are 0.5 and $r^p_{ji} = 1 - r^p_{ij}$ if $j \neq i$. Rewriting the matrix according to these facts, we have:

$$R^p = \begin{pmatrix} 0.5 & r^p_{12} & \cdots & r^p_{1n} \\ 1 - r^p_{12} & 0.5 & \cdots & r^p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 1 - r^p_{1n} & 1 - r^p_{2n} & \cdots & 0.5 \end{pmatrix}.$$

Under this setting, we assume that voters are able to distinguish whether they prefer one alternative to another or if they are totally indifferent between them. We also consider that voters may provide numerical degrees of preference among the alternatives by means of numbers within a bipolar scale in the unit interval. More specifically, given two alternatives $x_i$ and $x_j$, if voter $p$ is indifferent between these alternatives, then $r^p_{ij} = 0.5$. But if this individual prefers an alternative to the other, then he/she can show the intensity of preference in the following way: $r^p_{ij} = 0$, when $p$ absolutely prefers $x_j$ to $x_i$; $r^p_{ij} = 1$, when $p$ absolutely prefers $x_i$ to $x_j$. If $p$ does not declare extreme preferences or indifference, then $r_{ij}^p$ takes some value between 0 and 1 different to 0.5. If this voter somewhat prefers $x_i$ to $x_j$, then $0.5 < r_{ij}^p < 1$, and the closer this number is to 1, the more $x_i$ is preferred to $x_j$. A similar interpretation can be done for values located between 0 and 0.5: if $p$ somewhat prefers $x_j$ to $x_i$, then $0 < r_{ij}^p < 0.5$, and the closer this number is to 0, the more $x_j$ is preferred to $x_i$ (see Nurmi [99] and García-Lapresta and Llamazares [48]).

Notice that reciprocity extends two properties from the framework of ordinary preferences to the context of intensities of preference: asymmetry (see García-Lapresta and Meneses [54]) and completeness (see De Baets and De Meyer [24]).

With $\mathcal{R}(X)$ we denote the set of reciprocal preference relations on $X$. A *profile* is a vector $(R^1, \ldots, R^m)$ containing the individual reciprocal preference relations on $X$. Accordingly, the set of profiles is denoted by $\mathcal{R}(X)^m$.

An *ordinary preference relation* on $X$ is an *asymmetric* binary relation on $X$: if $x_i \, P \, x_j$, then does not happen $x_j \, P \, x_i$. The indifference relation associated with $P$ is defined as $x_i \, I \, x_j$ if neither $x_i \, P \, x_j$ nor $x_j \, P \, x_i$. With $\mathcal{P}(X)$ we denote the set of ordinary preference relations on $X$.

Notice that every ordinary preference relation $P$ can be considered as a reciprocal preference relation $R$:

$$\begin{array}{ccc} x_i \, P \, x_j & \Leftrightarrow & r_{ij} = 1 \\ x_j \, P \, x_i & \Leftrightarrow & r_{ij} = 0 \\ x_i \, I \, x_j & \Leftrightarrow & r_{ij} = 0.5 \, . \end{array}$$

On the other hand, a reciprocal preference relation $R^p$ is *crisp* if $r_{ij}^p \in \{0, 0.5, 1\}$ for all $i, j \in \{1, \ldots, n\}$. So, in practice, crisp reciprocal preference relations and ordinary preference relations are equivalent.

An ordinary preference relation $P \in \mathcal{P}(X)$ is *transitive* if for all $x_i, x_j, x_l \in X$ it holds that if $x_i \, P \, x_j$ and $x_j \, P \, x_l$, then it also holds $x_i \, P \, x_l$.

In Figure 1.1, the difference on the information reported by ordinary preferences in comparison with reciprocal preferences is shown.

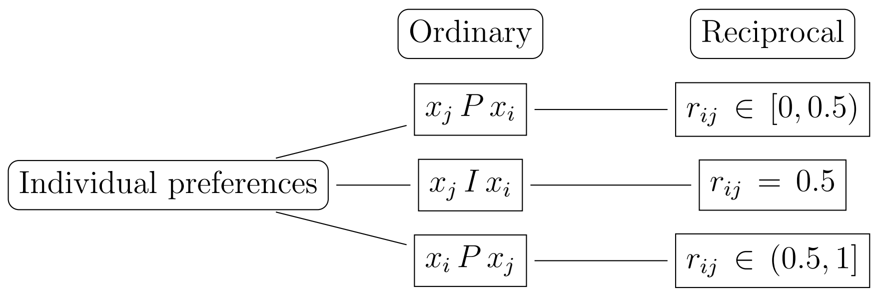


Figure 1.1: Ordinary *versus* reciprocal preferences.

### 1.2.1 Consistency on reciprocal preference relations

Within the framework of ordinary preference relations, the most well-known rationality assumption is transitivity. However, such a condition could be extended in many different ways when considering reciprocal preference relations or similar structures (see, for instance, Zadeh [123], Dubois and Prade [37], Tanino [112], Jain [75], De Baets *et al.* [28], De Baets and Van de Walle [27], Dasgupta and Deb [22], Van de Walle *et al.* [114], Świtalski [109, 110], Herrera-Viedma *et al.* [73], Díaz *et al.* [34, 35, 32, 33], De Baets and De Meyer [24], García-Lapresta and Meneses [53], De Baets *et al.* [26], García-Lapresta and Montero [55], Chiclana *et al.* [14, 15], Alonso *et al.* [1] or De Baets *et al.* [25]).

It is important to note that in 1973 Fishburn [39] provided some extensions of transitivity within the probabilistic choice framework. These extensions can be considered as precursors of some transitivity properties in the field of reciprocal relations.

We now introduce the notion of transitivity we use in the results (see also Figure 1.2).

**Definition 1.1.** A function $g : [0.5, 1]^2 \longrightarrow [0.5, 1]$ is a *monotonic operator* if it is satisfies the following conditions:

1. Continuity.
2. Increasingness: $g(a, b) \geq g(c, d)$ for all $a, b, c, d \in [0.5, 1]$ such that $a \geq c$ and $b \geq d$.

3. Symmetry: $g(a, b) = g(b, a)$ for all $a, b \in [0.5, 1]$.

**Definition 1.2.** Given a monotonic operator $g$, $R \in \mathcal{R}(X)$ is *$g$–transitive* if for all $i, j, l \in \{1, \ldots, n\}$ the following holds:

$$\big(r_{ij} > 0.5 \text{ and } r_{jl} > 0.5\big) \Rightarrow \big(r_{il} > 0.5 \text{ and } r_{il} \geq g(r_{ij}, r_{jl})\big).$$

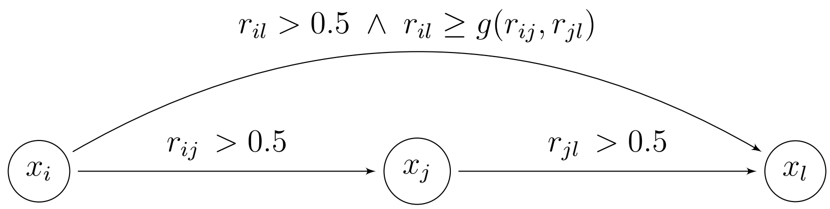


Figure 1.2: The notion of $g$–transitivity.

With $T_g$ we denote the set of all $g$–transitive reciprocal preference relations. Notice that if $f$ and $g$ are two monotonic operators such that $f \leq g$, i.e., $f(a, b) \leq g(a, b)$ for all $a, b \in [0.5, 1]$, then $T_g \subseteq T_f$.

Figure 1.2 illustrates $g$–transitivity. This definition of transitivity allows us to distinguish between different degrees of individual's rationality. Furthermore, an appropriate choice of $g$ allows us avoid situations where individuals' rationality could be questioned. For instance, consider the values $r^p_{ij} = 1$, $r^p_{jl} = 1$ and $r^p_{il} = 0.51$; that is, voter $p$ absolutely prefers $x_i$ to $x_j$ and $x_j$ to $x_l$, but only slightly $x_i$ to $x_l$. In this case, the rationality of individual $p$ could be questioned given that, in the ordinary case, the notion of transitivity means that if $x_i \, P \, x_j$ ($r^p_{ij} = 1$) and $x_j \, P \, x_l$ ($r^p_{jl} = 1$), then $x_i \, P \, x_l$ ($r^p_{il} = 1$).

We now consider three of the most commonly used transitivity conditions on reciprocal preference relations by means of the monotonic operators minimum, arithmetic mean and maximum:

1. $R$ is *min–transitive* if $R$ is $g$–transitive being $g(a, b) = \min\{a, b\}$ for all $(a, b) \in [0.5, 1]^2$.

2. $R$ is *am–transitive* if $R$ is $g$–transitive being $g(a,b) = (a+b)/2$ for all $(a,b) \in [0.5,1]^2$.

3. $R$ is *max–transitive* if $R$ is $g$–transitive being $g(a,b) = \max\{a,b\}$ for all $(a,b) \in [0.5,1]^2$.

We denote with $T_{\text{min}}$, $T_{\text{am}}$ and $T_{\text{max}}$ the sets of all min–transitive, am–transitive and max–transitive reciprocal preference relations, respectively. Clearly, $T_{\text{max}} \subset T_{\text{am}} \subset T_{\text{min}}$.

### 1.2.2 Majority rules

Among the wide variety of majority rules, *simple majority* holds a primary position. Recalling the rule, one alternative, say $x$, defeats another, say $y$, when the number of individuals who prefer $x$ to $y$ is greater than the number of individuals who prefer $y$ to $x$. Since it was first characterized by May [93], a wide research has been done with regard to analyzing its properties. In particular, simple majority is the most decisive majority rule when the alternatives are equally treated. On the one hand, this constitutes an advantage with respect to other rules. On the other hand, unfortunately, it also represents an important drawback: simple majority requires a really poor support for declaring an alternative as the winner. For instance, when all voters but one abstain, the winner is the alternative receiving that single vote.

In an attempt to get a better performance on that issue, other majorities have been introduced and studied in the literature. Among them one can find unanimous majority, qualified majorities and absolute majority (see Fishburn [40, chapter 6], Ferejohn and Grether [38], Saari [102, pp. 122–123], and García-Lapresta and Llamazares [49], among others).

According to *unanimous majority*, an alternative, say $x$, defeats another one, say $y$, when every voter involved in the election casts his/her ballot for the alternative $x$. Obviously reaching a winner is very difficult in practice under this rule; if there is just one discordant voter, a collective choice becomes impossible.

*Qualified majorities* require support for the winner alternative to be greater than or equal to a quota, fixed before the election, multiplied by the number of voters. So, an alternative $x$ defeats another alternative $y$ by a qualified majority of quota $\alpha > 0.5$, when at least $100\,\alpha\,\%$ of voters prefer $x$ to $y$. Common examples are three-fifths, two-thirds or three-quarters majorities. Note that when the required quota is 1, we are going back to unanimous majority.

Finally, when the required support for the winning alternative is greater than half the number of voters, we are looking at *absolute majority*. Obviously whenever every individual involved in the election has strict preferences over alternatives (that is, no individual is indifferent between distinct alternatives), simple majority and absolute majority are equivalent.

In García-Lapresta and Llamazares [49], another class of majorities have been introduced and analyzed: *majorities based on difference of votes* or $M_k$ *majorities*. According to these majorities, given two alternatives, $x$ and $y$, $x$ is collectively preferred to $y$, when the number of individuals who prefer $x$ to $y$ exceeds the number of individuals who prefer $y$ to $x$ by at least a fixed integer $k$ from 0 to $m-1$. These majorities have been axiomatically characterized by Llamazares [85] and subsequently by Houy [74].

We note that $M_k$ majorities are located between simple majority and unanimity. In particular, we face simple majority when the required threshold $k$ equals 0 whereas unanimity is reached when $k$ equals $m-1$. Therefore the above mentioned problem of support in the case of simple majority can be solved with these rules by using appropriate thresholds. Since voters are sometimes indifferent between some alternatives, $M_k$ majorities have an important role in group decision making.

Qualified majorities, however, are located between absolute majority and unanimity. It is interesting to note that $M_k$ majorities and qualified majorities become equivalent when indifference is ruled out from individual preferences.

In Figure 1.3 all these facts are summarized, and the difference between $M_k$-majorities and qualified majorities, when voters can declare indifference between alternatives, is emphasized.

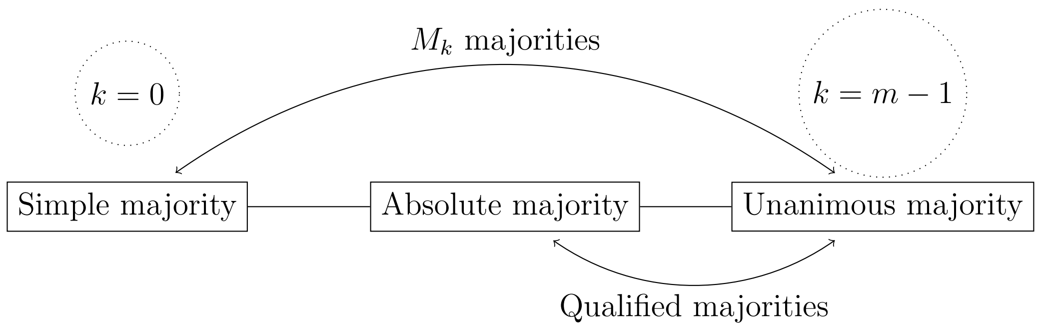


Figure 1.3: $M_k$ majorities *versus* qualified majorities.

### 1.2.3 Majorities based on difference in support

To finish this preliminary section, we now present the voting rules becoming goal of this research: majorities based on difference in support. These majorities extend the family of majorities based on difference of votes to the field of reciprocal preferences. Given two alternatives $x$ and $y$, $x$ defeats $y$ when the aggregated intensity of preference of $x$ over $y$ exceeds the aggregated intensity of $y$ over $x$ in a threshold $k$ fixed before the election, where $k$ is a real number located between 0 and the total number of voters.

**Definition 1.3.** Given a threshold $k \in [0, m)$ and $\mathcal{D} \subseteq \mathcal{R}(X)^m$, the $\widetilde{M}_k$ *majority* is the mapping $\widetilde{M}_k : \mathcal{D} \longrightarrow \mathcal{P}(X)$ defined by $\widetilde{M}_k(R^1, \ldots, R^m) = P_k$, where

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \sum_{p=1}^{m} r_{ji}^p + k.$$

Notice that $\widetilde{M}_k$ assigns a collective ordinary preference relation to each profile of individual reciprocal preference relations. It is easy to see (García-Lapresta and Llamazares [50]) that $P_k$ can be defined through the average of the individual intensities of preference:

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \frac{1}{m} \sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2m},$$

or, equivalently,

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2}. \tag{1.1}$$

We can rewrite $P_k$ by means of an $\alpha$–cut[1]. Let $\overline{R} : X \times X \longrightarrow [0,1]$ the reciprocal preference relation defined by the arithmetic mean of the individual intensities of preference, i.e.,

$$\overline{R}(x_i, x_j) = \frac{1}{m} \sum_{p=1}^{m} r_{ij}^p.$$

Then, $P_k = \overline{R}_\alpha$, with $\alpha = (m+k)/2m$.

The indifference relation associated with $P_k$ is defined by:

$$x_i \, I_k \, x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \sum_{p=1}^{m} r_{ji}^p \right| \leq k,$$

or, equivalently,

$$x_i \, I_k \, x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \frac{m}{2} \right| \leq \frac{k}{2}. \tag{1.2}$$

Going deeper on the behavior of $\widetilde{M}_k$ majorities, some interesting facts can be detailed.

*Remark* 1.1. For $\widetilde{M}_k$ majorities, the following statements hold:

1. $(\, x_i \, P_k \, x_j \;$ and $\; k' > k) \;\Rightarrow\; (\, x_i \, P_{k'} \, x_j \;$ or $\; x_i \, I_{k'} \, x_j \,)$.
2. $(\, x_i \, P_k \, x_j \;$ and $\; k' < k \,) \;\Rightarrow\; x_i \, P_{k'} \, x_j$.
3. $(\, \exists \, k \in [0,m) \; x_i \, P_k \, x_j \,) \;\Rightarrow\; (\forall \, k' \in [0,m) \; \neg(x_j \, P_{k'} \, x_i))$.

The first implication states that whenever an alternative is preferred to another one for a given threshold, that preference holds or, at most, turns into an indifference for any other threshold greater than the first considered

[1]If $R \in \mathcal{R}(X)$ and $\alpha \in [0.5, 1)$, the $\alpha$–cut of $R$ is the ordinary preference relation $R_\alpha$ defined by $x_i \, R_\alpha \, x_j \;\Leftrightarrow\; R(x_i, x_j) > \alpha$.

one. The second fact shows that the preference between two alternatives does not change when the threshold becomes smaller. The third statement is a consequence of the previous two: Whenever one alternative is preferred to another for a certain threshold, that preference cannot be reversed for neither a greater nor a smaller threshold.

## 1.3 Ensuring transitive collective decisions

This section includes the results and some illustrative examples. We establish necessary and sufficient conditions on thresholds $k$ for ensuring that majorities based on difference in support provide transitive collective preferences $P_k$ for every profile of several types of individual reciprocal preference relations. To be more specific, we assume different kinds of rationality assumptions to represent the voter's behavior.

In the following proposition we establish a necessary condition on thresholds $k$ in $\widetilde{M}_k$ majorities for having transitive collective preference relations for every profile of reciprocal preference relations satisfying any kind of $g$–transitivity. This necessary condition is very restrictive: $k$ should be at least $m-1$.

**Proposition 1.1.** *If $g$ is a monotonic operator, then there does not exist $k \in [0, m-1)$ such that $P_k$ is transitive for every profile of individual preferences $(R^1, \ldots, R^m) \in T_g^m$.*

*Proof.* Let $g$ be a monotonic operator, $k \in [0, m-1)$ and $R'$, $R''$ be the following reciprocal preference relations:

$$R' = \begin{pmatrix} 0.5 & 1 & 1 & \ldots \\ 0 & 0.5 & 1 & \ldots \\ 0 & 0 & 0.5 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}, \qquad R'' = \begin{pmatrix} 0.5 & 0.5 & 0 & \ldots \\ 0.5 & 0.5 & 0.5 & \ldots \\ 1 & 0.5 & 0.5 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where non clearly stated elements above take the value 0.5. It is not difficult to prove that $R', R'' \in T_g$.

Let consider the preference profile $(R^1, \ldots, R^m)$, where[2]

$$R^i = \begin{cases} R', & \text{if } i = 1, \ldots, \left\lfloor \frac{m+k}{2} \right\rfloor, \\ R'', & \text{if } i = \left\lfloor \frac{m+k}{2} \right\rfloor + 1, \ldots, m. \end{cases}$$

According to expression (1.1), $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ will happen if the following is true:

$$\left\lfloor \frac{m+k}{2} \right\rfloor + \frac{1}{2}\left(m - \left\lfloor \frac{m+k}{2} \right\rfloor\right) > \frac{m+k}{2},$$

or, equivalently, $\left\lfloor \frac{m+k}{2} \right\rfloor > k$. Let see that such condition is always fulfilled. Since $k \in [0, m-1)$, we distinguish two different cases:

1. If $k \in [m-2, m-1)$, then $\left\lfloor \frac{m+k}{2} \right\rfloor = m - 1 > k$.
2. If $k < m - 2$, then $2k < m + k - 2$; that is, $k < \frac{m+k}{2} - 1$. Therefore, $k < \left\lfloor \frac{m+k}{2} \right\rfloor$.

Consequently, $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$. If $P_k$ was transitive, then $x_1 \, P_k \, x_3$; that is,

$$\sum_{p=1}^{m} r_{13}^p = \left\lfloor \frac{m+k}{2} \right\rfloor > \frac{m+k}{2},$$

which is impossible. □

Proposition 1.1 establishes that is not possible to guarantee the transitivity of the collective preference relation for every profile of reciprocal preference relations whenever thresholds are smaller than $m - 1$. In spite of that result, transitivity can be satisfied in some profiles as it is shown in Example 1.1.

**Example 1.1.** Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 1 \\ 0.1 & 0 & 0.5 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 0.5 & 1 & 0.8 \\ 0 & 0.5 & 0.9 \\ 0.2 & 0.1 & 0.5 \end{pmatrix}, \quad R^3 = \begin{pmatrix} 0.5 & 0.9 & 0.8 \\ 0.1 & 0.5 & 1 \\ 0.2 & 0 & 0.5 \end{pmatrix}.$$

[2] Given $x \in \mathbb{R}$, $\lfloor x \rfloor$ means the integer part of $x$; that is, the highest integer lower than or equal to $x$.

It is easy to see that $R^1, R^2, R^3 \notin T_{\min}$.

Taking into account the definition of the collective ordinary preference relation for $\widetilde{M}_k$ majorities given by expression (1.1),

$$x_i \, P_k \, x_j \;\Leftrightarrow\; r_{ij}^1 + r_{ij}^2 + r_{ij}^3 > \frac{3+k}{2},$$

we have that

$$x_1 \, P_k \, x_2 \;\Leftrightarrow\; k < 2.8, \qquad x_2 \, P_k \, x_3 \;\Leftrightarrow\; k < 2.8, \qquad x_1 \, P_k \, x_3 \;\Leftrightarrow\; k < 2\,.$$

Table 1.1 contains the information about the collective preferences and indifferences (see expression (1.2)) for all the thresholds $k \in [0, 3)$. It can be checked that $P_k$ is transitive for every $k \in [0, 2) \cup [2.8, 3)$.

Table 1.1: Collective preferences in Example 1.1.

| | $x_1$ *vs.* $x_2$ | $x_2$ *vs.* $x_3$ | $x_1$ *vs.* $x_3$ |
|---|---|---|---|
| $0 \le k < 2$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_1 \, P_k \, x_3$ |
| $2 \le k < 2.8$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_1 \, I_k \, x_3$ |
| $2.8 \le k < 3$ | $x_1 \, I_k \, x_2$ | $x_2 \, I_k \, x_3$ | $x_1 \, I_k \, x_3$ |

In the following proposition we establish a sufficient condition on the individual rational behavior for ensuring transitive collective preference relations for every profile of individual reciprocal preferences if thresholds are at least $m - 1$. Specifically, this sufficient condition requires that each individual is $g$–transitive, $g$ being a monotonic operator whose values are not smaller than those given by the arithmetic mean.

**Proposition 1.2.** *For each monotonic operator $g$ such that $g(a, b) \ge (a + b)/2$ for all $(a, b) \in [0.5, 1]^2$ and each $k \in [m - 1, m)$, $P_k$ is transitive for every profile of individual preferences $(R^1, \ldots, R^m) \in T_g^m$.*

*Proof.* Let $g$ be a monotonic operator such that $g(a, b) \ge (a + b)/2$ for all $(a, b) \in [0.5, 1]^2$, $k \in [m - 1, m)$, $(R^1, \ldots, R^m) \in T_g^m$ and $x_i, x_j, x_l \in X$ such that $x_i\, P_k\, x_j$ and $x_j\, P_k\, x_l$. Since $k \geq m-1$, it happens that

$$\sum_{p=1}^{m} r_{ij}^{p} > \frac{m+k}{2} \geq m - \frac{1}{2} \quad \text{and} \quad \sum_{p=1}^{m} r_{jl}^{p} > \frac{m+k}{2} \geq m - \frac{1}{2}.$$

For inequalities above to be true, it is necessary that every addend be greater than 0.5; that is, $r_{ij}^{p} > 0.5$ and $r_{jl}^{p} > 0.5$ for all $p \in \{1, \dots, m\}$.

Since $(R^1, \dots, R^m) \in T_g^m$, it happens that $r_{il}^{p} \geq g(r_{ij}^{p}, r_{jl}^{p}) \geq (r_{ij}^{p} + r_{jl}^{p})/2$ for all $p \in \{1, \dots, m\}$. Therefore,

$$\begin{aligned}\sum_{p=1}^{m} r_{il}^{p} &\geq \sum_{p=1}^{m} \frac{r_{ij}^{p} + r_{jl}^{p}}{2} = \frac{1}{2}\left(\sum_{p=1}^{m} r_{ij}^{p} + \sum_{p=1}^{m} r_{jl}^{p}\right) \\ &> \frac{1}{2}\left(\frac{m+k}{2} + \frac{m+k}{2}\right) = \frac{m+k}{2};\end{aligned}$$

that is, $x_i\, P_k\, x_l$. □

Proposition 1.2 ensures the transitivity of the collective preference relation for every profile of reciprocal preference relations satisfying some transitivity conditions, whenever thresholds are at least $m-1$. Example 1.2 shows how inconsistencies diminish when thresholds increase.

**Example 1.2.** Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}, \quad R^2 = \begin{pmatrix} 0.5 & 0.1 & 0.1 \\ 0.9 & 0.5 & 0.9 \\ 0.9 & 0.1 & 0.5 \end{pmatrix}, \quad R^3 = \begin{pmatrix} 0.5 & 0.9 & 0.2 \\ 0.1 & 0.5 & 0.1 \\ 0.8 & 0.9 & 0.5 \end{pmatrix}.$$

It can be tested that $R^1, R^2, R^3 \in T_{\max}$.

Taking into account expression (1.1),

$$x_i\, P_k\, x_j \;\Leftrightarrow\; r_{ij}^{1} + r_{ij}^{2} + r_{ij}^{3} > \frac{3+k}{2},$$

we have that

$$x_1\, P_k\, x_2 \;\Leftrightarrow\; k < 1, \qquad x_2\, P_k\, x_3 \;\Leftrightarrow\; k < 1, \qquad x_3\, P_k\, x_1 \;\Leftrightarrow\; k < 0.4.$$

In Table 1.2 we present the information about the collective preferences and indifferences (see expression (1.2)) for all the thresholds $k \in [0, 3)$. In this case, $P_k$ is not transitive for every $k \in [0, 1)$. Specifically, if $k < 0.4$, we have a cycle, whereas if $k \in [0.4, 1)$ the cycle disappears but transitivity is not satisfied. If $k \geq 1$, all three alternatives are declared socially indifferent, so transitivity is trivially fulfilled.

Table 1.2: Collective preferences in Example 1.2.

| | $x_1$ *vs.* $x_2$ | $x_2$ *vs.* $x_3$ | $x_1$ *vs.* $x_3$ |
|---|---|---|---|
| $0 \leq k < 0.4$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_3 \, P_k \, x_1$ |
| $0.4 \leq k < 1$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_1 \, I_k \, x_3$ |
| $1 \leq k < 3$ | $x_1 \, I_k \, x_2$ | $x_2 \, I_k \, x_3$ | $x_1 \, I_k \, x_3$ |

In the following proposition we establish that the sufficient condition given in Proposition 1.2 is also necessary. Furthermore, if it is not fulfilled, then it is not possible to guarantee collective transitivity for every profile and any threshold.

**Proposition 1.3.** *For each monotonic operator $g$ such that $g(a, b) < (a + b)/2$ for all $(a, b) \in [0.5, 1]^2$ with $a \neq b$, there does not exist $k \in [m - 1, m)$ such that $P_k$ is transitive for every profile of individual preferences $(R^1, \ldots, R^m) \in T_g^m$.*

*Proof.* Let $g$ be a monotonic operator such that $g(a, b) < (a + b)/2$ for all $(a, b) \in [0.5, 1]^2$ with $a \neq b$, and let $h : [0.5, 1] \longrightarrow [0.5, 1]$ be such that $h(a) = g(a, 1)$ for all $a \in [0.5, 1]$. By construction, $h$ is continuous and $h(0.5) = g(0.5, 1) < (0.5 + 1)/2 = 0.75$. Moreover, given $k \in [m - 1, m)$, we have that $(m + k)/2m \geq (2m - 1)/2m$. Since $m \geq 2$, $(2m - 1)/2m \geq 0.75$. Therefore, $(m + k)/2m \geq 0.75$ and $h(0.5) < (m + k)/2m$.

We distinguish two cases:

1. If $h(1) \leq (m + k)/2m$, we consider the profile of individual preferences

$(R^1, \dots, R^m)$ such that, for all $i \in \{1, \dots, m\}$,

$$R^i = \begin{pmatrix} 0.5 & 1 & \frac{m+k}{2m} & \dots \\ 0 & 0.5 & 1 & \dots \\ \frac{m-k}{2m} & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Again, as in the proof of Proposition 1.1, the non clearly stated elements take the value 0.5. It is easy to check that $(R^1, \dots, R^m) \in T_g^m$. By expression (1.1), it is clear that $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$. Since

$$\sum_{p=1}^{m} r_{13}^p = m \frac{m+k}{2m} = \frac{m+k}{2},$$

we have that $\neg(x_1 \, P_k \, x_3)$, that is, $P_k$ is not transitive.

2. If $h(1) > (m+k)/2m$, since $h$ is a continuous function and $h(0.5) < (m+k)/2m$, there exists an $a_k \in (0.5, 1)$ such that $h(a_k) = g(a_k, 1) = (m+k)/2m$.

   Let $R'$, $R''$ and $R'''$ the reciprocal preference relations showed below:

$$R' = \begin{pmatrix} 0.5 & a_k & h(a_k) & \dots \\ 1 - a_k & 0.5 & 1 & \dots \\ 1 - h(a_k) & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R'' = \begin{pmatrix} 0.5 & 1 & h(a_k) & \dots \\ 0 & 0.5 & a_k & \dots \\ 1 - h(a_k) & 1 - a_k & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R''' = \begin{pmatrix} 0.5 & h(a_k) & h(a_k) & \dots \\ 1 - h(a_k) & 0.5 & h(a_k) & \dots \\ 1 - h(a_k) & 1 - h(a_k) & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where non clearly stated elements above take the value 0.5. It is obvious that $R', R'' \in T_g$. Moreover, since $g$ is continuous and $g(a,b) < (a+b)/2$ for all $(a,b) \in [0.5,1]^2$ with $a \neq b$, we have that

$$g(h(a_k), h(a_k)) \leq \frac{h(a_k) + h(a_k)}{2} = h(a_k).$$

Therefore $R''' \in T_g$. We distinguish two cases:

(a) If $m$ is even, we consider the profile of preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R', & \text{if } i = 1, \ldots, \frac{m}{2}, \\ R'', & \text{if } i = \frac{m}{2} + 1, \ldots, m. \end{cases}$$

According to expression (1.1), to $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ to be true it is necessary that

$$\frac{m}{2} a_k + \frac{m}{2} > \frac{m+k}{2}.$$

Let see that such condition is always fulfilled:

$$\frac{m}{2} a_k + \frac{m}{2} = \frac{m}{2}(a_k + 1) > \frac{m}{2}(2g(a_k, 1)) = m \, h(a_k) = \frac{m+k}{2}.$$

If $P_k$ was transitive, then $x_1 \, P_k \, x_3$; in other words,

$$\sum_{p=1}^{m} r_{13}^p = m \, h(a_k) = \frac{m+k}{2} > \frac{m+k}{2},$$

which is impossible.

(b) If $m$ is odd, we consider the profile of preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R', & \text{if } i = 1, \ldots, \frac{m-1}{2}, \\ R'', & \text{if } i = \frac{m+1}{2}, \ldots, m-1, \\ R''', & \text{if } i = m. \end{cases}$$

Having in mind expression (1.1), $x_1 \, P_k \, x_2$ and $x_2 \, P_k \, x_3$ happen if the following is fulfilled:

$$\frac{m-1}{2} a_k + \frac{m-1}{2} + h(a_k) > \frac{m+k}{2}.$$

Let see such condition is always satisfied:

$$\begin{aligned}\frac{m-1}{2}a_k + \frac{m-1}{2} + h(a_k) &= \frac{m-1}{2}(a_k+1) + h(a_k)\\ &> \frac{m-1}{2}(2g(a_k,1)) + h(a_k)\\ &= (m-1)\,h(a_k) + h(a_k) = \frac{m+k}{2}.\end{aligned}$$

If $P_k$ was transitive, then $x_1\,P_k\,x_3$; that is,

$$\sum_{p=1}^{m} r_{13}^p = m\,h(a_k) = \frac{m+k}{2} > \frac{m+k}{2},$$

which is impossible. □

Proposition 1.3 establishes that the transitivity of the collective preference relation cannot be guaranteed for every profile of reciprocal preference relations satisfying some low transitivity conditions, even when thresholds are at least $m-1$. As in Proposition 1.1, this result does not rule out the fact that transitivity can be satisfied in some profiles. To illustrate this, we provide the following example.

**Example 1.3.** Consider the profile of three reciprocal preference relations on $X = \{x_1, x_2, x_3\}$ given by the following matrices

$$R^1 = \begin{pmatrix} 0.5 & 1 & 0.9\\ 0 & 0.5 & 0.9\\ 0.1 & 0.1 & 0.5\end{pmatrix},\quad R^2 = \begin{pmatrix} 0.5 & 0.9 & 0.9\\ 0.1 & 0.5 & 1\\ 0.1 & 0 & 0.5\end{pmatrix},\quad R^3 = \begin{pmatrix} 0.5 & 1 & 0.9\\ 0 & 0.5 & 1\\ 0.1 & 0 & 0.5\end{pmatrix}.$$

It can be checked that $R^1, R^2, R^3 \notin T_{\text{am}}$.

By expression (1.1),

$$x_i\,P_k\,x_j \;\Leftrightarrow\; r_{ij}^1 + r_{ij}^2 + r_{ij}^3 > \frac{3+k}{2},$$

we have that

$$x_1\,P_k\,x_2 \;\Leftrightarrow\; k < 2.8, \qquad x_2\,P_k\,x_3 \;\Leftrightarrow\; k < 2.8, \qquad x_1\,P_k\,x_3 \;\Leftrightarrow\; k < 2.4.$$

Table 1.3: Collective preferences in Example 1.3.

| | $x_1$ *vs.* $x_2$ | $x_2$ *vs.* $x_3$ | $x_1$ *vs.* $x_3$ |
|---|---|---|---|
| $0 \leq k < 2.4$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_1 \, P_k \, x_3$ |
| $2.4 \leq k < 2.8$ | $x_1 \, P_k \, x_2$ | $x_2 \, P_k \, x_3$ | $x_1 \, I_k \, x_3$ |
| $2.8 \leq k < 3$ | $x_1 \, I_k \, x_2$ | $x_2 \, I_k \, x_3$ | $x_1 \, I_k \, x_3$ |

Table 1.3 contains the information about the collective preferences and indifferences (see expression (1.2)) for all the thresholds $k \in [0, 3)$. According to this situation, $P_k$ is transitive for every $k \in [0, 2.4) \cup [2.8, 3)$.

## 1.4 Concluding remarks

In this paper we have presented some necessary and sufficient conditions under which $\widetilde{M}_k$ majorities lead to transitive collective preference relations. These results highlight the importance of the arithmetic mean for aggregating individual intensities of preference.

Unfortunately, the results displayed above lead us almost to impossibility results. We have considered different assumptions about individuals' rationality but, under every specification, the way of ensuring transitive collective relations approaches unanimous support. In other words, a high threshold is required in order to get transitive collective preference relations for every profile of individual reciprocal preference relations. In practice, an almost unanimous support is needed for avoiding intransitivities.

Indeed, transitivity may be considered as a really strong condition. We left the study of the conditions required to get acyclic collective preference relations as an alternative way to study the consistency of collective preference relations for further research. With regard to this, we want to remark the results in Kramer [81] and in Slutsky [107]. In the first case, acyclic collective preference relations are reached by using a specific qualified majority, the *minmax voting rule*, and a particular structure of dichotomous collective preferences, called *Type I utility functions*. In the second case, the collective preference relation is built on the basis of a majority which takes into account the logarithmic difference between the number of individuals who prefer an alternative, say $x$, to another, say $y$, and the number of individuals who prefer $y$ to $x$. When this difference reaches a specific value, the collective preference relation is acyclic for any type of individual preferences (see also Craven [21] and Ferejohn and Grether [38]).

Moreover, we want to point out that there is a significant disparity between the possibility of having cycles and the empirical occurrence of them (see Gehrlein and Fishburn [61], Gehrlein [57] and Tangian [111], among others). We left the study of such occurrence in the case of $\widetilde{M}_k$ majorities for further research.

From a practical point of view, the class of majorities based on difference in support may be applied to several scenarios whenever one alternative has to be chosen with a desirable support with respect to other alternatives. Among international organizations, an interesting case is the European Commission, where voters are the different members states.

# Chapter 2

# Triple-acyclicity in majorities based on difference in support

[This chapter is under review (jointly with Bonifacio Llamazares).]

In this paper we study to what extent majorities based on difference in support leads to triple-acyclic collective decisions. These majorities, which take into account voters' intensities of preference between pairs of alternatives through reciprocal preference relations, require to the winner alternative to exceed the support for the other alternative in a difference fixed before the election. Depending on that difference, i.e., on the threshold of support, and on some requirements on the individual rationality of the voters, we provide necessary and sufficient conditions for avoiding cycles of three alternatives on the collective decision.

## 2.1 Introduction

The aggregation of individual preferences under the simple majority rule could lead to a cyclical collective preference. This fact was firstly pointed out by Condorcet [18] and known since then as the *Condorcet's paradox*. Recalling the classical example of this paradox, assume the following three voters' preferences over three alternatives $x_1$, $x_2$ and $x_3$:

$$x_1 \succ_1 x_2 \succ_1 x_3 \qquad x_2 \succ_2 x_3 \succ_2 x_1 \qquad x_3 \succ_3 x_1 \succ_3 x_2, \tag{2.1}$$

where $x_i \succ_p x_j$ $\big(i, j, p \in \{1, 2, 3\}\big)$ means that individual $p$ strictly prefers alternative $x_i$ to alternative $x_j$ when both alternatives are in comparison. Guided by the preferences given in (2.1), individuals cast a vote for their preferred alternative in pairwise contests. Aggregating these votes we obtain that alternatives $x_1$, $x_2$ and $x_3$ defeat $x_2$, $x_3$ and $x_1$, respectively, by two votes to one. Therefore, the collective preference relation $\succ$ is cyclical (and hence intransitive) given that $x_1 \succ x_2 \succ x_3 \succ x_1$.

In the above example, individual preferences are misrepresented. Every voter declares if he/she prefers an alternative to another one but nothing about the quantification of this preference. A wide variety of authors (see, for instance, Morales [97], Sen [106] and Nurmi [99]) have pointed out the importance for a voting system (for getting a representative aggregation of individual preferences) of taking into account the individuals' intensities of preference among the alternatives in comparison. Reciprocal preference relations formalize such idea. Through them, and by using values in the unit interval, every voter declares his/her intensity of preference between the alternatives compared by pairs. Following this approach, we assume that individuals' preferences are given by reciprocal preference relations and that they fulfill some kind of transitivity condition to avoid misleading preferences.

The introduction of the intensities of preference promotes the extension of several aggregation rules to the context of reciprocal preference relations. In the field of majority rules stand out the efforts done by García-Lapresta and Llamazares to extend some of them through different operators that aggregate individual reciprocal preferences (see García-Lapresta and Llamazares [48, 87, 88], and Llamazares [84, 86]). They have also introduced *majorities based on difference in support* or $\widetilde{M}_k$ *majorities* (see García-Lapresta and Llamazares [50]). Under them, an alternative $x_i$ defeats another one $x_j$

if the sum of the voters' intensities of preference for $x_i$ exceeds the sum of the intensities of preferences for $x_j$ in a given quantity, a threshold $k$, fixed before the election process. These rules extend *majorities based on difference of votes* (see García-Lapresta and Llamazares [49] and, for the axiomatic characterization, Llamazares [85] and Houy [74]) from the context of ordinary individual preferences to that of intensities of preference.

In the area of ordinary individual preferences, studies about the consistency of several majority rules have been previously done by Greenberg [66], Coughlin [19, 20], Caplin and Nalebuff [10] and Weber [116].

In the area of individual reciprocal preferences, Llamazares *et al.* [90] have introduced conditions that ensure collective transitivity decisions under majorities based on difference in support. Unfortunately, such conditions require a high support to declare an alternative as a winner regarding highly rational reciprocal preference relations.

In this paper we establish the thresholds $k$ such that majorities based on difference in support do not generate cycles of three alternatives on the collective preference. Relaxing the collective consistency condition to the weakest one, i.e., triple-acyclicity, promotes the obtaining of more reasonable requirements than in the case of the collective transitivity. To be specific, we will show the following results:

1. The collective preference is triple-acyclic, taking into account 0.5-transitive reciprocal preference relations, if the threshold is equal to or greater than the integer part of two thirds the total number of voters.

2. The collective preference is triple-acyclic, taking into account min-transitive or max-transitive reciprocal preference relations, if the threshold is equal to or greater than one third the number of voters.

The paper is organized as follows. Section 2.2 is devoted to introduce the basic technical concepts we deal with. Our main results are stated in sections 2.3 and 2.4. Specifically, in Section 2.3, we set the conditions for triple-acyclic collective decisions under majorities based on difference in support when the rationality of individual preferences is the weakest that it can be, whereas in Section 2.4 stronger individual rationality conditions than that are taken into account. Finally, Section 2.5 is dedicated to compare the results obtained here with those on transitive collective decisions under majorities based on difference in support stated in Llamazares *et al.* [90].

## 2.2 Preliminaries

This section is organized in three different parts: the first one deals with the types of preference relations concerned, in particular, reciprocal preference relations and ordinary preference relations. The second one is about the consistency conditions asked to these preference relations. Finally, the third one is dedicated to explain the aggregation rule, i.e., majorities based on difference in support.

### 2.2.1 Preference relations

Consider a set of $m$ voters, $V = \{1, \ldots, m\}$, who show their intensities of preferences on a set of alternatives, $X = \{x_1, \ldots, x_n\}$, through *reciprocal preference relations* $R^p : X \times X \longrightarrow [0, 1]$, $p = 1, \ldots, m$; i.e., $r^p_{ij} + r^p_{ji} = 1$ for all $i, j \in \{1, \ldots, n\}$, where $r^p_{ij} = R^p(x_i, x_j)$. In the context we consider, it is usual to represent $R^p$ by an $n \times n$ matrix which coefficients in the unit interval,

$$R^p = \begin{pmatrix} r^p_{11} & r^p_{12} & \ldots & r^p_{1n} \\ r^p_{21} & r^p_{22} & \ldots & r^p_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ r^p_{n1} & r^p_{n2} & \ldots & r^p_{nn} \end{pmatrix}.$$

By reciprocity, all the main diagonal elements are 0.5 and $r^p_{ji} = 1 - r^p_{ij}$ if $j > i$. Therefore,

$$R^p = \begin{pmatrix} 0.5 & r^p_{12} & \ldots & r^p_{1n} \\ 1 - r^p_{12} & 0.5 & \ldots & r^p_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ 1 - r^p_{1n} & 1 - r^p_{2n} & \ldots & 0.5 \end{pmatrix}.$$

In this framework, voters can declare their preferences between alternatives, ordered by pairs, but also the degree with which they prefer one alternative to other one through numbers in $[0, 1]$. Obviously, they also could declare themselves indifferent between the alternatives. To be more concrete, given two alternatives $x_i$ and $x_j$, if voter $p$ is indifferent between these two alternatives, then $r_{ij}^p = 0.5$. If he/she absolutely prefers the alternative $x_i$ to the alternative $x_j$, then $r_{ij}^p = 1$; on the contrary, if he/she absolutely prefers the alternative $x_j$ to $x_i$, then $r_{ij} = 0$. So far, the preferences described above can be viewed as a representation of ordinary preferences.

Notice that an *ordinary preference relation* over $X$ is an *asymmetric* binary relation on $X$: if $x_i P x_j$, then does not happen $x_j P x_i$. The *indifference relation* associated with $P$ is defined as $x_i I x_j$ and it means that neither $x_i$ is preferred to $x_j$ nor $x_j$ is preferred to $x_i$.

So, every ordinary preference relation $P$ can be considered as a reciprocal preference relation $R$. Furthermore, a reciprocal preference relation $R$ is *crisp* if $r_{ij}^p \in \{0, 0.5, 1\}$ for all $i, j \in \{1, \ldots, n\}$; so, we can consider ordinary preferences and crisp preferences as equivalent for practical purposes.

Coming back to reciprocal preference relations[1], they allow voters to describe not so extreme preferences as the absolute preference or the indifference stated above. Specifically, if a voter somewhat prefers alternative $x_i$ to $x_j$, then $0.5 < r_{ij}^p < 1$ and the closer is this number to 1, the more $x_i$ is preferred to $x_j$. On the contrary, if a voter somewhat prefers alternative $x_j$ to $x_i$, then $0 < r_{ij}^p < 0.5$ and the closer is this number to 0, the more $x_j$ is preferred to $x_i$.

Throughout the paper, $\mathcal{R}(X)$ denotes the set of reciprocal preference relations on $X$ and $\mathcal{P}(X)$ denotes the set of ordinary preference relations on $X$.[2] A *profile* is a vector $(R_1, \ldots, R_m)$ which contains the individual preference relations on $X$. The set of profiles is denoted by $\mathcal{R}(X)^m$. Moreover, given a set $A$, $\#A$ will denote the cardinality of $A$. Lastly, given $a \in \mathbb{R}$, $\lfloor a \rfloor$ will denote the integer part of $a$; that is, the highest integer lower than or equal to $a$.

[1]Note that the property of reciprocity extends the properties of asymmetry (see Lapresta and Meneses [53]) and completeness (see De Baets and De Meyer [24]) from the framework of ordinary preferences to the context of intensities of preference.

[2]Unlike in this paper, the notation $\mathcal{P}(X)$ is also used in Ensemble Theory for denoting the set of all subsets of $X$.

### 2.2.2 Consistency on preference relations

In the context of ordinary preference relations, it is usual to relate consistency to transitivity condition, but, as said before, in this paper we focus on a weaker consistency property than that, i.e., triple-acyclicity. Triple-acyclicity is obtained by considering the acyclicity condition restricted to three alternatives. This property has been widely studied in the framework of Social Choice theory, specifically in the area of social choice functions (see, among others Sen [106], Suzumura [108], Schwartz [104] and Cato and Hirata [11]). It provides the minimum consistency requirement for social decisions, that is, to avoid cycles of three alternatives. In the following definitions, we formally recall transitivity, acyclicity and triple-acyclicity conditions.

**Definition 2.1.** An ordinary preference relation $P \in \mathcal{P}(X)$ is

1. *transitive* if for all $x_i, x_j, x_l \in X$ it holds that if $x_i P x_j$ and $x_j P x_l$, then it also holds $x_i P x_l$.
2. *acyclic* if for all $x_{i_1}, \ldots, x_{i_s} \in X$ it holds that if $x_{i_1} P x_{i_2}, \ldots, x_{i_{s-1}} P x_{i_s}$, then it does not happen $x_{i_s} P x_{i_1}$.
3. *triple-acyclic* if for all $x_i, x_j, x_l \in X$ it holds that if $x_i P x_j$ and $x_j P x_l$, then it does not happen $x_l P x_i$.

Given that transitivity is a stronger condition than acyclicity and that one is a stronger requirement than triple-acyclicity, if a preference relation is not triple-acyclic, it is also not acyclic and, consequently, not transitive. Obviously, acyclicity and triple-acyclicity are equivalent when the social decision involves three alternatives.

In the context of reciprocal preference relations, the notion of transitivity is not as clear as it is in the context of ordinary preference relations (see, for instance, Chiclana *et al.* [15, 16], Dasgupta and Deb [22], De Baets and De Meyer [24], De Baets *et al.* [26], Dubois and Prade [37], Freson *et al.* [45], Freson *et al.* [46], García-Lapresta and Meneses [53], García-Lapresta and Montero [55], Genç *et al.* [65], Herrera-Viedma *et al.* [73], Jain [75], Świtalski [109, 110], Tanino [112], Xu *et al.* [118], Zadeh [123]).

In our case, we make use of a monotonic operator to define the transitivity conditions for the reciprocal preference relations. Proper definitions are stated below.

**Definition 2.2.** A function $g : [0.5, 1]^2 \longrightarrow [0.5, 1]$ is a *monotonic operator* if it satisfies the following conditions:

1. Continuity.
2. Increasingness: $g(a, b) \geq g(c, d)$ for all $a, b, c, d \in [0.5, 1]$ such that $a \geq c$ and $b \geq d$.
3. Symmetry: $g(a, b) = g(b, a)$ for all $a, b \in [0.5, 1]$.

**Definition 2.3.** Given a monotonic operator $g$, $R \in \mathcal{R}(X)$ is *g–transitive* if for all $i, j, l \in \{1, \dots, n\}$ the following holds:

$$(r_{ij} > 0.5 \text{ and } r_{jl} > 0.5) \Rightarrow \big(r_{il} > 0.5 \text{ and } r_{il} \geq g(r_{ij}, r_{jl})\big).$$

$T_g$ denotes the set of all $g$–transitive reciprocal preference relations. Notice that, given two monotonic operators, say $f$ and $g$, such that $f \leq g$, i.e., $f(a, b) \leq g(a, b)$ for all $a, b \in [0.5, 1]$, if a reciprocal preference relation $R$ is $g$–transitive, then $R$ is also $f$–transitive; in other words, $T_g \subseteq T_f$.

We only consider three monotonic operators in order to model the consistency on individual reciprocal preference relations: the constant function 0.5, the minimum and the maximum.

1. $R$ is 0.5*–transitive* if $R$ is $g$–transitive, with $g(a, b) = 0.5$ for all $a, b \in [0.5, 1]$.
2. $R$ is *min-transitive* if $R$ is $g$–transitive, with $g(a, b) = \min\{a, b\}$ for all $a, b \in [0.5, 1]$.
3. $R$ is *max-transitive* if $R$ is $g$–transitive, with $g(a, b) = \max\{a, b\}$ for all $a, b \in [0.5, 1]$.

We denote with $T_{0.5}$, $T_{\min}$, $T_{\max}$ the sets of all 0.5–transitive, min–transitive and max–transitive reciprocal preference relations, respectively. Obviously, $T_{\max} \subseteq T_{\min} \subseteq T_{0.5}$.

### 2.2.3 Majorities based on difference in support

Majorities based on difference in support (also called $\widetilde{M}_k$ majorities), aggregate individual intensities of preference, i.e., reciprocal preference relations, into collective ordinary preferences. When we compare two alternatives, they declare an alternative as the winner if the sum of the intensities for that alternative exceeds the sum of the intensities for the other one in a threshold, fixed before the election process. Such threshold varies in a continuous space given that intensities also do it. $\widetilde{M}_k$ majorities were introduced and axiomatically characterized by García-Lapresta and Llamazares [50] and they generalize majorities based on difference of votes (see García-Lapresta and Llamazares [49] and Llamazares [85]), which ask to the winner alternative for a positive difference of votes with respect to the other one, also fixed before the election process. In the following definition we formally present these majorities.

**Definition 2.4.** Given a threshold $k \in [0, m)$, the $\widetilde{M}_k$ *majority* is the mapping $\widetilde{M}_k : \mathcal{R}(X)^m \longrightarrow \mathcal{P}(X)$ defined by $\widetilde{M}_k(R^1, \ldots, R^m) = P_k$, where

$$x_i P_k x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \sum_{p=1}^{m} r_{ji}^p + k.$$

As we just show, $\widetilde{M}_k$ majorities assign a collective ordinary preference relation to each profile of individual reciprocal preference relations. It is easy to check (see García-Lapresta and Llamazares [50]) that $P_k$ can be defined through the average of the individual intensities of preference:

$$x_i P_k x_j \;\Leftrightarrow\; \frac{1}{m} \sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2m},$$

or, equivalently,

$$x_i P_k x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2}. \tag{2.2}$$

The indifference relation associated with $P_k$ is defined by:

$$x_i I_k x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \sum_{p=1}^{m} r_{ji}^p \right| \leq k,$$

or, equivalently,

$$x_i I_k x_j \;\Leftrightarrow\; \left|\sum_{p=1}^{m} r_{ij}^p - \frac{m}{2}\right| \leq \frac{k}{2}.$$

Some interesting facts could be stated about the behavior of $\widetilde{M}_k$ majorities. Assume that an alternative is preferred to another one for a given threshold. If the threshold becomes smaller, then the preference does not change. And if such a threshold becomes greater than before, the preference holds or, at most, turns into indifference. Due to both facts, we have that whenever an alternative is preferred to another one for a certain threshold, such preference cannot be reverse for neither a greater, nor a smaller threshold (see Remark 1 in Llamazares *et al.* [90]).

## 2.3 Triple-acyclicity when individuals are 0.5–transitive

This section includes the conditions on thresholds $k$ for triple-acyclic collective decisions $P_k$ when individual reciprocal relations fulfill 0.5–transitivity. In such a case, these thresholds depend on the number of voters involved on the election process, which is reflected in the following results.

In Theorem 2.1 we show that for any threshold smaller than $\lfloor 2m/3 \rfloor$ we can find profiles of 0.5–transitive reciprocal preferences for which the triple-acyclicity on collective decision fails.

**Theorem 2.1.** *There does not exist $k \in \big[0, \lfloor 2m/3 \rfloor\big)$ such that $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, \ldots, R^m) \in T_{0.5}^m$.*

*Proof.* Let $k \in \big[0, \lfloor 2m/3 \rfloor\big)$ and let $R^{\mathrm{I}}$, $R^{\mathrm{II}}$, $R^{\mathrm{III}}$ and $R^{\mathrm{IV}}$ be the following reciprocal preference relations:

$$R^{\mathrm{I}} = \begin{pmatrix} 0.5 & 1 & \frac{3}{4} - \frac{k}{4\lfloor 2m/3 \rfloor} & \dots \\ 0 & 0.5 & 1 & \dots \\ \frac{1}{4} + \frac{k}{4\lfloor 2m/3 \rfloor} & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R^{\mathrm{II}} = \begin{pmatrix} 0.5 & 1 & 0 & \dots \\ 0 & 0.5 & \frac{1}{4} + \frac{k}{4\lfloor 2m/3 \rfloor} & \dots \\ 1 & \frac{3}{4} - \frac{k}{4\lfloor 2m/3 \rfloor} & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R^{\mathrm{III}} = \begin{pmatrix} 0.5 & \frac{1}{4} + \frac{k}{4\lfloor 2m/3 \rfloor} & 0 & \dots \\ \frac{3}{4} - \frac{k}{4\lfloor 2m/3 \rfloor} & 0.5 & 1 & \dots \\ 1 & 0 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

$$R^{\mathrm{IV}} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & \dots \\ 0.5 & 0.5 & 0.5 & \dots \\ 0.5 & 0.5 & 0.5 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix},$$

where non clearly stated elements above take the value of 0.5. It is easy to check that the previous reciprocal relations belong to $T_{0.5}$. We distinguish three cases:

1. If $m = 3q$, with $q \in \mathbb{N}$, we consider the profile of individual preferences $(R^1, \dots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \dots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \dots, 2q, \\ R^{\mathrm{III}} & \text{if } i = 2q+1, \dots, 3q. \end{cases}$$

   According to equivalence (2.2) and given that $\lfloor 2m/3 \rfloor = 2q$, $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$q\left(1 + 1 + \frac{1}{4} + \frac{k}{8q}\right) > \frac{3q + k}{2}.$$

Since

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q}\right) > \frac{3q+k}{2} \Leftrightarrow 18q+k > 12q+4k$$
$$\Leftrightarrow k < 2q,$$

$2q = \lfloor 2m/3 \rfloor$, and $k < \lfloor 2m/3 \rfloor$ is satisfied by hypothesis, we get $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$, and, consequently, $P_k$ is not triple-acyclic.

2. If $m = 3q+1$, with $q \in \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \ldots, 2q, \\ R^{\mathrm{III}} & \text{if } i = 2q+1, \ldots, 3q, \\ R^{\mathrm{IV}} & \text{if } i = 3q+1. \end{cases}$$

According to equivalence (2.2) and given that $\lfloor 2m/3 \rfloor = 2q$, $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q}\right) + 0.5 > \frac{3q+1+k}{2}.$$

Since

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q}\right) + 0.5 > \frac{3q+1+k}{2} \Leftrightarrow 18q+k > 12q+4k$$
$$\Leftrightarrow k < 2q,$$

$2q = \lfloor 2m/3 \rfloor$, and $k < \lfloor 2m/3 \rfloor$ is satisfied by hypothesis, we get $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$, and, consequently, $P_k$ is not triple-acyclic.

3. If $m = 3q+2$, with $q \in \{0\} \cup \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q+1, \\ R^{\mathrm{II}} & \text{if } i = q+2, \ldots, 2q+2, \\ R^{\mathrm{III}} & \text{if } i = 2q+3, \ldots, 3q+2. \end{cases}$$

According to equivalence (2.2) and given that $\lfloor 2m/3 \rfloor = 2q+1$, $x_1 P_k x_2$ will happen if

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q+4}\right)+2 > \frac{3q+2+k}{2}.$$

On the other hand, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q+4}\right)+1+\frac{1}{4}+\frac{k}{8q+4} > \frac{3q+2+k}{2}.$$

Since

$$\begin{aligned} q\left(\frac{9}{4}+\frac{k}{8q+4}\right)+\frac{5}{4}+\frac{k}{8q+4} > \frac{3q+2+k}{2} \\ &\Leftrightarrow\ q\left(9+\frac{k}{2q+1}\right)+5+\frac{k}{2q+1} > 6q+4+2k \\ &\Leftrightarrow\ 3q+1 > k\left(2-\frac{q+1}{2q+1}\right) \\ &\Leftrightarrow\ 3q+1 > k\frac{3q+1}{2q+1}\ \Leftrightarrow\ k < 2q+1, \end{aligned}$$

$2q+1 = \lfloor 2m/3 \rfloor$, and $k < \lfloor 2m/3 \rfloor$ is satisfied by hypothesis, we have $x_2 P_k x_3$ and $x_3 P_k x_1$. Moreover, given that

$$q\left(1+1+\frac{1}{4}+\frac{k}{8q+4}\right)+2 > q\left(1+1+\frac{1}{4}+\frac{k}{8q+4}\right)+1+\frac{1}{4}+\frac{k}{8q+4},$$

we also have $x_1 P_k x_2$. Therefore, $P_k$ is not triple-acyclic. □

Triple-acyclic collective decisions are guaranteed when the threshold is greater than or equal to $\lfloor 2m/3 \rfloor$. Before establishing this result, we specify in the following lemma the minimum number of 0.5–transitive individuals who have to prefer an alternative to another one to reach a particular collective intensity of preference for the first alternative over the second one.

**Lemma 2.1.** *Let* $(R^1, \ldots, R^m) \in T^m_{0.5}$*. Given* $a \in \mathbb{R}$ *and* $i, j \in \{1, \ldots, n\}$*, if* $\sum_{p=1}^m r^p_{ij} > a$*, then there are at least* $\lfloor 2a - m \rfloor + 1$ *individuals for which* $r^p_{ij} > 0.5$*.*

*Proof.* The following case provides the minimum number of individuals for which $r_{ij}^p > 0.5$:

1. If $r_{ij}^p > 0.5$, then $r_{ij}^p = 1$.
2. If $r_{ij}^p \leq 0.5$, then $r_{ij}^p = 0.5$.

Therefore, in this case, if $z$ is the number of individuals for which $r_{ij}^p > 0.5$, we have

$$\sum_{p=1}^{m} r_{ij}^p > a \;\Leftrightarrow\; 1z + 0.5(m-z) > a \;\Leftrightarrow\; 0.5z > a - 0.5m$$
$$\Leftrightarrow\; z > 2a - m \;\Leftrightarrow\; z \geq \lfloor 2a - m \rfloor + 1. \qquad \square$$

**Theorem 2.2.** *If $k \in \big[\lfloor 2m/3 \rfloor, m\big)$, then $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, \ldots, R^m) \in T_{0.5}^m$.*

*Proof.* We are going to prove that if $P_k$ is not triple-acyclic, then $k < \lfloor 2m/3 \rfloor$. Suppose there exist $(R^1, \ldots, R^m) \in T_{0.5}^m$ and $i, j, l \in \{1, \ldots, n\}$ such that $x_i P_k x_j$, $x_j P_k x_l$ and $x_l P_k x_i$. According to equivalence (2.2) we have

$$\sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2}, \qquad \sum_{p=1}^{m} r_{jl}^p > \frac{m+k}{2} \qquad \text{and} \qquad \sum_{p=1}^{m} r_{li}^p > \frac{m+k}{2}.$$

Then, by Lemma 2.1, we get

$$\#\big\{p \in \{1, \ldots, m\} \mid r_{ij}^p > 0.5\big\} + \#\big\{p \in \{1, \ldots, m\} \mid r_{jl}^p > 0.5\big\}$$
$$+ \#\big\{p \in \{1, \ldots, m\} \mid r_{li}^p > 0.5\big\} \geq 3\big(\lfloor k \rfloor + 1\big).$$

On the other hand, $R^p \in T_{0.5}$ for every $p \in \{1, \ldots, m\}$. Therefore, for every $p \in \{1, \ldots, m\}$, at most two of the values $r_{ij}^p$, $r_{jl}^p$ and $r_{li}^p$ are greater than 0.5. So,

$$\#\big\{p \in \{1, \ldots, m\} \mid r_{ij}^p > 0.5\big\} + \#\big\{p \in \{1, \ldots, m\} \mid r_{jl}^p > 0.5\big\}$$
$$+ \#\big\{p \in \{1, \ldots, m\} \mid r_{li}^p > 0.5\big\} \leq 2m.$$

Consequently,

$$3\big(\lfloor k \rfloor + 1\big) \leq 2m \;\Leftrightarrow\; \lfloor k \rfloor \leq \frac{2m}{3} - 1 \;\Leftrightarrow\; k < \left\lfloor \frac{2m}{3} \right\rfloor. \qquad \square$$

## 2.4 Triple-acyclicity when individuals are $g$–transitive ($g \geq \min$)

Now, we explore the conditions for triple-acyclic collective decisions under majorities based on difference in support when reciprocal preference relations are $g$–transitive being $g$ a function greater than or equal to the minimum operator.

Next lemma states that whenever an individual is endowed with the just described reciprocal preference relations over three alternatives, say $x_i$, $x_j$ and $x_l$, then the sum of the intensities $r_{ij}$, $r_{jl}$ and $r_{li}$ reaches at maximum the value of 2.

**Lemma 2.2.** *Let $g$ be a monotonic operator such that $g \geq \min$. If $R \in T_g$, then $r_{ij} + r_{jl} + r_{li} \leq 2$ for all $i, j, l \in \{1, \ldots, n\}$.*

*Proof.* Assume, by reduction to absurdity, that there exist $i, j, l \in \{1, \ldots, n\}$ such that $r_{ij} + r_{jl} + r_{li} > 2$. From this inequality we get that at least two of the above addends are greater than 0.5. But, since $R$ is $g$–transitive (being $g$ a function greater than or equal to the minimum), only two of the above addends are greater than 0.5. Assume that $r_{ij}, r_{jl} > 0.5$. Then $r_{il} \geq \min\{r_{ij}, r_{jl}\}$ or, in the same way, $r_{li} \leq \max\{r_{ji}, r_{lj}\} = \max\{1 - r_{ij}, 1 - r_{jl}\}$. Therefore,

$$r_{ij} + r_{jl} + r_{li} \leq r_{ij} + r_{jl} + \max\{1 - r_{ij}, 1 - r_{jl}\} \leq 2,$$

which contradicts $r_{ij} + r_{jl} + r_{li} > 2$.

With the other two possible cases, say $r_{ij}, r_{li} > 0.5$ and $r_{jl}, r_{li} > 0.5$, the contradiction is also achieved with a similar reasoning as the one just used for the case $r_{ij}, r_{jl} > 0.5$. □

Now, we can establish a general result for the individual preferences that fulfil the types of transitivities included in this section.

**Theorem 2.3.** *For each monotonic operator $g$ such that $g \geq \min$ and each $k \in [m/3, m)$, $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, \ldots, R^m) \in T_g^m$.*

*Proof.* Assume, by reduction to absurdity, that $P_k$ is not triple-acyclic. Then, it exists a profile of preferences $(R^1, \ldots, R^m) \in T_g^m$ and $i, j, l \in \{1, \ldots, n\}$ such that $x_i P_k x_j$, $x_j P_k x_l$ and $x_l P_k x_i$; that is,

$$\sum_{p=1}^{m} r_{ij}^p > \frac{m+k}{2}, \qquad \sum_{p=1}^{m} r_{jl}^p > \frac{m+k}{2} \qquad \text{and} \qquad \sum_{p=1}^{m} r_{li}^p > \frac{m+k}{2}.$$

Adding member to member the three inequalities above and taking into account that $k \geq m/3$, we have

$$\sum_{p=1}^{m} r_{ij}^p + \sum_{p=1}^{m} r_{jl}^p + \sum_{p=1}^{m} r_{li}^p > \frac{3}{2}(m+k) \geq \frac{3}{2}\left(m + \frac{m}{3}\right) = 2m. \tag{2.3}$$

But, by Lemma 2.2, we have

$$\sum_{p=1}^{m} r_{ij}^p + \sum_{p=1}^{m} r_{jl}^p + \sum_{p=1}^{m} r_{li}^p = \sum_{p=1}^{m} (r_{ij}^p + r_{jl}^p + r_{li}^p) \leq 2m,$$

which contradicts inequality (2.3). □

The previous theorem allows us to guarantee triple-acyclic collective decisions when the threshold is greater than or equal to $m/3$. In what follows, the remaining values of the threshold are analyzed according to whether the reciprocal preference relations fulfil min–transitivity or max–transitivity.

### 2.4.1 The case $g = \min$

As we establish in the following theorem, when the threshold is smaller than $m/3$, we can find profiles of min–transitive reciprocal preferences for which the triple-acyclicity on collective decision fails.

**Theorem 2.4.** *There does not exist $k \in [0, m/3)$ such that $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, \ldots, R^m) \in T_{\min}^m$.*

*Proof.* Let $k \in [0, m/3)$ and let $R^{\mathrm{I}}$, $R^{\mathrm{II}}$, $R^{\mathrm{III}}$, $R^{\mathrm{IV}}$ and $R^{\mathrm{V}}$ be the following reciprocal preference relations:

$$R^{\mathrm{I}} = \begin{pmatrix} 0.5 & 1 & \frac{2}{3} & \cdots \\ 0 & 0.5 & \frac{2}{3} & \cdots \\ \frac{1}{3} & \frac{1}{3} & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad R^{\mathrm{II}} = \begin{pmatrix} 0.5 & \frac{1}{3} & 0 & \cdots \\ \frac{2}{3} & 0.5 & \frac{2}{3} & \cdots \\ 1 & \frac{1}{3} & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$R^{\mathrm{III}} = \begin{pmatrix} 0.5 & 1 & 1 & \cdots \\ 0 & 0.5 & 1 & \cdots \\ 0 & 0 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad R^{\mathrm{IV}} = \begin{pmatrix} 0.5 & 1 & 0 & \cdots \\ 0 & 0.5 & 0 & \cdots \\ 1 & 1 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$R^{\mathrm{V}} = \begin{pmatrix} 0.5 & 0 & 0 & \cdots \\ 1 & 0.5 & 1 & \cdots \\ 1 & 0 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where non clearly stated elements above take the value of 0.5. It is easy to check that the previous reciprocal relations belong to $T_{\min}$. We distinguish two cases according to whether $m$ is even or odd.

1. If $m = 2q$, with $q \in \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \ldots, 2q. \end{cases}$$

Suppose $(i,j) \in \{(1,2),(2,3),(3,1)\}$. Note that $r_{ij}^p + r_{ij}^{q+p} = 4/3$ for all $p \in \{1, \ldots, q\}$. Therefore,

$$\sum_{p=1}^{m} r_{ij}^p = \sum_{p=1}^{q} (r_{ij}^p + r_{ij}^{q+p}) = \frac{m}{2}\frac{4}{3} = \frac{2m}{3}.$$

Now, according to equivalence (2.2), $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$\frac{2m}{3} > \frac{m+k}{2} \Leftrightarrow k < \frac{m}{3},$$

which is satisfied by hypothesis. Consequently, $P_k$ is not triple-acyclic.

2. If $m = 2q + 3$, with $q \in \{0\} \cup \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\text{I}} & \text{if } i = 1, \ldots, q, \\ R^{\text{II}} & \text{if } i = q+1, \ldots, 2q, \\ R^{\text{III}} & \text{if } i = 2q+1, \\ R^{\text{IV}} & \text{if } i = 2q+2, \\ R^{\text{V}} & \text{if } i = 2q+3. \end{cases}$$

Suppose $(i,j) \in \{(1,2),(2,3),(3,1)\}$. Note that $r_{ij}^p + r_{ij}^{q+p} = 4/3$ for all $p \in \{1, \ldots, q\}$ and $r_{ij}^{2q+1} + r_{ij}^{2q+2} + r_{ij}^{2q+3} = 2$. Therefore,

$$\sum_{p=1}^{m} r_{ij}^p = \sum_{p=1}^{q} (r_{ij}^p + r_{ij}^{q+p}) + 2 = \frac{m-3}{2}\frac{4}{3} + 2 = \frac{2m}{3}.$$

Now, according to equivalence (2.2), $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$\frac{2m}{3} > \frac{m+k}{2} \Leftrightarrow k < \frac{m}{3},$$

which is satisfied by hypothesis. So, $P_k$ is not triple-acyclic. □

### 2.4.2 The case $g = \max$

Analogously to the case $g = \min$, our aim is to analyze what happens when the threshold is smaller than $m/3$. But, in this case, we can only show that there exist profiles of max–transitive reciprocal preferences for which the triple-acyclicity on collective decision fails when the threshold is smaller than $\lfloor 2m/3 \rfloor / 2$.

**Theorem 2.5.** *There does not exist $k \in \big[0, \lfloor 2m/3 \rfloor / 2\big)$ such that $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, \ldots, R^m) \in T^m_{\max}$.*

*Proof.* Let $k \in \big[0, \lfloor 2m/3 \rfloor / 2\big)$ and let $R^{\text{I}}$, $R^{\text{II}}$, $R^{\text{III}}$, $R^{\text{IV}}$, $R^{\text{V}}$ and $R^{\text{VI}}$ be the following reciprocal preference relations:

$$R^{\mathrm{I}} = \begin{pmatrix} 0.5 & 1 & 1 & \cdots \\ 0 & 0.5 & 1 & \cdots \\ 0 & 0 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad R^{\mathrm{II}} = \begin{pmatrix} 0.5 & 1 & 0 & \cdots \\ 0 & 0.5 & 0 & \cdots \\ 1 & 1 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$R^{\mathrm{III}} = \begin{pmatrix} 0.5 & 0 & 0 & \cdots \\ 1 & 0.5 & 1 & \cdots \\ 1 & 0 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad R^{\mathrm{IV}} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & \cdots \\ 0.5 & 0.5 & 0.5 & \cdots \\ 0.5 & 0.5 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

$$R^{\mathrm{V}} = \begin{pmatrix} 0.5 & 0.75 & 0.75 & \cdots \\ 0.25 & 0.5 & 0.75 & \cdots \\ 0.25 & 0.25 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \qquad R^{\mathrm{VI}} = \begin{pmatrix} 0.5 & 0.5 & 0 & \cdots \\ 0.5 & 0.5 & 0.5 & \cdots \\ 1 & 0.5 & 0.5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

where non clearly stated elements above take the value of 0.5. It is easy to check that the previous reciprocal relations belong to $T_{\max}$. We distinguish three cases:

1. If $m = 3q$, with $q \in \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \ldots, 2q, \\ R^{\mathrm{III}} & \text{if } i = 2q+1, \ldots, 3q. \end{cases}$$

   According to equivalence (2.2), $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$2q > \frac{3q+k}{2}.$$

   Since

$$2q > \frac{3q+k}{2} \;\Leftrightarrow\; k < q,$$

   $q = \lfloor 2m/3 \rfloor / 2$, and $k < \lfloor 2m/3 \rfloor / 2$ is satisfied by hypothesis, we get $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$, and, consequently, $P_k$ is not triple-acyclic.

2. If $m = 3q + 1$, with $q \in \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \ldots, 2q, \\ R^{\mathrm{III}} & \text{if } i = 2q+1, \ldots, 3q, \\ R^{\mathrm{IV}} & \text{if } i = 3q+1. \end{cases}$$

   According to equivalence (2.2), $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$2q + 0.5 > \frac{3q + 1 + k}{2}.$$

   Since

$$2q + 0.5 > \frac{3q + 1 + k}{2} \;\Leftrightarrow\; k < q,$$

   $q = \lfloor 2m/3 \rfloor /2$, and $k < \lfloor 2m/3 \rfloor /2$ is satisfied by hypothesis, we get $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$, and, consequently, $P_k$ is not triple-acyclic.

3. If $m = 3q + 2$, with $q \in \{0\} \cup \mathbb{N}$, we consider the profile of individual preferences $(R^1, \ldots, R^m)$, where

$$R^i = \begin{cases} R^{\mathrm{I}} & \text{if } i = 1, \ldots, q, \\ R^{\mathrm{II}} & \text{if } i = q+1, \ldots, 2q, \\ R^{\mathrm{III}} & \text{if } i = 2q+1, \ldots, 3q, \\ R^{\mathrm{V}} & \text{if } i = 3q+1, \\ R^{\mathrm{VI}} & \text{if } i = 3q+2. \end{cases}$$

   According to equivalence (2.2), $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$ will happen if

$$2q + 1.25 > \frac{3q + 2 + k}{2}.$$

   Since

$$2q + 1.25 > \frac{3q + 2 + k}{2} \;\Leftrightarrow\; k < q + 0.5,$$

   $q + 0.5 = \lfloor 2m/3 \rfloor /2$, and $k < \lfloor 2m/3 \rfloor /2$ is satisfied by hypothesis, we get $x_1 P_k x_2$, $x_2 P_k x_3$ and $x_3 P_k x_1$, and, consequently, $P_k$ is not triple-acyclic. □

The results obtained in Theorems 2.3 and 2.5 do not include all possible values for the threshold $k$. So, we do not know what happens when $k \in \big[\lfloor 2m/3 \rfloor/2, m/3\big)$. In the case $m = 3q$, with $q \in \mathbb{N}$, such interval is empty whereas in the cases $m = 3q + 1$, with $q \in \mathbb{N}$, and $m = 3q + 2$, with $q \in \{0\} \cup \mathbb{N}$, it is not. Specifically, the problematic interval in the case $m = 3q + 1$ is $\big[m/3 - 1/3, m/3\big)$, and $\big[m/3 - 1/6, m/3\big)$ when $m = 3q + 2$. Our conjecture in these intervals is that $P_k$ is also triple-acyclic for every profile of max–transitive individual preferences. Below, we show that it is the case when $m = 2$, that is, when $m = 3q + 2$ with $q = 0$. Nevertheless, the mathematical complexity of the proof seems to predict the impossibility of getting similar proofs for the general case $m = 3q + 2$ (the same comment can be made for the case $m = 3q + 1$).

**Theorem 2.6.** *If $m = 2$ and $k \in [0.5, 2)$, then $P_k$ is triple-acyclic for every profile of individual preferences $(R^1, R^2) \in T^2_{\max}$.*

*Proof.* Assume, by reduction to absurdity, that $P_k$ is not triple-acyclic. Then, there exist $(R^1, R^2) \in T^2_{\max}$, $i, j, l \in \{1, \ldots, n\}$ such that $x_i P_k x_j$, $x_j P_k x_l$ and $x_l P_k x_i$. Therefore,

$$r^1_{ij} + r^2_{ij} > 1 + \frac{k}{2} \geq 1.25, \quad r^1_{jl} + r^2_{jl} > 1 + \frac{k}{2} \geq 1.25, \quad r^1_{li} + r^2_{li} > 1 + \frac{k}{2} \geq 1.25.$$

It is valuable to highlight that the last condition is equivalent to $r^1_{il} + r^2_{il} < 0.75$. Let distinguish three cases depending on the cardinality of the following set:

$$\mathcal{P} = \left\{p \in \{1, 2\} \mid \min\{r^p_{ij}, r^p_{jl}\} > 0.5\right\}.$$

1. If $\#\mathcal{P}$=2, then, by the max–transitivity condition, we have $r^p_{il} \geq \max\{r^p_{ij}, r^p_{jl}\}$ for all $p \in \{1, 2\}$. Therefore, $r^1_{il} + r^2_{il} \geq r^1_{ij} + r^2_{ij} > 1.25$, which contradicts $r^1_{il} + r^2_{il} < 0.75$.

2. If $\#\mathcal{P}$=1, we can assume, without loss of generality, that $\mathcal{P} = \{1\}$. So, by the max–transitivity condition, we get $r^1_{il} \geq \max\{r^1_{ij}, r^1_{jl}\} > 0.5$. Given that $r^1_{il} + r^2_{il} < 0.75$, we have $r^2_{il} < 0.25$ and $\max\{r^1_{ij}, r^1_{jl}\} < 0.75 - r^2_{il}$. We distinguish two cases:

   (a) If $r^2_{ij} \leq 0.5$, then, $0.75 - r^2_{il} > r^1_{ij} > 1.25 - r^2_{ij}$; so, $r^2_{ij} - r^2_{il} > 0.5$, which is an absurdity given that $r^2_{ij}, r^2_{il} \in [0, 0.5]$.

(b) If $r^2_{ij} > 0.5$, then, given that $\mathcal{P} = \{1\}$, we have that $r^2_{jl} \leq 0.5$. In that case, $0.75 - r^2_{il} > r^1_{jl} > 1.25 - r^2_{jl}$; so, $r^2_{jl} - r^2_{il} > 0.5$, which is an absurdity given that $r^2_{jl}, r^2_{il} \in [0, 0.5]$.

3. If $\#\mathcal{P}$=0, given that $r^1_{ij} + r^2_{ij} > 1.25$ and $r^1_{jl} + r^2_{jl} > 1.25$, we can assume, without lost of generality, that $r^1_{ij} > 0.5$, $r^1_{jl} \leq 0.5$, $r^2_{ij} \leq 0.5$ and $r^2_{jl} > 0.5$. Given that $r^1_{li} + r^2_{li} > 1.25$, we distinguish three cases:

   (a) If $r^1_{li} > 0.5$ and $r^2_{li} > 0.5$, then, as $R^1, R^2 \in T_{\max}$, we have $r^1_{lj} \geq \max\{r^1_{li}, r^1_{ij}\}$ and $r^2_{ji} \geq \max\{r^2_{jl}, r^2_{li}\}$. The first condition is equivalent to $r^1_{jl} \leq \min\{r^1_{ji}, r^1_{il}\}$. Therefore,

$$r^1_{ji} + r^2_{ji} \geq r^1_{jl} + r^2_{jl} > 1.25,$$

   which is an absurdity because $r^1_{ij} + r^2_{ij} > 1.25$.

   (b) If $r^1_{li} > 0.5$ and $r^2_{li} \leq 0.5$, then, as $R^1 \in T_{\max}$, we have $r^1_{lj} \geq \max\{r^1_{li}, r^1_{ij}\}$, which is equivalent to $r^1_{jl} \leq \min\{r^1_{ji}, r^1_{il}\}$. Therefore,

$$r^2_{jl} - r^2_{il} \geq (r^1_{jl} + r^2_{jl}) - (r^1_{il} + r^2_{il}) > 1.25 - 0.75 = 0.5,$$

   which is an absurdity because $r^2_{jl}, r^2_{il} \in [0.5, 1]$.

   (c) If $r^1_{li} \leq 0.5$, then, given that $r^1_{li} + r^2_{li} > 1.25$, we have $r^2_{li} > 0.75$. As $R^2 \in T_{\max}$, then $r^2_{ji} \geq \max\{r^2_{jl}, r^2_{li}\}$, which is equivalent to $r^2_{ij} \leq \min\{r^2_{il}, r^2_{lj}\}$. Therefore,

$$r^1_{ij} - r^1_{il} \geq (r^1_{ij} + r^2_{ij}) - (r^1_{il} + r^2_{il}) > 1.25 - 0.75 = 0.5,$$

   which is an absurdity because $r^1_{ij}, r^1_{il} \in [0.5, 1]$. □

## 2.5 Discussion

In this paper we have determined the values of the threshold $k$ to ensure triple-acyclic collective preference relations when we consider $\widetilde{M}_k$ majorities on three types of $g$–transitive reciprocal preference relations. On the one hand, for 0.5–transitive reciprocal preference relations, we have found that needed thresholds are, at minimum, around two thirds of the voters involved in the election process. On the other hand, for min–transitive and max–transitive reciprocal preferences relations, the needed threshold fails to around one third of the voters. Therefore, the harder the rationality condition over individual preferences is, the smaller the threshold required for triple-acyclic collective decisions is.

It is worth noting that a study of the consistency of the collective decisions under majorities based on difference in support was carried out by Llamazares *et al.* [90]. In that work, consistency was understand as transitivity. The main conclusions there were somewhat disappointing. On the one hand, for any $k \in [0, m-1)$ and any monotonic operator $g$, we can find profiles of $g$–transitive reciprocal preferences for which the collective preference decision is not transitive. The same result is obtained when $k \in [m-1, m)$ and $g(a,b) < (a+b)/2$ for all $a, b \in [0.5, 1], a \neq b$. On the other hand, for $g$–transitive reciprocal preferences, with $g(a,b) \geq (a+b)/2$ for all $a, b \in [0.5, 1]$, transitive collective preferences can be ensured for thresholds located in $[m-1, m)$. Therefore, it is required almost unanimity in individual preferences for arriving to a transitive collective decision and only when individual preference relations fulfill $g$–transitivity, being $g$ greater than or equal to the arithmetic mean operator.

We summarize these results in Table 2.1, where we show the individual rationality conditions considered in the analysis of triple-acyclicity; that is, 0.5–transitivity, min–transitivity and max–transitivity.

Table 2.1: Values of $k$ for collective transitivity and triple-acyclicity.

| Individual $g$–transitivity | Transitivity | Triple-acyclicity |
|---|---|---|
| $g = 0.5$ | $\varnothing$ | $\big[\lfloor 2m/3 \rfloor, m\big)$ |
| $g = \min$ | $\varnothing$ | $[m/3, m)$ |
| $g = \max$ | $[m-1, m)$ | $[m/3, m)$ |

Notice that the conditions for consistent collective decisions are setting on the thresholds of support and the requirements on them depend on how rational individuals are; i.e., on the transitivity condition that fulfil the individual preferences. So, the more rational the individuals are, the less the needed support for getting consistent collective decisions is. In other words, the stronger the transitivity condition on individual preferences is, the easier to reach consistent collective decisions is. Moreover, the required thresholds look more feasible in the case of triple-acyclicity than in the case of transitivity, given that no so extreme support is required. That is coherent with the fact that triple-acyclicity is a weaker rationality condition than transitivity.

The aggregation of individual reciprocal preferences under majorities based on difference in support can be understood as the aggregation of such individual preference relations through the arithmetic mean operator. Under such view, the preference $P_k$ is reached by means of an $\alpha$-cut[3]; i.e., an alternative is preferred to another one if the arithmetic mean of the intensities of preferences for that alternative over the other one exceeds the value of $\alpha$. To rewrite $P_k$ by means of an $\alpha$-cut, let $\overline{R} : X \times X \longrightarrow [0, 1]$ the reciprocal preference relation defined by the arithmetic mean of the individual intensities of preference, i.e.,

$$\overline{R}(x_i, x_j) = \frac{1}{m} \sum_{p=1}^{m} r_{ij}^p.$$

Then, $P_k = \overline{R}_\alpha$, with $\alpha = (m + k)/2m$.

Notice that *cut relations* ($\alpha$-cuts of valued binary relations) have been used by different authors. For instance, Fodor and Roubens [43] give some relationships between valued binary relations and the cut relations associated with them. Świtalski [109], for his part, analyzes relationships between the transitivity and the acyclicity of crisp relations (cut relations and others) obtained from reciprocal preference relations.

It is worth noting that the arithmetic mean operator has been widely used in the literature for aggregating individual intensities of preference into collective intensities of preference. But, as it has been pointed out by some authors, the choice of an alternative has to be unambiguous. Quoting Barret *et al.* [5]: 'In real life, people often have vague preferences. . . However, when confronted with an actual choice situation, where an alternative has to be chosen from a given feasible set of alternatives, the decision maker must make an unambiguous choice, even when his preferences are fuzzy; there cannot be any vagueness about the actual act of choice itself.'

In this sense, $\alpha$-cuts are a valuable tool for obtaining unambiguous choices

[3]If $R \in \mathcal{R}(X)$ and $\alpha \in [0.5, 1)$, the $\alpha$-cut of $R$ is the ordinary preference relation $R_\alpha$ defined by $x_i R_\alpha x_j \Leftrightarrow R(x_i, x_j) > \alpha$.

from collective intensities of preference. The results given in this paper, together with those given by Llamazares *et al.* [90], allow us to know the values of $\alpha$ for which the collective decision is transitive or triple-acyclic. These values, calculated by applying the relation between $\alpha$ and $k$ to the values in Table 2.1, are shown in Table 2.2.

Table 2.2: Values of $\alpha$ for collective transitivity and triple-acyclicity.

| Individual $g$–transitivity | Transitivity | Triple-acyclicity |
|---|---|---|
| $g = 0.5$ | $\varnothing$ | $[5/6, 1)$, when $m = 3q$<br>$[5/6 - 1/3m, 1)$, when $m = 3q + 1$<br>$[5/6 - 1/6m, 1)$, when $m = 3q + 2$ |
| $g = \min$ | $\varnothing$ | $[2/3, 1)$ |
| $g = \max$ | $[1 - 1/2m, 1)$ | $[2/3, 1)$ |

Finally, previous results on the consistency under majorities based on difference in support rely on the transitivity requirements on the reciprocal preference relations. The impact on the consistency regarding other rationality conditions on reciprocal preference relations (see for instance, Tanino [112], Świtalski [110], Herrera-Viedma *et al.* [73], De Baets and De Meyer [24] and De Baets *et al.* [26]) is an open question. Moreover, it could be a significant disparity between the possibility of having cycles and the empirical occurrence of them (see, in similar contexts, Gehrlein and Fishburn [61], Tangian [111] and Gehrlein [59], among others). For this reason, an empirical analysis of such occurrences will be considered in future research.

# Chapter 3

# Consistent collective decisions under majorities based on differences

[This chapter is submitted (jointly with Mostapha Diss).]

The main criticism to the aggregation of individual preferences under majority rules refers to the possibility of reaching inconsistent collective decisions from the election process. In these cases, the collective preference includes cycles and even could prevent the election of any alternative as the collective choice. The likelihood of consistent outcomes under two classes of majority rules constitutes the aim of this paper. Specifically, we focus on majority rules that require certain consensus in individual preferences to declare an alternative as the winner. In the case of majorities based on difference of votes, such requirement asks to the winner alternative to obtain a difference in votes with respect to the loser alternative taken into account that individuals are endowed with weak preference orderings. Same requirement is asked to the restriction of these rules to individual linear preferences, whereas in the case of majorities based on difference in support, the requirement has to do with the difference in the sum of the intensities for the alternatives in contest.

## 3.1 Introduction

Since Condorcet [18] introduced The Voting Paradox, it is well known that the aggregation of transitive individual preferences under simple majority rule could lead to inconsistent collective preferences. Recalling the classical example, consider a three-alternative election with alternatives $x_1, x_2, x_3$ and three individuals endowed with the following $x_1x_2x_3$, $x_2x_3x_1$ and $x_3x_1x_2$, where, for instance, $x_1x_2x_3$ means that $x_1$ is preferred to $x_2$, $x_2$ is preferred to $x_3$ and $x_1$ is preferred to $x_3$. For each pair of alternatives, each individual casts a vote for her/his preferred alternative following just assumed orderings. Adding up these votes, alternatives $x_1, x_2$ and $x_3$ defeat $x_2, x_3$ and $x_1$ respectively, by two votes to one. In that voting situation, there is a cycle on the ordering induced by the strict collective preference. In such a case, that preference fails on transitivity and on triple-acyclicity given the requirements of such conditions. To illustrate, assume that alternative $x_1$ defeats $x_2$ and $x_2$ defeats $x_3$; $x_1$ defeats $x_3$ whenever the strict collective preference is transitive whereas $x_3$ does not defeat $x_1$ whenever the strict collective preference is triple-acyclic.

Consider now the following voting process' outcome: $x_1$ defeats $x_2$ and it is indifferent to $x_3$, and $x_2$ is also indifferent to $x_3$. In this case, the weak collective preference fails on consistency. Notice that the strict preference associated with that weak preference behaves right but the indifference relation associated with the weak preference fails on transitivity.

The idea that The Voting Paradox 'should rarely be observed in any real three-candidate elections with large electorates' stated by Gehrlein [60], promotes the probabilistic study of the occurrence of that paradox and of their consequences under different aggregation rules. In several studies, it is assumed an a priori probability model to estimate the likelihood of different voting situations, derived the conditions under which the paradox or the effects of that appear and reached probabilities through combinatoric calculus. In this context, stand out the studies about simple majority rule (Gehrlein and Fishburn [61], Fishburn and Gerhlein [42] and Gehrlein [56]), supermajority rules (Balasko and Crès [4] and Tovey [113]) or scoring rules (Gehrlein and Fishburn [62, 63, 64] and Cervone *et al.* [12]), among others. In other ones, these probabilities are calculated following the Montecarlo simulation methodology. Specifically, the study of the cyclical and intransitive collective decisions under the simple majority rule are carried out in Campbell and Tullock [9], Klahr [80], DeMeyer and Plott [31] and Jones [76].

This paper is devoted to analyze and compare the probabilities of consistent collective decisions over three alternatives for two different classes of majorities rules: majorities based on difference of votes (García-Lapresta and Llamazares [49], Llamazares [85] and Houy [74]) and majorities based on difference in support (García-Lapresta and Llamazares [50]). Given two alternatives, these majorities based on differences focus on requiring to an alternative, to be declared the winner, to reach a number of votes or a support that exceeds the number of votes or the support for the other alternative in a quantity fixed before the voting process. Therefore, the difference between these two classes of majorities restricts to the types of individual preferences considered. In the first case, individual preferences are understood as crisp preferences, i.e. given a pair of alternatives individuals declare if they prefer an alternative to another one or if they are indifferent between them. Here, we distinguish between the case where individuals are endowed with weak preferences and the case where individuals are endowed with linear orderings. In the second case, individual preferences are understood as reciprocal preference relations, i.e. given two alternatives, individuals declare the degree with which they prefer an alternative to another one, in other words their intensities of preference, by means of numerical values in the unit interval.

Coming back to the consistent collective decisions analyzed here, we specifically calculate the probabilities of transitive and triple-acyclic strict collective preferences and the corresponding ones of transitive weak collective preferences for the three specifications of majorities based on differences stated before, as the proportion of collective decisions that fulfill each of such consistency conditions over the total number of possible collective decisions.

To calculate the probabilities of consistent outcomes under majorities based on difference of votes taken into account weak and linear individual preferences respectively, we consider that each individual preference ordering is equiprobable by embracing the Impartial Anonymous Culture (IAC) condition (Gehrlein and Fishburn [61]) to describe the likelihood of the possible individual orderings. On the one hand, assuming weak or linear individual orderings jointly with the IAC condition allow to know the total number of possible collective preferences (again, Gehrlein and Fishburn [61]). On the other hand, the number of consistent profiles is calculated by means of Ehrhart polynomials, a method recently introduced in the social choice literature by Wilson and Pritchard [117] and Lepelley *et al.* [82] in order to estimate the probabilities of some voting paradoxes under the IAC condition.

In the case of the calculations of the probabilities of consistent outcomes under majorities based on difference in support, we propose to apply the Montecarlo simulation methodology to estimate such probabilities inspired by the studies in Campbell and Tullock [9], Klahr [80], DeMeyer and Plott [31] and Jones [76].

Specifically, we generate the individual reciprocal preference relations for the case of three alternatives. Each individual intensity of preference is understood as a continuous random variable in the unit interval consistently built with a specific transitivity condition over the individual's reciprocal preference relations. Then, we fix the required difference in support and aggregate these individual preferences with the corresponding majority based on difference in support. We derive the resultant collective ordering of alternatives and evaluate its consistency. Finally, we iterate that procedure to estimate desired probabilities as the number of consistent orderings over the total number of simulated collective orderings.

The methodology proposed here allows us to hypothesize about a relationship between the type of individual preferences assumed under each rule and the likelihood of inconsistent collective decisions. Moreover, we set forth our results for majorities based on difference of votes with previous ones on simple majority (Gehrlein [58] and Lepelley and Martin [83]). We also are able to analyze the impact of the types of transitivity conditions for individual reciprocal preferences on the probability of consistent collective decisions under majorities based on difference in support. In addition, we compare our results on probabilities with some theoretical ones about the consistency of the collective preferences under majorities based on difference in support (Llamazares *et al.* [90] and Llamazares and Pérez-Asurmendi [89]).

The paper is organized as follows. Section 3.2 describes the theoretical framework followed in this paper and introduces majorities based on difference of votes and majorities based on difference in support. Sections 3.3 and 3.4 provide the results about the probability of consistent collective decisions under majorities based on difference of votes with linear preferences and with weak preferences, respectively. Section 3.5 is devoted to the simulated probabilities in the case of majorities based on difference in support.

Section 3.6 discusses the results and concludes.

## 3.2 Preliminaries

Consider a set of alternatives $X = \{x_1, x_2, x_3\}$ in an election with $m$ individuals. Let $S$ be a binary relation on $X$, i.e. a subset of the cartesian product $X \times X$. In what follows, $x_i S x_j$ stands for $(x_i, x_j) \in S$, i.e. when $x_i$ is in the relation $S$ with $x_j$. $S^{-1}$ is the inverse relation of $S$ defined by $x_i S^{-1} x_j \Leftrightarrow x_j S x_j$ and $S^c$ is the complement relation of $S$ defined by $x_i S^c x_j \Leftrightarrow \neg(x_i S x_j)$. Given two binary relations $S$ and $T$, the intersection of $S$ and $T$ is also a binary relation defined by $x_i(S \cap T)x_j \Leftrightarrow (x_i S x_j \wedge x_i T x_j)$. A binary relation $S$ on $X$ is

1. *reflexive* if $\forall x \in X,\ xSx$,
2. *symmetric* if $\forall x_i, x_j \in X,\ x_i S x_j \Rightarrow x_j S x_i$,
3. *asymmetric* if $\forall x_i, x_j \in X,\ x_i S x_j \Rightarrow \neg(x_j S x_i)$,
4. *antisymmetric* if $\forall x_i, x_j \in X,\ x_i S x_j \wedge x_j S x_i \Rightarrow x_i = x_j$,
5. *complete* if $\forall x_i, x_j \in X,\ x_i S x_j \vee x_j S x_i$,
6. *transitive* if $\forall x_i, x_j, x_l \in X,\ x_i S x_j \wedge x_j S x_l \Rightarrow x_i S x_l$,
7. *triple-acyclic* if $\forall x_i, x_j, x_l \in X,\ x_i S x_j \wedge x_j S x_l \Rightarrow \neg(x_l S x_i)$.

A *weak preference* $R$ is a complete binary relation on the set of alternatives $X$. The *ordinary preference* $P$ associated with $R$ is the asymmetric binary relation on the set $X$ defined by $P = (R^{-1})^c$ and the corresponding *indifference relation* $I$ is the reflexive and symmetric binary relation on $X$ defined by $I = R \cap R^{-1}$. $\mathcal{P}(X)$ is the set of ordinary preferences. A *weak ordering* is a transitive weak preference whereas a *linear ordering* is also antisymmetric.

From definitions above it is well know that any weak ordering implies a transitive ordinary preference relation and a transitive indifference relation. Moreover, any transitive ordinary preference is also a triple-acyclic preference relation. Notice that the converse is not true.

Given that the social decision between two alternatives is given by either an ordinary preference relation or an indifference relation, and that three alternatives are in contest, we consider the 27 cases in Table 3.1 as possible social outcomes.

Table 3.1: Possible social outcomes in a three alternative election.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $x_1Px_2$ | $x_2Px_3$ | $x_1Px_3$ | 14. | $x_1Px_2$ | $x_2Ix_3$ | $x_1Ix_3$ |
| 2. | $x_1Px_3$ | $x_3Px_2$ | $x_1Px_2$ | 15. | $x_1Ix_2$ | $x_2Ix_3$ | $x_1Px_3$ |
| 3. | $x_2Px_1$ | $x_1Px_3$ | $x_2Px_3$ | 16. | $x_1Ix_2$ | $x_2Ix_3$ | $x_3Px_1$ |
| 4. | $x_2Px_3$ | $x_3Px_1$ | $x_2Px_1$ | 17. | $x_1Ix_3$ | $x_3Px_2$ | $x_1Ix_2$ |
| 5. | $x_3Px_1$ | $x_1Px_2$ | $x_3Px_2$ | 18. | $x_2Px_1$ | $x_1Ix_3$ | $x_2Ix_3$ |
| 6. | $x_3Px_2$ | $x_2Px_1$ | $x_3Px_1$ | 19. | $x_1Ix_2$ | $x_2Px_3$ | $x_1Ix_3$ |
| 7. | $x_1Px_2$ | $x_2Ix_3$ | $x_1Px_3$ | 20. | $x_1Px_2$ | $x_2Px_3$ | $x_1Ix_3$ |
| 8. | $x_2Px_1$ | $x_1Ix_3$ | $x_2Px_3$ | 21. | $x_3Px_1$ | $x_1Px_2$ | $x_2Ix_3$ |
| 9. | $x_3Px_1$ | $x_1Ix_2$ | $x_3Px_2$ | 22. | $x_2Px_3$ | $x_3Px_1$ | $x_1Ix_2$ |
| 10. | $x_1Ix_2$ | $x_2Px_3$ | $x_1Px_3$ | 23. | $x_3Px_2$ | $x_1Px_3$ | $x_1Ix_2$ |
| 11. | $x_2Ix_3$ | $x_3Px_1$ | $x_2Px_1$ | 24. | $x_2Px_1$ | $x_1Px_3$ | $x_2Ix_3$ |
| 12. | $x_1Ix_3$ | $x_3Px_2$ | $x_1Px_2$ | 25. | $x_3Px_2$ | $x_2Px_1$ | $x_1Ix_3$ |
| 13. | $x_1Ix_2$ | $x_2Ix_3$ | $x_1Ix_3$ | 26. | $x_1Px_2$ | $x_2Px_3$ | $x_3Px_1$ |
| | | | | 27. | $x_2Px_1$ | $x_1Px_3$ | $x_3Px_2$ |

Our interest focuses on the frequency of consistent social outcomes given the 27 possible outcomes above. We distinguish among three cases of consistent outcomes; the case of weak orderings corresponding to the first thirteen outcomes, the case of transitive ordinary preferences corresponding to the first nineteen and the case of triple-acyclic ordinary preferences corresponding to the first twenty-fifth outcomes.

### 3.2.1 Individual preferences

We consider that individuals compare the alternatives on $X$ by pairs and declare their preferences by means of values $r_{ij}^p \in [0, 1]$, where $r_{ij}^p > 0.5$ means that alternative $x_i$ is somewhat preferred to $x_j$ by the individual $p$, whereas $r^p_{ij} < 0.5$ signifies that alternative $x_j$ is somewhat preferred to $x_i$ by the individual $p$. At that point, we distinguish between the general case in which preferences are represented by *reciprocal preferences* and the particular case of that, referred to as *crisp preferences*.

1. Crisp preferences: the values of $r^p_{ij}$ are restricted to the set of discrete values $\{0, 0.5, 1\}$. If $r^p_{ij} = 1$, individual $p$ prefers alternative $x_i$ to alternative $x_j$, whereas if $r^p_{ij} = 0$, individual $p$ prefers $x_j$ to $x_i$. If $r^p_{ij} = 0.5$, individual $p$ is indifferent between both alternatives. Condition $r^p_{ij} + r^p_{ji} = 1$ guarantees that the preference of individual $p$ is a weak preference. Moreover, the conditions

$$(r^p_{ij} = 1 \wedge r^p_{jl} = 1) \;\Rightarrow\; r^p_{il} = 1,$$
$$(r^p_{ij} = 0.5 \wedge r^p_{jl} = 0.5) \;\Rightarrow\; r^p_{il} = 0.5,$$

   assure that the preference of individual $p$ is a weak ordering. Individual linear orderings could also be represented in this framework by including the following condition: $r^p_{ij} \in \{0, 1\}\ \forall i \neq j$. Thus, individuals could only be indifferent between an alternative and itself.

2. Reciprocal preferences: the values of $r^p_{ij}$ belong to the unit interval $[0, 1]$ with the following interpretation: $r^p_{ij} > 0.5$ indicates that the individual $p$ prefers the alternative $x_i$ to the alternative $x_j$, the more the nearer is the value of $r^p_{ij}$ to 1 that represents the maximum degree of preference for $x_i$ over $x_j$; conversely, $r^p_{ij} < 0.5$, means that individual $p$ prefers alternative $x_j$ to $x_i$, the more the nearer is the value of $r^p_{ij}$ to 0 that represents the maximum degree of preference for $x_j$ over $x_i$; finally, $r^p_{ij} = 0.5$ stands for the indifference between $x_i$ and $x_j$ for individual $p$. The reciprocity of these preferences is described by the condition $r^p_{ij} + r^p_{ji} = 1$. To avoid the possibility of having incoherent individual preferences, we need to assume some kind of rationality condition. But, in this framework, several concepts could be taken to ensure such rationality requirement (see, among others Zadeh [123], Dubois and Prade [37], Dasgupta and Deb [22] and García-Lapresta and Meneses [53]). Here, we consider the following transitivity conditions for reciprocal preference relations.

**Definition 3.1.** We say that individual $p$ is

(a) 0.5–*transitive* if $\forall\, i, j, l \in \{1, 2, 3\}$

$$(r^p_{ij} > 0.5 \wedge r^p_{jl} > 0.5) \;\Rightarrow\; r^p_{il} > 0.5,$$

(b) min–*transitive* if $\forall\, i, j, l \in \{1, 2, 3\}$

$$(r^p_{ij} > 0.5 \wedge r^p_{jl} > 0.5) \;\Rightarrow\; r^p_{il} \geq \min\{r^p_{ij}, r^p_{jl}\},$$

(c) am–*transitive* if $\forall\, i, j, l \in \{1, 2, 3\}$

$$(r^p_{ij} > 0.5 \wedge r^p_{jl} > 0.5) \;\Rightarrow\; r^p_{il} \geq \big(r^p_{ij} + r^p_{jl}\big)/2,$$

(d) max–*transitive* if $\forall\, i, j, l \in \{1, 2, 3\}$

$$(r^p_{ij} > 0.5 \wedge r^p_{jl} > 0.5) \;\Rightarrow\; r^p_{il} \geq \max\{r^p_{ij}, r^p_{jl}\}.$$

The preferences of each individual over the alternatives in $X = \{x_1, x_2, x_3\}$ can be represented using a $3 \times 3$ matrix $R^p = \big(r^p_{ij}\big)$ as follows:

$$R^p = \begin{pmatrix} 0.5 & r^p_{12} & r^p_{13} \\ 1 - r^p_{12} & 0.5 & r^p_{23} \\ 1 - r^p_{13} & 1 - r^p_{23} & 0.5 \end{pmatrix}. \qquad (3.1)$$

Individual preferences are collected in a vector where each vector-element represents the preferences of an individual. Assuming $m$ individuals[1] and taking into account the above distinction among linear, weak and reciprocal preferences, a profile of linear orderings is a vector $(R^1, \ldots, R^m) \in \mathcal{L}(X)^m$, being $\mathcal{L}(X)$ the set of all linear orderings; a profile of weak orderings is a vector $(R^1, \ldots, R^m) \in \mathcal{W}(X)^m$, being $\mathcal{W}(X)$ the set of all weak orderings; and a profile of reciprocal preferences is a vector $(R^1, \ldots, R^m) \in \mathcal{R}(X)^m$, being $\mathcal{R}(X)$ the set of all reciprocal preference relations.

[1] To calculate the probabilities presented here, $m$ takes the following values: 3, 4, 5, 10, 100, 1 000 and 100 000.

### 3.2.2 Majorities based on differences

Given the nature of the three types of individual preferences considered in Subsection 3.2.1, we consider also three different specifications of majorities based on differences relying on the types of individuals preferences that we take into account in each case. In what follows, we refer to these majorities as majorities based on difference of votes with linear orderings, majorities based on difference of votes with weak orderings and majorities based on difference in support for reciprocal preference relations.

The concept of majorities based on difference of votes was introduced in García-Lapresta and Llamazares [49] and was later axiomatically characterized in Llamazares [85], and subsequently in Houy [74]. These rules involves crisp preferences, i.e. given a pair of alternatives, individuals could declare their preference for one of them or their indifference between both alternatives.

Under these majorities, an alternative say $x_i$, is declared the winner if the number of individuals who prefer that alternative, to the other one, say $x_j$, exceeds the number of individuals who prefer $x_j$ to $x_i$ in a difference in votes, fixed before the election process. Assuming $m$ individuals, that difference could take any integer value in $\{0,\ldots,m-1\}$. These majorities are located between simple majority rule where the difference of votes is zero and unanimity where the difference of votes is the total number of individuals $m$ minus one. Moreover, if the indifference state is ruled out from individual preferences, these majorities are equivalent to supermajority rules.

Taking into account the former majorities based on difference of votes, we introduce in Definition 3.2 the majorities based on difference of votes with linear orderings and in Definition 3.3 the majorities based on difference of votes with weak orderings. In what follows, the symbol # stands for the cardinality of a set.

**Definition 3.2** (Majorities based on difference of votes with linear orderings or $M^L_{k'}$ majorities)**.** Given $k' \in \{0,1,\ldots,m-1\}$, the majority based on difference of votes with linear orderings or $M^L_{k'}$ *majority* is the mapping $M^L_{k'} : \mathcal{L}(X)^m \longrightarrow \mathcal{P}(X)$ defined by $M^L_{k'}(R^1,\ldots,R^m) = P^L_{k'}$, where

$$x_i \, P^L_{k'} \, x_j \;\Leftrightarrow\; \#\{p \mid r^p_{ij} = 1\} > \#\{p \mid r^p_{ji} = 1\} + k'.$$

The indifference relation associated to $P^L_{k'}$ is as follows:

$$x_i \, I^L_{k'} \, x_j \;\Leftrightarrow\; \big|\#\{p \mid r^p_{ij} = 1\} - \#\{p \mid r^p_{ji} = 1\}\big| \leq k'.$$

**Example 3.1.** Let $R^I$ and $R^{II}$ be the following individual linear preference orderings over the alternatives on $X = \{x_1, x_2, x_3\}$.

$$R^I = \begin{pmatrix} 0.5 & 1 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}, \qquad R^{II} = \begin{pmatrix} 0.5 & 0 & 1 \\ 1 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

Consider the profile $(R^1, R^2, R^3, R^4, R^5)$ where

$$R^p = \begin{cases} R^I & \text{if } p = 1, 2, 3, \\ R^{II} & \text{if } p = 4, 5. \end{cases}$$

Assuming a required difference of votes $k'$ equal to 1 and applying the corresponding $M^L_1$ majority we have

$$\begin{aligned} |\#\{p \mid r^p_{12} = 1\} - \#\{p \mid r^p_{21} = 1\}| = |3 - 2| \leq 1 &\Rightarrow x_1 \, I^L_1 \, x_2, \\ \#\{p \mid r^p_{23} = 1\} = 5 > \#\{p \mid r^p_{32} = 1\} + 1 = 0 + 1 &\Rightarrow x_2 \, P^L_1 \, x_3, \\ \#\{p \mid r^p_{13} = 1\} = 5 > \#\{p \mid r^p_{31} = 1\} + 1 = 0 + 1 &\Rightarrow x_1 \, P^L_1 \, x_3. \end{aligned}$$

**Definition 3.3** (Majorities based on difference of votes with weak orderings or $M_{k'}$ majorities)**.** Given $k' \in \{0, \ldots, m-1\}$, the majority based on difference of votes with weak orderings or $M_{k'}$ *majority* is the mapping $M_{k'} : \mathcal{W}(X)^m \longrightarrow \mathcal{P}(X)$ defined by $M_{k'}(R^1, \ldots, R^m) = P_{k'}$, where

$$x_i \, P_{k'} \, x_j \;\Leftrightarrow\; \#\{p \mid r^p_{ij} = 1\} > \#\{p \mid r^p_{ji} = 1\} + k'.$$

The indifference relation associated to $P_{k'}$ is as follows:

$$x_i \, I_{k'} \, x_j \;\Leftrightarrow\; \big|\#\{p \mid r^p_{ij} = 1\} - \#\{p \mid r^p_{ji} = 1\}\big| \leq k'.$$

**Example 3.2.** Let $R^I$ and $R^{II}$ be the following individual weak preference orderings over the alternatives on $X = \{x_1, x_2, x_3\}$.

$$R^I = \begin{pmatrix} 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0 \\ 0.5 & 1 & 0.5 \end{pmatrix}, \qquad R^{II} = \begin{pmatrix} 0.5 & 0.5 & 1 \\ 0.5 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}.$$

Consider the profile $(R^1, R^2, R^3, R^4, R^5)$ where

$$R^p = \begin{cases} R^I & \text{if } p = 1, 2, 3, \\ R^{II} & \text{if } p = 4, 5. \end{cases}$$

Assuming a required difference of votes $k'$ equal to 2 and applying the corresponding $M_2$ majority we have

$$\#\{p \mid r_{12}^p = 1\} = 3 > \#\{p \mid r_{21}^p = 1\} + 2 = 0 + 2 \Rightarrow x_1 \, P_2 \, x_2,$$
$$|\#\{p \mid r_{23}^p = 1\} - \#\{p \mid r_{32}^p = 1\}| = |2 - 3| \leq 2 \Rightarrow x_2 \, I_2 \, x_3,$$
$$|\#\{p \mid r_{13}^p = 1\} - \#\{p \mid r_{31}^p = 1\}| = |2 - 0| \leq 2 \Rightarrow x_1 \, I_2 \, x_3.$$

In García-Lapresta and Llamazares [50], majorities based on difference of votes were extended to the framework of reciprocal preferences allowing individuals to declare their degrees of preferences over pairs of alternatives. Majorities based on difference in support allow us to aggregate each profile of reciprocal preferences into a strict collective preference $P_k$ over the set of alternatives. Under these rules, the winner alternative is required to reach a support that exceeds the support for the other alternative in a quantity, fixed before the voting process. Formal definition for these majorities is as follows.

**Definition 3.4** (Majorities based on difference in support or $\widetilde{M}_k$ majorities (García-Lapresta and Llamazares [50])). Given $k \in [0, m)$, the majority based on difference in support or $\widetilde{M}_k$ *majority* is the mapping $\widetilde{M}_k : \mathcal{R}(X)^m \longrightarrow \mathcal{P}(X)$ defined by $\widetilde{M}_k(R^1, \ldots, R^m) = P_k$, where

$$x_i \, P_k \, x_j \;\Leftrightarrow\; \sum_{p=1}^{m} r_{ij}^p > \sum_{p=1}^{m} r_{ji}^p + k. \tag{3.2}$$

The indifference relation associated with $P_k$ is defined by:

$$x_i \, I_k \, x_j \;\Leftrightarrow\; \left| \sum_{p=1}^{m} r_{ij}^p - \sum_{p=1}^{m} r_{ji}^p \right| \leq k. \tag{3.3}$$

Notice that when considering crisp preferences, the expression in (3.2) goes for defining the majorities based on difference of votes with linear and weak preference orderings.

**Example 3.3.** Let $R^I$ and $R^{II}$ be the following reciprocal preference relations over the alternatives on $X = \{x_1, x_2, x_3\}$.

$$R^I = \begin{pmatrix} 0.5 & 1 & 0.9 \\ 0 & 0.5 & 0.6 \\ 0.1 & 0.4 & 0.5 \end{pmatrix}, \qquad R^{II} = \begin{pmatrix} 0.5 & 0.8 & 1 \\ 0.2 & 0.5 & 0.7 \\ 0 & 0.3 & 0.5 \end{pmatrix}.$$

Consider the profile $(R^1, R^2, R^3, R^4, R^5)$ where

$$R^p = \begin{cases} R^I & \text{if } p = 1, 2, 3, \\ R^{II} & \text{if } p = 4, 5. \end{cases}$$

Assuming a required difference in support $k$ equal to 1.75 and applying the corresponding $\widetilde{M}_{1.75}$ majority we have

$$\sum_{p=1}^{5} r_{12}^p = 4.6 > \sum_{p=1}^{5} r_{21}^p + 1.75 = 0.4 + 1.75 \Rightarrow x_1 \, P_{1.75} \, x_2,$$

$$\left| \sum_{p=1}^{5} r_{23}^p - \sum_{p=1}^{5} r_{32}^p \right| = |3.2 - 1.8| \leq 1.75 \Rightarrow x_2 \, I_{1.75} \, x_3,$$

$$\sum_{p=1}^{5} r_{13}^p = 4.7 > \sum_{p=1}^{5} r_{31}^p + 1.75 = 0.3 + 1.75 \Rightarrow x_1 \, P_{1.75} \, x_3.$$

## 3.3 Probability of consistent collective decisions under majorities based on difference of votes with linear orderings

In this section the results about the probability of consistent collective decisions under $M_{k'}^L$ majorities are introduced under IAC assumption. Given that voters are endowed with complete linear preference orderings, there are six possible preference orders that they might have,

$$\begin{array}{lll} x_1x_2x_3 \ (m_1) & x_1x_3x_2 \ (m_2) & x_2x_1x_3 \ (m_3) \\ x_2x_3x_1 \ (m_4) & x_3x_1x_2 \ (m_5) & x_3x_2x_1 \ (m_6) \end{array} \tag{3.4}$$

where $m_i$ is the number of voters with the associated linear preference ordering. In this framework, a voting situation is a vector $\boldsymbol{m} = (m_1, m_2, m_3, m_4, m_5, m_6)$ such that $\sum_{i=1}^{6} m_i = m$. As the IAC condition is assumed, all possible voting situations $\boldsymbol{m}$ are equally liked to be observed. Gehrlein and Fishburn [61] showed that for $m$ agents and 3 alternatives, the total number of voting situations $\boldsymbol{m}$ is given by the expression:

$$\psi(m) = \frac{(m+1)(m+2)(m+3)(m+4)(m+5)}{120}. \tag{3.5}$$

### 3.3.1 Probabilities of triple-acyclic ordinary preferences under majorities based on difference of votes with linear orderings

To calculate the probability of triple-acyclic strict preferences under $M_{k'}^L$ majorities we focus on the cases from 1 to 25 in Table 3.1. Specifically, we first calculate the probability of cyclic strict preferences, i.e. the probability of having preferences like the ones described in the cases 26 and 27 (see again Table 3.1). Thereafter, we obtain the probability of triple-acyclic cases as 1 minus the probability of cyclic strict preferences.

Going deeper on the strict preference described in the case 26, we notice that for such preference to exist, the numbers of voters associated with the linear orderings described in (3.4) have to fulfil the following conditions: $m_1 + m_2 - m_3 - m_4 + m_5 - m_6 > k'$, $m_1 - m_2 + m_3 + m_4 - m_5 - m_6 > k'$ and $-m_1 - m_2 - m_3 + m_4 + m_5 + m_6 > k'$.

In other words, the strict preference in the case 26 requires a voting situation $\boldsymbol{m} = (m_1, m_2, m_3, m_4, m_5, m_6)$ that fulfils the conditions given by the system of inequalities below.

$$(x_1 P_{k'}^L x_2, x_2 P_{k'}^L x_3 \text{ and } x_3 P_{k'}^L x_1) \Rightarrow \begin{cases} m_1 + m_2 - m_3 - m_4 + m_5 - m_6 > k', \\ m_1 - m_2 + m_3 + m_4 - m_5 - m_6 > k', \\ -m_1 - m_2 - m_3 + m_4 + m_5 + m_6 > k', \\ m_i \geq 0 \text{ for } i \in \{1, \ldots, 6\}, \\ m - 1 \geq k' \geq 0, \\ m_1 + m_2 + m_3 + m_4 + m_5 + m_6 = m. \end{cases}$$

Therefore, to calculate the probability of cyclic strict preferences, we need to solve the system of linear inequalities derived from conditions that the numbers of voters associated with the linear orderings in (3.4) have to hold for strict preferences like the ones in the cases 26 and 27 to exist.

We compute the number of voting situations that fulfil these conditions by means of the Parameterized Barvinok's algorithm (Verdoolaege *et al.* [115])[2]. Such algorithm allows to quantify the number of integer solutions for systems of inequalities with parameters. The connection of such algorithm to Social Choice Theory was recently pointed out by Wilson and Pritchard [117] and Lepelley *et al.* [82].

Given the two parameters $m$ and $k'$, the number of voting situations $\boldsymbol{m}$ for our system is given by bivariate quasi polynomials in $m$ and $k'$ with 2-periodic coefficients meaning that such coefficients depend on the parity of the parameters $m$ and $k'$. Following the notation introduced in Lepelley *et al.* [82], we represent these coefficients by a list of 2 rational numbers enclosed in square brackets. To illustrate, assume the bracketed list $[[a,b]_m, [b,a]_m]_{k'}$. In the case of even $k'$, the relevant list corresponds to $[a,b]_m$. The coefficient will be either $a$ when $m$ is even or $b$ when $m$ is odd. Accordingly, in the case of odd $k'$, the relevant list is $[b,a]_m$ and therefore, the coefficient will be either $b$ when $m$ is even or $a$ when $m$ is odd. Thus, the coefficient will be $a$ when $m$ and $k'$ have the same the parity and $b$ otherwise.

Notice that in the case of the cyclical strict preferences depicted in the cases 26 and 27 (Table 3.1), the number of solutions in the system of inequalities derived from the strict preference in the case 26 is the same as in the system derived from the strict preference in the case 27 given the symmetry of such cases. The program indicates that the corresponding quasi

[2]The free software to calculate the integer points under the Parameterized Barvinok's algorithm can be found in http://freecode.com/projects/barvinok.

polynomial for each of these cases is as follows:

$$
\begin{aligned}
&-\frac{81}{1280}k'^5+\left(\frac{27}{256}m+\left[\left[0,\frac{81}{256}\right]_m,\left[\frac{81}{256},0\right]_m\right]_{k'}\right)k'^4+\\
&\left(-\frac{9}{128}m^2+\left[\left[0,-\frac{27}{64}\right]_m,\left[-\frac{27}{64},0\right]_m\right]_{k'}m+\right.\\
&\left.\left[\left[\frac{9}{64},-\frac{63}{128}\right]_m,\left[-\frac{63}{128},\frac{9}{64}\right]_m\right]_{k'}\right)k'^3+\\
&\left(\frac{3}{128}m^3+\left[\left[0,\frac{27}{128}\right]_m,\left[\frac{27}{128},0\right]_m\right]_{k'}m^2+\right.\\
&\left.\left[\left[-\frac{9}{64},\frac{63}{128}\right]_m,\left[\frac{63}{128},-\frac{9}{64}\right]_m\right]_{k'}m+\left[\left[0,\frac{27}{128}\right]_m,\left[\frac{27}{128},0\right]_m\right]_{k'}\right)k'^2+\\
&\left(-\frac{1}{256}m^4+\left[\left[0,-\frac{3}{64}\right]_m,\left[-\frac{3}{64},0\right]_m\right]_{k'}m^3+\right.\\
&\left[\left[\frac{3}{64},-\frac{21}{128}\right]_m,\left[-\frac{21}{128},\frac{3}{64}\right]_m\right]_{k'}m^2+\left[\left[0,-\frac{9}{64}\right]_m,\left[-\frac{9}{64},0\right]_m\right]_{k'}m+\\
&\left.\left[\left[-\frac{1}{20},\frac{71}{1280}\right]_m,\left[\frac{71}{1280},-\frac{1}{20}\right]_m\right]_{k'}\right)k'+\\
&\frac{1}{3840}m^5+\left[\left[0,\frac{1}{256}\right]_m,\left[\frac{1}{256},0\right]_m\right]_{k'}m^4+\\
&\left[\left[-\frac{1}{192},\frac{7}{384}\right]_m,\left[\frac{7}{384},-\frac{1}{192}\right]_m\right]_{k'}m^3+\\
&\left[\left[0,\frac{3}{128}\right]_m,\left[\frac{3}{128},0\right]_m\right]_{k'}m^2+\left[\left[\frac{1}{60},-\frac{71}{3840}\right]_m,\left[-\frac{71}{3840},\frac{1}{60}\right]_m\right]_{k'}m+\\
&\left[\left[0,-\frac{7}{256}\right]_m,\left[-\frac{7}{256},0\right]_m\right]_{k'}.
\end{aligned}
$$

In addition, the program points out that this relation holds only if $k' \leq (m-3)/3$. Otherwise, the number of voting situations is zero.

We simplify[3] the quasi polynomial above by considering different values of $m$ and $k'$. Thus, it can be deduced that the number of voting situations corresponding to each of the strict preferences represented by the cases 26 and 27 in Table 3.1 is given by $F_1(m,k')$ if both $m$ and $k'$ are odd (or even) and by $F_2(m,k')$ if one of the parameters ($m$ or $k'$) is odd and the other one is even such that:

$$\begin{aligned}F_1(m,k') &= \frac{1}{3840}\Big((m-3\,k'-4)\,(m-3\,k')\,(m-3\,k'+4)\\ &\quad (m-3\,k'-2)\,(m-3\,k'+2)\Big).\\ F_2(m,k') &= \frac{1}{3840}\Big((m-3\,k'+3)\,(m-3\,k'+7)\,(m-3\,k'+1)\\ &\quad (m-3\,k'+5)\,(m-3\,k'-1)\Big).\end{aligned}$$

As a consequence of the above number of voting situations and taking into account the symmetry of the strict preferences of the cases 26 and 27 in Table 3.1 and the total number of voting situations $\psi(m)$ in (3.5), we introduce the probabilities of having triple-acyclic strict preferences under $M^L_{k'}$ majorities in the following result.

**Proposition 3.1.** *Consider a three candidate election with $m$ voters under $M^L_{k'}$ majority rules where each individual vote consists of a linear preference ordering on the candidates. Assuming that all voting situations are equally likely (IAC), if $k' \leq (m-3)/3$, the probability of triple-acyclic strict preference is as follows:*

- *If both $m$ and $k'$ are odd (or even):*

$$1 - \frac{2F_1(m,k')}{\psi(m)}.$$

- *If one of the parameters ($m$ or $k'$) is odd and the other one is even:*

$$1 - \frac{2F_2(m,k')}{\psi(m)}.$$

[3] Such simplification is done with Maple software.

Computed values of this probability are listed in Table 3.2. The values of the difference of votes $k'$ correspond to the ones that provide a probability of triple-acyclic strict preferences equal to 1. As it is previously mentioned, these probabilities indicate that the number of solutions in the systems of inequalities corresponding to the cyclic strict preferences (cases 26 and 27 in Table 3.1) are equal to zero when $k' > (m-3)/3$.

Table 3.2: Probability of triple-acyclic $P^L_{k'}$.

| $m \rightarrow$ <br> $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9643 | 1 | 0.9524 | 0.9860 | 0.9462 | 0.9384 | 0.9375 |
| 1 | 1 | | 1 | 0.9860 | 0.9462 | 0.9384 | 0.9375 |
| 2 | | | | 1 | 0.9605 | 0.9403 | 0.9375 |
| 32 | | | | | 1 | 0.9628 | 0.9378 |
| 332 | | | | | | 1 | 0.9406 |
| 33 332 | | | | | | | 1 |

Some interesting facts could be emphasized from the probabilities in Table 3.2.

First, $M^L_0$ majority provide a probability of triple-acyclic strict preferences equal to 1 for the case of $m = 4$ whereas a difference of votes equal to 1 is necessary in the case of $m = 3$ and $m = 5$ to achieve such probability. We conjecture that this odd result attends to the fact that the likelihood of having ties is greater in the case of an even number of individuals than in the case of an odd number of individuals when these voters are endowed with linear orderings.

Second, the weight of the difference of votes necessary to achieve a probability of triple-acyclic strict preferences over the total number of votes equal to 1 increases with the number of individuals from $m = 10$ to $m = 100\,000$. In the case of $m = 10$ required difference signifies a 20% of the value of $m$, a 32% in the case of $m = 100$, a 33.2% in the case of $m = 1\,000$ and a 33.332% in the case of $m = 100\,000$.

Third, the probabilities do not reflect small changes in the magnitude of the thresholds. See for instance, the probabilities attached to the cases $m = 10$, $m = 100$ and $m = 1\,000$ for the values of the difference $k'$ equal to 0 and 1 and for the case of $m = 100\,000$ in the case of $k'$ equal 0, 1 and 2.

Finally, triple-acyclic strict preferences under $M^L_{k'}$ majorities can be guaranteed with a probability of 1 for not too demanding differences of votes.

### 3.3.2 Probabilities of transitive strict preferences under majorities based on difference of votes with linear orderings

To study the probability of transitive strict preferences under $M^L_{k'}$ majorities, we follow the same methodology as the one applied in Subsection 3.3.1 adding the cases 20, 21, 22, 23, 24 and 25 in Table 3.1 to the outcomes 26 and 27 analyzed in the case of triple-acyclic strict preferences. Once we calculate the probability of non transitive strict preferences collected in the cases from 20 to 27, we determine the probability of transitive strict preferences as 1 minus the previous probability.

Ordinary preferences collected in cases from 20 to 25 are similar and hence, the number of integer solutions given by Barvinok's algorithm is the same in each of the six systems of inequalities representing these strict preferences.

For these cases, two validity domains can be distinguished. On the one hand, if $k' \leq (m-3)/3$, the number of voting situations is given by $G_1(m, k')$ if both $m$ and $k'$ are odd (or even) and by $G_2(m, k')$ if one of the parameters ($m$ or $k'$) is odd and the other one is even such that:

$$G_1(m,k') = \frac{1}{1920}\,(k'+1)\Big(121\,k'^4 - 116\,k'^3 - 200\,mk'^3 + 180\,mk'^2 - 164\,k'^2 + 130\,m^2k'^2 + 144\,k' - 40\,m^3k' - 100\,m^2k' + 120\,mk' - 20\,m^2 - 80\,m + 5\,m^4 + 20\,m^3\Big).$$

$$G_2(m,k') = \frac{1}{1920}\,k'\Big(121\,k'^4 - 200\,mk'^3 - 600\,k'^3 + 130\,m^2k'^2 + 780\,mk'^2 + 910\,k'^2 - 40\,m^3k' - 360\,m^2k' - 840\,mk' - 360\,k' + 5\,m^4 + 60\,m^3 + 210\,m^2 + 180\,m - 71\Big).$$

On the other hand, if $(m-2)/3 \leq k' \leq m-2$, the number of voting situations is given by $G_3(m,k')$ if both $m$ and $k'$ are odd (or even) and by $G_4(m,k')$ if one of the parameters ($m$ or $k'$) is odd and the other one is even such that:

$$G_3(m,k') = \frac{1}{3840}\Big((m-k'-2)\,(m-k'+4)\,(m-k')\,(m-k'+6)\,(m-k'+2)\Big).$$

$$G_4(m,k') = \frac{1}{3840}\Big((m-k'+7)\,(m-k'+3)\,(m-k'-1)\,(m-k'+5)\,(m-k'+1)\Big).$$

Bearing in mind above numbers of voting situations, the results in Proposition 3.1 and the total number of voting situations $\psi(m)$ in (3.5), we derive the probability of transitive strict preferences under $M^L_{k'}$ majorities as follows.

**Proposition 3.2.** *Consider a three candidate election with $m$ voters under $M^L_{k'}$ majority rule where each individual vote consists of a linear preference ordering on the candidates. Assuming that all voting situations are equally likely (IAC), the probability of transitive strict preferences is as follows:*

1. *If $k' \leq (m-3)/3$*

   - *If both $m$ and $k'$ are odd (or even):*

$$1 - \frac{2F_1(m,k') + 6G_1(m,k')}{\psi(m)}.$$

   - *If one of the parameters ($m$ or $k'$) is odd and the other is even:*

$$1 - \frac{2F_2(m,k') + 6G_2(m,k')}{\psi(m)}.$$

*2. If $(m-2)/3 \leq k' \leq m-2$*

- *If both $m$ and $k'$ are odd (or even):*

$$1 - \frac{6G_3(m,k')}{\psi(m)}.$$

- *If one of the parameters ($m$ or $k'$) is odd and the other is even:*

$$1 - \frac{6G_4(m,k')}{\psi(m)}.$$

Computed values of this probability are listed in Table 3.3. Going deeper on them, the weight of the required difference of votes $k'$ to guarantee a probability of transitive strict preferences equal to 1 with respect to the total number of individuals increases as the number of individuals does. To

Table 3.3: Probability of transitive $P^L_{k'}$.

| $m \rightarrow$ <br> $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9643 | 0.9524 | 0.9524 | 0.9161 | 0.9293 | 0.9366 | 0.9375 |
| 1 | 1 | 0.9524 | 0.9762 | 0.9161 | 0.9293 | 0.9366 | 0.9375 |
| 2 | | 1 | 0.9762 | 0.9580 | 0.9175 | 0.9348 | 0.9375 |
| 3 | | | 1 | 0.9580 | 0.9175 | 0.9348 | 0.9375 |
| 8 | | | | 1 | 0.9088 | 0.9298 | 0.9374 |
| 98 | | | | | 1 | 0.9101 | 0.9366 |
| 998 | | | | | | 1 | 0.9290 |
| 99 998 | | | | | | | 1 |

illustrate, in the case of $m = 3$ the required $k' = 1$ represents around a 33.33% of the value of $m$ whereas in the case of $m = 100\,000$ the required $k'$ represents a 99.998% of the value of $m$. In fact, the required differences are very large for all the considered cases with the exception of $m = 3$. Even for $m = 4$, the difference signifies a 50% of the value of $m$. For not too demanding differences, the probabilities increase in the cases of $m = 4$, $m = 5$ and

$m = 10$; this is not the case for $m = 100$, $m = 1\,000$ and $m = 100\,000$ where asking reasonable differences of votes decreases the probability of transitive strict preferences.

Moreover, as in the case of the probabilities stated in Table 3.2, small variations in the magnitude of the differences of votes do not change, at least in a significant way, the probabilities. To illustrate, look at the probabilities of $m = 100\,000$ with differences $k'$ equal to 0, 1, 2 and 3.

### 3.3.3 Probabilities of transitive weak preferences under majorities based on difference of votes with linear orderings

To derive the probability of transitive weak preferences under $M^L_{k'}$ majorities, we need to consider, in addition with the cases analyzed in Proposition 3.2, the cases from 14 to 19 in Table 3.1. With that, we calculate the probability of non transitive weak preferences and therefore, the probability of transitive weak preferences is determined as 1 minus the probability of non transitive weak preferences.

By symmetry arguments, the weak preferences represented in cases from 14 to 19 are similar and therefore the number of integer solutions of the six systems of inequalities corresponding to such cases is the same. Using again the Barvinok's algorithm, two validity domains can be considered. If $k' \leq (m-2)/3$, the number of voting situations inside each system is given by $H_1(m,k')$ if both $m$ and $k'$ are odd (or even), and by $H_2(m,k')$ if one of the parameters ($m$ or $k'$) is odd and the other one is even such that:

$$\begin{aligned} H_1(m,k') = -\frac{1}{240}\Big( &(k'+1)\left(17\,k'^4 - 30\,mk'^3 - 22\,k'^3 + 20\,m^2k'^2 - \right.\\ &28\,k'^2 + 30\,mk'^2 + 50\,mk' + 48\,k' - 5\,m^3k' - 5\,m^2k' - \\ &\left. 5\,m^3 - 30\,m^2 - 40\,m\right)\Big). \end{aligned}$$

$$H_2(m,k') = -\frac{1}{240}\,k'\Big(17\,k'^4 - 30\,mk'^3 - 90\,k'^3 + 20\,m^2k'^2 + 120\,mk'^2 + 140\,k'^2 - 5\,m^3k' - 45\,m^2k' - 100\,mk' - 30\,k' - 37 - 5\,m^2 - 30\,m\Big).$$

For the second validity domain, if $(m-1)/3 \le k' \le m-1$, this number is given by $H_3(m,k')$ if both $m$ and $k'$ are odd (or even) and by $H_4(m,k')$ if one of the parameters ($m$ or $k'$) is odd and the other one is even such that:

$$H_3(m,k') = \frac{1}{3840}\Big((m-k'+2)\,(m-k')\,(m-k'+4)\,\big(29\,k'^2 + 12\,mk' + 94\,k' + 72 - m^2 + 6\,m\big)\Big).$$

$$H_4(m,k') = \frac{1}{3840}\Big((m-k'+1)\,(m-k'+5)\,(m-k'+3)\,\big(29\,k'^2 + 12\,mk' + 36\,k' - 6\,m + 7 - m^2\big)\Big).$$

Taking into consideration the intersections between the different validity domains and using the results in Propositions 3.1 and 3.2, the probability of transitive weak preferences is as follows.

**Proposition 3.3.** *Consider a three candidate election with $m$ voters under $M^L_{k'}$ majority rule where each individual vote consists of a linear preference ordering on the candidates. Assuming that all voting situations are equally likely (IAC), the probability of transitive weak preferences is as follows:*

1. *If $k' \le (m-3)/3$*

   - *If both $m$ and $k'$ are odd (or even):*

   $$1 - \frac{2F_1(m,k') + 6G_1(m,k') + 6H_1(m,k')}{\psi(m)}.$$

   - *If one of the parameters ($m$ or $k'$) is odd and the other one is even:*

   $$1 - \frac{2F_2(m,k') + 6G_2(m,k') + 6H_2(m,k')}{\psi(m)}.$$

2. *If $(m-1)/3 \le k' \le m-2$*

- *If both $m$ and $k'$ are odd (or even):*

$$1 - \frac{6G_3(m,k') + 6H_3(m,k')}{\psi(m)}.$$

- *If one of the parameters ($m$ or $k'$) is odd and the other one is even:*

$$1 - \frac{6G_4(m,k') + 6H_4(m,k')}{\psi(m)}.$$

3. *If $k' = (m-2)/3$*

- *Either both $m$ and $k'$ are odd or both are even:*

$$1 - \frac{6H_1(m,k') + 6G_3(m,k')}{\psi(m)}.$$

4. *If $k' = m-1$*

- *One of the parameters ($m$ or $k'$) is odd and the other one is even:*

$$1 - \frac{6H_4(m,k')}{\psi(m)}.$$

Analyzing the probabilities of transitive weak preferences displayed in Table 3.4, $M_0^L$ majority provides the highest values for the probability of having transitive weak preferences for almost all the considered values of $m$. In fact, any difference of votes can be asked to guarantee a probability value of 1. Only in the cases of $m = 1\,000$ and $m = 100\,000$ the probability arrives to the value of 1.0000, i.e. the probability approximates to the value of 1 without reaching it[4]. Even so, in both cases the required difference of votes is extremely large. Specifically, it represents a 99.9% of the value of $m$ in the case of $m = 1\,000$ and a 99.999% in the case of $m = 100\,000$.

Finally, as in the previous cases stated in Tables 3.2 and 3.3, the probabilities do not significantly change with small variations of the magnitude of the difference in votes. On this, see for instance the cases of $k'$ equal 2 and 3 for $m$ equal 4, 10, 100, 1 000 and 100 000.

[4]Notice that this value also appears in Tables 3.5, 3.7, 3.12, 3.13, 3.14 and 3.15 with the same meaning.

Table 3.4: Probability of transitive $R^L_{k'}$.

| $m \rightarrow$ $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9643 | 0.7619 | 0.9524 | 0.8462 | 0.9280 | 0.9366 | 0.9375 |
| 2 | 0.6786 | 0.7143 | 0.6667 | 0.6703 | 0.9062 | 0.9346 | 0.9375 |
| 3 | | 0.7143 | 0.7619 | 0.6703 | 0.9062 | 0.9346 | 0.9375 |
| 4 | | | 0.7619 | 0.6503 | 0.8818 | 0.9327 | 0.9375 |
| 9 | | | | 0.9101 | 0.8292 | 0.9287 | 0.9374 |
| 99 | | | | | 0.9997 | 0.8136 | 0.9366 |
| 999 | | | | | | 1.0000 | 0.9276 |
| 99 999 | | | | | | | 1.0000 |

## 3.4 Probabilities of consistent collective decisions under majorities based on difference of votes with weak orderings

In this section the results about the probabilities of consistent collective decisions under $M_{k'}$ majorities are introduced under the IAC assumption. Given that voters could be indifferent between the alternatives, we have to take into account the six linear preference orderings in (3.4), the six possible orderings that collect the partial indifference and the one that represents the complete indifference among three alternatives. Therefore,

$$\begin{array}{lll} x_1x_2x_3\ (m_1) & x_1x_3x_2\ (m_2) & x_2x_1x_3\ (m_3) \\ x_2x_3x_1\ (m_4) & x_3x_1x_2\ (m_5) & x_3x_2x_1\ (m_6) \\ \{x_1x_2\}x_3\ (m_7) & \{x_1x_3\}x_2\ (m_8) & \{x_2x_3\}x_1\ (m_9) \\ x_1\{x_2x_3\}\ (m_{10}) & x_2\{x_1x_3\}\ (m_{11}) & x_3\{x_1x_2\}\ (m_{12}) \\ \{x_1x_2x_3\}\ (m_{13}) & & \end{array} \tag{3.6}$$

where $m_i$ represents the number of voters with the associated preference ordering and $\{x_ix_j\}$ stands for the indifference between the alternatives $x_i$ and $x_j$. As the IAC condition is assumed, all possible voting situations $\boldsymbol{m}$ are equally liked to be observed. For $m$ individuals and 3 alternatives, if the indifference between alternatives is allowed, the total number of voting situations $\boldsymbol{m}$ is given by the expression:

$$\Psi(m) = \frac{(m+1)(m+2)\cdots(m+12)}{12!}. \tag{3.7}$$

Using the same approach applied in Section 3.3, the probability of consistent outcomes is calculated by means of the computation of the probability of inconsistent outcomes. As there, such probabilities are given by Ehrhart polynomials that provide the number of integer points inside the systems of inequalities that characterize each of the analyzed inconsistent outcomes.

In the framework of the preferences represented in (3.6), the complexity of the conditions makes impossible the derivation of a general mathematical representation as the one provided in Section 3.3. This is because for each considered consistency condition, the number of validity domains and the length of the polynomials are greater than in the cases of Section 3.3.

Fortunately, when the number of individuals $m$ and the threshold $k'$ are fixed, the probabilities can be calculated for the given number of voting situations $\boldsymbol{m}$.

### 3.4.1 Probabilities of triple-acyclic strict preferences under majorities based on difference of votes with weak orderings

In Table 3.5, the probabilities of triple-acyclic strict preferences under $M_{k'}$ majorities are displayed.

The following facts can be pointed our from these results. The probabilities of having triple-acyclic strict preferences when no difference of votes is required, i.e. when $M_0$ majority is applied, reach very high values. Specifically, they are located between 0.9571 and 0.9989. For $m = 3$, $m = 4$ and $m = 5$, the needed difference of votes to achieve a probability value of 1 equals 1. Therefore, it represents a one third of the value of $m$ in the case of $m = 3$, a 25% in the case of $m = 4$ and a 20% in the case of $m = 5$. For the remaining considered values, the weight of the required differences represent around one third of the value of $m$ which means that we can guarantee with a

Table 3.5: Probability of triple-acyclic $P_{k'}$.

| $m \rightarrow$ <br> $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9956 | 0.9989 | 0.9929 | 0.9873 | 0.9627 | 0.9577 | 0.9571 |
| 1 | 1 | 1 | 1 | 0.9992 | 0.9720 | 0.9589 | 0.9572 |
| 2 | | | | 1.0000 | 0.9794 | 0.9601 | 0.9572 |
| 3 | | | | 1 | 0.9851 | 0.9613 | 0.9572 |
| 33 | | | | | 1 | 0.9852 | 0.9576 |
| 333 | | | | | | 1 | 0.9611 |
| 33 333 | | | | | | | 1 |

probability of 1 the triple-acyclicity of strict preferences under $M_{k'}$ majorities for reasonable values of the difference of votes.

### 3.4.2 Probabilities of transitive strict preferences under majorities based on difference of votes with weak orderings

In Table 3.6, the probabilities of transitive strict preferences under $M_{k'}$ majorities are presented.

It is remarkable that to reach a probability of transitive strict preferences equal to 1, the weight of the required difference of votes $k'$ with respect to the total number of individuals increases as the number of individuals does. For instance, in the case of $m = 4$, it represents a 50% of the value of $m$ whereas in the case of $m = 1\,000$ it does a 99.8% of the value of $m$.

Moreover, the required differences are too demanding for all the cases with the exception of the case of $m = 3$ where it represents one third of the value of $m$.

Table 3.6: Probability of transitive $P_{k'}$.

| $m \rightarrow$ <br> $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9692 | 0.9626 | 0.9531 | 0.9422 | 0.9519 | 0.9565 | 0.9571 |
| 1 | 1 | 0.9901 | 0.9884 | 0.9632 | 0.9443 | 0.9554 | 0.9571 |
| 2 | | 1 | 0.9971 | 0.9851 | 0.9399 | 0.9542 | 0.9571 |
| 3 | | | 1 | 0.9942 | 0.9383 | 0.9531 | 0.9571 |
| 8 | | | | 1 | 0.9516 | 0.9484 | 0.9570 |
| 98 | | | | | 1 | 0.9597 | 0.9560 |
| 998 | | | | | | 1 | 0.9471 |
| 99 998 | | | | | | | 1 |

### 3.4.3 Probabilities of transitive weak preferences under majorities based on difference of votes with weak orderings

Probabilities of transitive weak preferences under $M_{k'}$ majorities defined for weak orderings are displayed in Table 3.7.

Table 3.7: Probability of transitive $R_{k'}$.

| $m \rightarrow$ <br> $k' \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9801 | 0.8538 | 0.8920 | 0.9072 | 0.9511 | 0.9565 | 0.9571 |
| 2 | 0.7890 | 0.7429 | 0.7014 | 0.6980 | 0.9218 | 0.9540 | 0.9571 |
| 3 | | 0.8648 | 0.8167 | 0.6814 | 0.9049 | 0.9527 | 0.9571 |
| 4 | | | 0.9176 | 0.7234 | 0.8871 | 0.9514 | 0.9571 |
| 9 | | | | 0.9919 | 0.7954 | 0.9444 | 0.9570 |
| 99 | | | | | 1.0000 | 0.7782 | 0.9559 |
| 999 | | | | | | 1.0000 | 0.9436 |
| 99 999 | | | | | | | 1.0000 |

The probability value of 1 is almost achieved for the values of $m$ equal to 100, 1 000 and 100 000. In these cases, the required differences in votes $k'$ are so high that signify a 98% of the value of $m$ in the case of $m = 100$, a 99.8% in the case of $m = 1\,000$ and a 99.998% in the case of $m = 100\,000$.

## 3.5 Probabilities of consistent collective decisions under majorities based on difference in support

In this section, we provide the probabilities of reaching consistent collective decisions under $\widetilde{M}_k$ majorities for three alternatives.

As long as the intensities of preference between each pair of alternatives can take any value in the continuous interval $[0, 1]$, the IAC model can not be applied to provide an a priori probability to each possible voting situation. Therefore the probabilistic analysis carried out in Sections 3.3 and 3.4 turns impossible to study the case of $\widetilde{M}_k$ majorities.

Consequently, we perform a simulation with the software Matlab to estimate these probabilities as the proportion of the number of consistent outcomes in the simulation over the total number of simulated outcomes. We generate for each of these values 100 000 outcomes to guarantee our results with a confidence level of 99% and a sampling error of less than a 0.0041%[5].

Below, we describe the methodology applied in the simulations to es-

[5]Assuming a proportion of consistent outcomes $P$ on the population of a 50%, the proportion $p$ in a random sample of size $n \geq 30$ for a confidence level of 99%, diverges from the one of the population in an error of less than $\epsilon$:

$$Prob(|P - p| \leq \epsilon) \geq 0.99.$$

Taken into account that the sample proportion $p$ is distributed as $N\left(P, \sqrt{P(1-P)/n}\right)$, the sampling error $\epsilon$ is as follows:

$$\epsilon = z_{\alpha/2}\sqrt{P(1-P)/n}.$$

In our case, $n = 100\,000$ and the corresponding percentile of the normal distribution for a confidence level of 99% is $z_{\alpha/2} = 2.57$. Thus, $\epsilon \leq 0.00407$.

timate the probability for the considered three types of consistent collective decisions under $\widetilde{M}_k$ majorities, i.e. transitive weak preferences, transitive and triple-acyclic strict preferences. We follow that scheme taking into account each type of individual transitive reciprocal relations, i.e. 0.5–transitive, min–transitive, am–transitive and max–transitive reciprocal preference relations. Notice that the matrix in (3.1) representing a reciprocal preference relation is determined by the vector composed of the intensities $r_{12}, r_{23}$ and $r_{13}$.

1. We randomly generate $m$ vectors representing the transitive reciprocal preference relations of the $m$ individuals. Such vectors are built bearing in mind one of the considered transitivity conditions for reciprocal preference relations.

2. We compute the sum of the individuals' intensities of preference over each pair of alternatives through a vector $\boldsymbol{S} = (S_{12}, S_{23}, S_{13})$ where $S_{ij} = \sum\limits_{p=1}^{m} r_{ij}^{p}$.

3. Having in mind the conditions in equations (3.2) and (3.3) and the value of $k$, the collective decision is evaluated over each pair of alternatives in the vector $\boldsymbol{S}$.

4. The collective decision in $\boldsymbol{S}$ is classified following the cases of possible collective outcomes displayed in Table 3.1. If it is one of the cases 26 or 27, the strict preference $P_k$ is not triple-acyclic. If it is one of the cases from 19 to 27, the strict preference $P_k$ is not transitive. Finally, if it is one of the cases from 14 to 27, the weak preference $R_k$ is not transitive.

5. This four steps are iterated 100 000 times to obtain the number of inconsistent collective decisions. Specifically, the number of simulated outcomes in which the weak preference $R_k$ is not transitive, in which the strict preference $P_k$ is not transitive and in which $P_k$ is not triple-acyclic, respectively.

6. The number of each considered type of consistent social outcomes is computed as the total number of simulated outcomes, i.e. 100 000, minus the number of inconsistent ones computed in the previous step.

7. Finally, each of the desired probabilities, i.e. the probability of transitive $R_k$ and the probability of transitive and triple-acyclic $P_k$, is calculated as the number of consistent outcomes over the total number of simulated outcomes.

In the following, the simulated probabilities of consistent collective decisions under $\widetilde{M}_k$ majorities are listed in tables.

### 3.5.1 Probability of transitive weak preferences under majorities based on difference in support

Tables 3.8, 3.9, 3.10 and 3.11 provide the probabilities of transitive weak preferences when reciprocal preference relations fulfil 0.5–transitivity, min–transitivity, am–transitivity and max–transitivity, respectively.

Table 3.8: Probabilities of transitive $R_k$ for 0.5–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---:|---:|---:|---:|---:|---:|---:|---:|
| 0 | 0.8835 | 0.8781 | 0.8763 | 0.8751 | 0.8706 | 0.8705 | 0.8728 |
| 2.97 | 1 | 0.9836 | 0.9485 | 0.7630 | 0.5927 | 0.7826 | 0.8655 |
| 3.81 | | 1 | 0.9964 | 0.9065 | 0.5481 | 0.7544 | 0.8633 |
| 4.70 | | | 1 | 0.9748 | 0.5312 | 0.7248 | 0.8609 |
| 7.95 | | | | 1 | 0.6729 | 0.6266 | 0.8518 |
| 26.40 | | | | | 1 | 0.7012 | 0.7954 |
| 84.83 | | | | | | 1 | 0.6156 |
| 95.32 | | | | | | | 0.5913 |

When the required difference in support equals zero, the harder the transitivity condition over the reciprocal preference relations is, the higher the

Table 3.9: Probabilities of transitive $R_k$ for min–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9563 | 0.9513 | 0.9505 | 0.9487 | 0.9468 | 0.9476 | 0.9455 |
| 2.96 | 1 | 0.9831 | 0.9528 | 0.8015 | 0.7220 | 0.8814 | 0.9400 |
| 3.84 | | 1 | 0.9966 | 0.9205 | 0.6776 | 0.8576 | 0.9383 |
| 4.80 | | | 1 | 0.9809 | 0.6552 | 0.8307 | 0.9366 |
| 8.32 | | | | 1 | 0.7559 | 0.7432 | 0.9291 |
| 27.23 | | | | | 1 | 0.7702 | 0.8831 |
| 95.32 | | | | | | 1 | 0.7189 |

Table 3.10: Probabilities of transitive $R_k$ for am–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9751 | 0.9710 | 0.9712 | 0.9694 | 0.9676 | 0.9670 | 0.9692 |
| 2.97 | 1 | 0.9812 | 0.9492 | 0.8087 | 0.7738 | 0.9117 | 0.9650 |
| 3.91 | | 1 | 0.9971 | 0.9225 | 0.7295 | 0.8904 | 0.9636 |
| 4.91 | | | 1 | 0.9813 | 0.7043 | 0.8683 | 0.9622 |
| 7.48 | | | | 1 | 0.7370 | 0.8112 | 0.9582 |
| 27.90 | | | | | 1 | 0.7873 | 0.9180 |
| 83.01 | | | | | | 1 | 0.7967 |
| 95.32 | | | | | | | 0.7731 |

probabilities of having transitive weak preferences are. To illustrate, notice that the probabilities vary in between 0.8705 and 0.8835 considering 0.5–transitive reciprocal preference relations, between 0.9455 and 0.9563 taking into account min–transitive reciprocal preference relations, between 0.9670 and 0.9751 bearing in mind am–transitive reciprocal preference relations and between 0.9726 and 0.9777 in the case of max–transitive reciprocal relations.

Table 3.11: Probabilities of transitive $R_k$ for max–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9777 | 0.9766 | 0.9751 | 0.9741 | 0.9727 | 0.9726 | 0.9744 |
| 2.98 | 1 | 0.9842 | 0.9561 | 0.8219 | 0.7867 | 0.9189 | 0.9708 |
| 3.84 | | 1 | 0.9964 | 0.9241 | 0.7450 | 0.9003 | 0.9696 |
| 4.72 | | | 1 | 0.9782 | 0.7188 | 0.8805 | 0.9681 |
| 7.56 | | | | 1 | 0.7495 | 0.8190 | 0.9637 |
| 25.47 | | | | | 1 | 0.7720 | 0.9298 |
| 80.00 | | | | | | 1 | 0.8030 |
| 95.32 | | | | | | | 0.7744 |

In the case of $m = 100\,000$ again, the stronger the required transitivity condition over the reciprocal preference relations is, the higher the probabilities of having transitive weak preferences are. See for instance, the probability values attached to the difference in support $k = 95.32$. In the case of 0.5–transitive reciprocal preference relations, the probability reaches a value of 0.5913, whereas it does a value of 0.7189 in the case of min–transitive reciprocal preference relations. In the case of am–transitive reciprocal preference relations the probability achieves a value of 0.7731 while it does a value of 0.7744 in the case of max–transitive reciprocal preference relations.

We have found some unexpected results with respect to the needed thresholds to reach probability values equal to 1 for the considered values of $m$ from 3 to 1 000. In some of these cases, the thresholds are almost the same ones with independence of the required transitivity condition over the reciprocal preference relations (on this, see the cases of $m = 3$, $m = 4$ and $m = 5$). In the case of $m = 100$, the highest thresholds are the ones required when reciprocal preference relations are am–transitive and min–transitive, while the lowest ones are the corresponding thresholds to max–transitive and 0.5–transitive reciprocal preference relations. Finally, in the case of $m = 1\,000$, the highest threshold needed to achieve a probability value of 1 corresponds to min–transitive reciprocal preference relations. The second highest one corresponds to 0.5–transitive reciprocal preference relations, the third one to am–transitive reciprocal preference relations and the lowest one to max–transitive reciprocal preference relations. Therefore, in these situations we can not establish a clear relationship between the strength of the transitivity condition fulfilled by the reciprocal preference relations and the size of the needed threshold to achieve a probability value of 1 of having transitive weak preferences.

### 3.5.2 Probabilities of transitive strict preferences under majorities based on difference in support

Tables 3.12, 3.13, 3.14 and 3.15 display the probabilities of transitive strict preferences $P_k$ when reciprocal preference relations fulfil 0.5–transitivity, min–transitivity, am–transitivity and max–transitivity, respectively.

Table 3.12: Probabilities of transitive $P_k$ for 0.5–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.8835 | 0.8781 | 0.8763 | 0.8751 | 0.8706 | 0.8705 | 0.8728 |
| 2.58 | 1 | 0.9999 | 0.9995 | 0.9916 | 0.8512 | 0.8287 | 0.8668 |
| 2.97 | | 1 | 1.0000 | 0.9970 | 0.8661 | 0.8268 | 0.8659 |
| 3.24 | | | 1 | 0.9985 | 0.8759 | 0.8257 | 0.8654 |
| 5.17 | | | | 1 | 0.9465 | 0.8267 | 0.8611 |
| 14.51 | | | | | 1 | 0.9295 | 0.8440 |
| 47.79 | | | | | | 1 | 0.8293 |

At a first glance, the probability of transitive strict preferences goes to value of 1 for non zero values of the threshold $k$ for all the considered type of transitive reciprocal preference relations.

The rhythm of the convergence of these probabilities to 1 depends on

Table 3.13: Probabilities of transitive $P_k$ for min–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9563 | 0.9513 | 0.9505 | 0.9487 | 0.9468 | 0.9476 | 0.9455 |
| 1.96 | 1 | 0.9999 | 0.9997 | 0.9968 | 0.9377 | 0.9271 | 0.9421 |
| 2.49 | | 1 | 1.0000 | 0.9993 | 0.9514 | 0.9255 | 0.9414 |
| 2.61 | | | 1 | 0.9995 | 0.9533 | 0.9254 | 0.9413 |
| 3.47 | | | | 1 | 0.9704 | 0.9248 | 0.9398 |
| 12.51 | | | | | 1 | 0.9793 | 0.9291 |
| 35.24 | | | | | | 1 | 0.9251 |
| 47.79 | | | | | | | 0.9311 |

Table 3.14: Probabilities of transitive $P_k$ for am–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9751 | 0.9710 | 0.9712 | 0.9694 | 0.9676 | 0.9670 | 0.9692 |
| 1.56 | 1 | 0.9999 | 0.9997 | 0.9973 | 0.9605 | 0.9551 | 0.9672 |
| 1.85 | | 1 | 1.0000 | 0.9990 | 0.9655 | 0.9542 | 0.9668 |
| 2.07 | | | 1 | 0.9995 | 0.9697 | 0.9538 | 0.9665 |
| 3.10 | | | | 1 | 0.9845 | 0.9538 | 0.9653 |
| 9.17 | | | | | 1 | 0.9828 | 0.9600 |
| 33.10 | | | | | | 1 | 0.9570 |
| 47.79 | | | | | | | 0.9629 |

the number of individuals $m$ because in each of the individual transitivity specifications, the weight of the threshold with respect to the considered number of individuals decreases when $m$ increases.

Notice that the strength of the transitivity condition over the reciprocal preference relations seems to play a role in that convergence. It looks that the

Table 3.15: Probabilities of transitive $P_k$ for max–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9777 | 0.9766 | 0.9751 | 0.9741 | 0.9727 | 0.9726 | 0.9744 |
| 1.51 | 1 | 1.0000 | 0.9999 | 0.9981 | 0.9689 | 0.9619 | 0.9727 |
| 1.66 | | 1 | 0.9999 | 0.9989 | 0.9712 | 0.9614 | 0.9725 |
| 1.93 | | | 1 | 0.9996 | 0.9758 | 0.9608 | 0.9723 |
| 2.78 | | | | 1 | 0.9870 | 0.9607 | 0.9715 |
| 9.72 | | | | | 1 | 0.9897 | 0.9662 |
| 26.10 | | | | | | 1 | 0.9626 |
| 47.79 | | | | | | | 0.9690 |

more rational the individuals are, the smaller the thresholds different from 0 needed to induce transitive strict preferences are. The unique exception to that behavior is found when $m = 100$, where the threshold with attached probability value of 1 is slightly lower in the case of am–transitive reciprocal preference relations (see Table 3.14) than in the case of max–transitive ones (see Table 3.15).

To illustrate these general facts, we focus on the required thresholds for $m = 3$ and $m = 1\,000$ in the four cases. In the case of 0.5–transitive reciprocal relations in Table 3.12, the required threshold equals 2.58 and consequently represents a 85% of the value of $m$. In the case of $m = 1\,000$, the required threshold represents less than a 4.8% of the value of $m$. In the case of min–transitive reciprocal relations in Table 3.13, the threshold for reaching a probability of 1 is 1.96 representing less than a 66% in the case of $m = 3$ whereas it symbolizes around a 3.5% in the case of $m = 1\,000$. In the case of am–transitive reciprocal relations in Table 3.14, the thresholds are 1.56 and 33.10, representing a 52% and around a 3.3% of the considered numbers of individuals, respectively. Finally, in the case of max–transitive reciprocal relations in Table 3.15, the threshold for $m = 3$ equals 1.51 and therefore represents around a 50% of the number of voters and in the case of $m = 1\,000$ represents around a 2.6% of the number of voters.

### 3.5.3 Probabilities of triple-acyclic strict preferences under majorities based on difference in support

In Tables 3.16, 3.17, 3.18 and 3.19, we present the results for the simulated probabilities of having triple-acyclic strict preferences when 0.5–transitive, min–transitive, am–transitive and max–transitive individual preferences are considered.

Table 3.16: Probabilities of triple-acyclic $P_k$ for 0.5–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.8835 | 0.8781 | 0.8763 | 0.8751 | 0.8706 | 0.8705 | 0.8728 |
| 1.32 | 1 | 0.9999 | 0.9998 | 0.9967 | 0.9479 | 0.9013 | 0.8763 |
| 1.50 | | 1 | 1.0000 | 0.9983 | 0.9546 | 0.9052 | 0.8768 |
| 1.91 | | | 1 | 0.9998 | 0.9681 | 0.9135 | 0.8778 |
| 2.65 | | | | 1 | 0.9837 | 0.9270 | 0.8794 |
| 8.29 | | | | | 1 | 0.9834 | 0.8926 |
| 26.60 | | | | | | 1 | 0.9291 |

Conclusions are similar to the case depicted in Subsection 3.5.2. First, computed probabilities are high and go to 1 for non zero values of $k$ in all considered the cases.

Second, the weight of the needed threshold to reach a probability value of 1 relative to the number of individuals involved in the voting process decreases when the value of $m$ increases, with the exception of the events of $m = 4$ and $m = 5$ in the case of 0.5-transitive reciprocal preference relations. For example, consider the probabilities for the case of am-transitive reciprocal preference relations in Table 3.18. There, for $m = 3$, the probability achieves the value of 1 for a $k = 0.56$ that represents less than a 19% of the value

Table 3.17: Probabilities of triple-acyclic $P_k$ for min–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9563 | 0.9513 | 0.9505 | 0.9487 | 0.9468 | 0.9476 | 0.9455 |
| 0.67 | 1 | 0.9998 | 0.9997 | 0.9971 | 0.9812 | 0.9583 | 0.9467 |
| 0.88 | | 1 | 1.0000 | 0.9993 | 0.9817 | 0.9612 | 0.9471 |
| 0.90 | | | 1 | 0.9993 | 0.9906 | 0.9614 | 0.9472 |
| 1.40 | | | | 1 | 0.9906 | 0.9675 | 0.9481 |
| 4.56 | | | | | 1 | 0.9913 | 0.9540 |
| 14.99 | | | | | | 1 | 0.9687 |
| 26.60 | | | | | | | 0.9799 |

Table 3.18: Probabilities of triple-acyclic $P_k$ for am–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9751 | 0.9710 | 0.9712 | 0.9694 | 0.9676 | 0.9670 | 0.9692 |
| 0.56 | 1 | 1.0000 | 0.9998 | 0.9986 | 0.9854 | 0.9743 | 0.9698 |
| 0.67 | | 1 | 1.0000 | 0.9995 | 0.9876 | 0.9758 | 0.9700 |
| 0.83 | | | 1 | 0.9999 | 0.9905 | 0.9775 | 0.9702 |
| 1.11 | | | | 1 | 0.9941 | 0.9804 | 0.9706 |
| 3.34 | | | | | 1 | 0.9936 | 0.9730 |
| 11.68 | | | | | | 1 | 0.9811 |
| 26.60 | | | | | | | 0.9911 |

of $m$. Instead, it represents a a 11.1% for $m = 10$ and less than a 1.2% for $m = 1\,000$.

Third, the thresholds that provide a probability of triple-acyclic strict preferences equal to one are lower considering am–transitive reciprocal preference relations than considering min–transitive reciprocal preference rela-

Table 3.19: Probabilities of triple-acyclic $P_k$ for max–transitive reciprocal preference relations.

| $m \rightarrow$ <br> $k \downarrow$ | 3 | 4 | 5 | 10 | 100 | 1 000 | 100 000 |
|---|---|---|---|---|---|---|---|
| 0 | 0.9777 | 0.9766 | 0.9751 | 0.9741 | 0.9727 | 0.9726 | 0.9744 |
| 0.67 | 1 | 1.0000 | 0.9999 | 0.9998 | 0.9909 | 0.9804 | 0.9753 |
| 0.72 | | 1 | 1.0000 | 0.9999 | 0.9918 | 0.9808 | 0.9753 |
| 0.75 | | | 1 | 0.9999 | 0.9922 | 0.9811 | 0.9754 |
| 0.99 | | | | 1 | 0.9951 | 0.9834 | 0.9758 |
| 3.58 | | | | | 1 | 0.9963 | 0.9786 |
| 9.95 | | | | | | 1 | 0.9845 |
| 26.60 | | | | | | | 0.9943 |

tions. The last ones are also lower than the ones corresponding to the cased of 0.5–transitive reciprocal preference relations (see Tables 3.16, 3.17 and 3.18). The thresholds considering max–transitive reciprocal preference relations are lower than or equal to the ones corresponding to the case of min–transitive reciprocal preference relations. To illustrate, look at the thresholds of support $k$ in the cases of $m = 3$ and $m = 10$ in Tables 3.17 and 3.19. In the case of am–transitive reciprocal relations instead, some thresholds are higher than and some others lower than the ones resulting from taking into account max–transitive reciprocal preference relations (see, for instance, the cases of $m = 4$ and $m = 10$ in Tables 3.18 and 3.19). Therefore, it seems that the strength of the transitivity condition over reciprocal preference relations plays a role in the size of the threshold to achieve a probability of triple-acyclic strict preferences but the relation between such strength and the size of the threshold is dubious.

## 3.6 Discussion

Since now, we have computed the theoretical probabilities of consistent preferences under majorities based on difference of votes defined for both individual linear orderings and weak orderings, and the simulated probabilities of consistent preferences under majorities based on difference in support.

Notice that the results on the probabilities of triple-acyclic strict preferences under $M_0^L$ majority in Table 3.2 and under $M_0$ majority are consistent with the corresponding ones in Gehrlein [58] and in Lepelley and Martin [83].

Focusing on the results for majorities based on difference of votes with linear orderings (Section 3.3) and with weak orderings (Section 3.4) we have the following. In the cases of transitive and triple-acyclic strict preferences, the probabilities are higher considering weak than linear orderings (see Tables 3.2, 3.3, 3.5 and 3.6). In the case of transitive weak preferences, the same is true when $k'$ equals 1 and 2 and, with the exception of the case in which $m = 4$, also when $k' = 0$.

Looking at the results of Subsection 3.5.1, notice that the probabilities under $\widetilde{M}_0$ remains the same for each type of transitive reciprocal preference relation with independence of the type of consistency condition required to the collective preference. That counterintuitive result contrasts with the remaining ones in which the probability increases together with the increase of the number of social outcomes considered as consistent. That oddity dues to the following. Taking into account expression (3.3) the absolute value of the difference between the sum of the intensities of preference $r_{ij}^p$ and the sum of the intensities $r_{ij}^p$ has to be null to the indifference between alternatives to be declared. As far as Matlab generates random numbers with 15 decimal positions, the indifference between alternatives is almost impossible.

Recently some analytical studies about the consistency of majorities based on difference in support have been developed. These theoretical results rely on the needed threshold to ensure transitive and triple-acyclic strict preferences for different types of transitive reciprocal preference relations.

On the one hand, the case of transitive strict preferences is studied in Llamazares *et al.* [90]. The results can be summarized as follows:

1. The transitivity of the strict preference can not be ensured for any threshold of support $k$ less than $m - 1$.
2. The transitivity of the strict preference can not be ensured for any threshold of support $k$ less than $m$ if the reciprocal preference relations are less demanding than am–transitive ones.

3. The strict preference is transitive for any threshold of support such that $k \in [m-1, m)$ if the reciprocal preference relations are at least am–transitive ones.

On the other hand, the case of triple-acyclic strict preferences is analyzed in Llamazares and Pérez-Asurmendi [89] with the following results:

1. The triple-acyclicity of the strict preference, in the case of 0.5–transitive reciprocal preference relations, can be guaranteed if the threshold of support $k$ is located in $\big[\lfloor 2m/3 \rfloor, m\big)$ where $\lfloor a \rfloor$ stands for the integer part of $a$.

2. The triple-acyclicity of the strict preference, in the case of min–transitive and max–transitive reciprocal preference relations, can be guaranteed if the threshold of support $k$ belongs to $[m/3, m)$.

3. In the case of max–transitive reciprocal preference relations, it conjectures that strict preference relations are triple-acyclic if the threshold $k$ belongs to $\big[\lfloor 2m/3 \rfloor/2, m\big)$.

The probabilistic results setting here complement the above theoretical ones by the following reasons. First, thresholds with associated probabilities of consistent strict preferences equal to 1 are found for all the considered types of transitive reciprocal preference relations. Second, reasonable thresholds are required to certify the consistency of the strict preference with a probability value of 1 in those cases where theoretical results asked a very high threshold to guarantee such consistency. Third, the conjecture about the needed thresholds in the case of max–transitive reciprocal preference relations seems to be true.

Specifically, in the case of transitive strict preferences with 0.5–transitive and min–transitive reciprocal preference relations, the probabilities achieve the value of 1 for the considered values of $m$ (see Tables 3.12 and 3.13, respectively) whereas as it is said before, the theoretical result asserts that no threshold guarantees the transitivity of the strict preference for such types of reciprocal preference relations.

In the cases of am–transitive and max–transitive reciprocal preference relations, the thresholds that provide a probability value of transitive strict preference relations equal to 1 are lower than the ones that guarantee the transitivity of the strict preference in the theoretical framework. To illustrate, assume $m = 1\,000$. The theoretical result asserts that the threshold $k$ has to belong to $[999, 1\,000)$. By contrast, a probability value of 1 is achieved with a threshold of 33.10 in the case of am-transitive reciprocal preference relations and of 26.10 in the case of max–transitive reciprocal preference relations (see Tables 3.14 and 3.15, respectively).

In the case of triple-acyclic strict preferences, the thresholds to reach a probability value of 1 again are much lower than the ones required in the theoretical setting. For instance, assume $m = 5$. The needed thresholds to certify triple-acyclic strict preferences are at least 3 in the case of 0.5–transitive reciprocal preference relations and 5/3 in the cases of min–transitive and max–reciprocal preference relations. Attending to the probabilistic analysis, the needed thresholds to achieve a probability value of 1 are 1.91 in the case of 0.5-transitive reciprocal preference relations (see Table 3.16), 0.90 in the case of min–transitive reciprocal preference relations (see Table 3.17) and 0.75 in the case of max-transitive reciprocal preference relations (see Table 3.19).

Attending to the results under the probabilistic approach, the conjecture in Llamazares and Pérez-Asurmendi [89] seems to be true. In Table 3.20, we provide some other examples that support such idea. For instance, look at the case of $m = 100$. The probability of triple-acyclic strict preference relations equals 1 for $k = \lfloor (2 \cdot 100)/3 \rfloor /2 = 33$.

Table 3.20: Probabilities of triple-acyclic $P_k$ in the case of max-transitive reciprocal preference relations with $k = \big[\lfloor 2m/3 \rfloor /2, m\big)$.

| $m$ | 3 | 4 | 5 | 6 | 7 | 8 | 48 | 49 | 50 | 99 | 100 | 101 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | 1 | 1 | 1.5 | 2 | 2 | 2.5 | 16 | 16 | 16.5 | 33 | 33 | 33.5 |
| Probability | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

We conclude with a comment about the comparison among the probabilities of consistent preferences under the considered three specifications of the majorities based on differences. Notice that the method used to calculate these probabilities differs from the cases of the two specifications of majorities based on difference of votes to the case of majorities based on difference in support. In the first cases, the probability is calculated by counting the integer points given by systems of inequalities whereas in the second case the probability is simulated by Montecarlo techniques. Accordingly, we can not make any quantitative comparison between the results obtained in the first two cases and in the third one.

# Chapter 4

# Linguistic majorities based on differences

[This chapter has been accepted for publication (jointly with Francisco Chiclana) in the journal *Applied Soft Computing*.]

In social choice voting, majorities based on difference of votes and their extension, majorities based on difference in support, implement the crisp preference values (votes) and the intensities of preference provided by voters when comparing pairs of alternatives, respectively. The aim of these rules is declaring which alternative is socially preferred and to that, they require the winner alternative to reach a certain positive difference in its social valuation with respect to the one reached by the loser alternative. This paper introduces a new aggregation rule that extends majorities based on difference of votes from the context of crisp preferences to the framework of linguistic preferences. Under linguistic majorities with difference in support, voters express their intensities of preference between pairs of alternatives using linguistic labels and an alternative defeats another one when a specific support, fixed before the election process, is reached. There exist two main representation methodologies of linguistic preferences: the cardinal one based on the use of fuzzy sets, and the ordinal one based on the use of the 2-tuples. Linguistic majorities with difference in support are formalised in both representation settings, and conditions are given to guarantee that fuzzy linguistic majorities and 2-tuple linguistic majorities are mathematically isomorphic. Finally, linguistic majorities based on difference in support are proved to

verify relevant normative properties: anonymity, neutrality, monotonicity, weak Pareto and cancellativeness.

## 4.1 Introduction

Decision making problems deal with the social choice of the best alternative among all the possible alternatives taking into account the views and opinions, i.e. the preferences, of all the individuals of a particular social group (Fishburn [40], Nurmi [100] and Sen [105]). Two approaches are possible to address these problems (Kacprzyk [77] and Kacprzyk *et al.* [79]): a direct approach that derives a social choice from the sole manipulation and processing of the information provided by all the individuals without the intermediate derivation of any kind of collective information using a fusion or aggregation operator, which is characteristic of the indirect approach. Obviously, the type of aggregation rule implemented in the second approach is crucial in designing the corresponding social choice rule, and ultimately in the final social solution to the decision making problem. This paper deals with this specific issue, and it is devoted to the introduction of a new aggregation rule for individual preferences.

A comparison study between different alternative preference elicitation methods is reported in Millet [96], where it was concluded that pairwise comparison methods are more accurate than non-pairwise methods. The main advantage of pairwise comparison methods is that facilitates individuals expressing their preferences because they focus exclusively on two alternatives at a time. Given two alternatives, an individual either prefers one to the other or is indifferent between them, which can be represented using a preference relation whose elements represent the preference of one alternative over another one. There exist two main mathematical models to represent pairwise comparison of alternatives based on the concept of preference relation (Fishburn [40] and Roubens and Vincke [101]): in the first one, a preference relation is defined for each one of the above three possible preference states, which is usually referred to as a preference structure on the set of alternatives; the second one integrates the three possible preference states into a single preference relation. This paper deals with the second type of relations, for which reciprocity of preferences is usually assumed in order to guarantee the following basic rationality properties in making paired comparisons (Saaty [103]): indifference between any alternative and itself, and asymmetry of preferences, i.e. if an individual prefers alternative $x$ to $y$, that individual does not simultaneously prefer $y$ to $x$.

In classical voting systems the set of numerical values $\{1, 0.5, 0\}$, or its equivalent $\{1, 0, -1\}$ (Fishburn [40]), is used to represent when the first alternative is preferred to the second alternative, when both alternatives are considered equally preferred (indifference), and when the second alternative is preferred to the first one, respectively. This classical preference modelling constitutes the simplest numeric discrimination model of preferences, and it proves insufficient in many decision making situations as the following example illustrates: Let $x, y, z$ be three alternatives of which we know that one individual prefers $x$ to $y$ and $y$ to $z$, and another individual prefers $z$ to $y$ and $y$ to $x$; then using the above numerical values it may be difficult or impossible to decide which alternative is the best. As Fishburn points out in [40], if alternative $y$ is closer to the best alternative than to the worst one for both individuals then it might seem appropriate to 'elect' it as the social choice, whilst if it is closer to the worst than to the best, then it might be excluded from the choice set. Thus, in many cases it might be necessary the implementation of some kind of 'intensity of preference' between alternatives.

The concept of fuzzy set, which extends the classical concept of set, when applied to a classical relation leads to the concept of a fuzzy relation, which in turn allows the implementation of intensity of preferences (Zadeh [122]). In Bezdek *et al.* [6], we can find for the first time the fuzzy interpretation of intensity of preferences via the concept of a reciprocal fuzzy preference relation, which was later reinterpreted by Nurmi in [99]. In this approach, the numeric scale to evaluate intensity of preferences is the whole unit interval [0, 1] instead of $\{1, 0.5, 0\}$, which it is argued though to assume unlimited computational abilities and resources from the individuals (Chiclana *et al.* [15]).

Subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications, and individuals usually find difficult to evaluate their preference using exact numbers. Individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences (Zadeh [125]). Furthermore, humans exhibit a remarkable capability to manipulate perceptions and other characteristics of physical and mental objects, without any exact numerical measurements and complex computations (Chen and Hwan [13], Fodor and Roubens [44], Kacprzyk and Fedrizzi [78], Lu *et al.* [91] and Zadeh [126]). Therefore, in this paper, the individuals' preferences between pair of alternatives will be assumed to be given in the form of linguistic labels.

It was mentioned before that the type of aggregation rule implemented is crucial in designing the corresponding social choice rule. This paper focuses on the majority voting rules, which are very easy to understand by voters and therefore, when comparing two alternatives, they are seen as very attractive and appropriate to aggregate individual preferences into a collective one. *Simple majority rule* (May [93]) stands out among the different majority rules. Under this rule, an alternative defeats another one when the number of votes cast for the first one exceeds the number of votes cast for the second one. In fact, the requirement to declare indifference between two alternatives is quite strong given that both alternatives have to receive exactly the same number of favourable votes. Furthermore, under the simple majority rule, the support required for an alternative to be the winner is minimum because it is only required to exceed the defeated alternative in just one vote. Being the most decisive aggregation rule turns out to become a drawback because the collective decision is very unstable, i.e. it could be reverted with the change of just one vote. In an attempt to overcome this shortcoming, tougher requirements for declaring an alternative as the winner have been defined and studied. Among these rules, it is worth mentioning the following: *unanimous majority*, *absolute majority* and *qualified majorities* (Fishburn [40], Ferejohn and Grether [38], Saari [102]).

*Majorities based on difference of votes* ($M_k$) (García-Lapresta and Llamazares [49], Llamazares [85] and Houy [74]) constitute another general approach to majority voting rules. These majorities allow to calibrate the amount of support required for the winner alternative by means of a difference of votes fixed before the election process. At the extreme cases, i.e. no difference and maximum difference of votes, majorities based on difference of votes become the simple majority and unanimous majority, respectively. Moreover, if indifference is ruled out from individual preferences, they are equivalent to qualified majorities. With these rules, indifference between two alternatives is possible to be declared for more cases than under the simple majority rule. In fact, the indifference state could be enlarged as much as desired. The application of the majorities based on difference of votes to the case of [0,1]–valued reciprocal fuzzy preference relations is known as the *majorities based on difference in support* ($\widetilde{M}_k$) (García-Lapresta and Llamazares [50]).

The aim of this paper is to fill the gap between majorities based on dif-

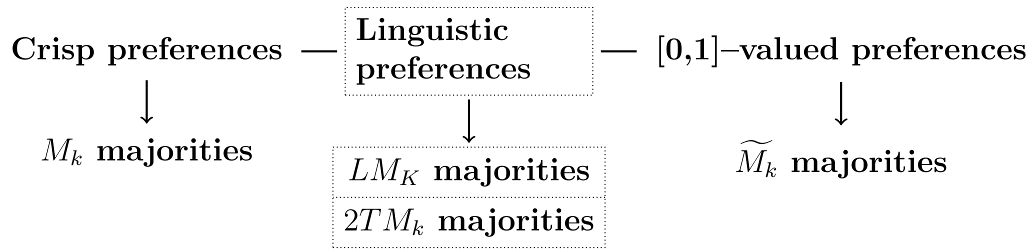


Figure 4.1: Preferences and majorities based on differences.

ference of votes and majorities based on difference in support by providing new majority rules based on difference of support in the linguistic framework. Linguistic majorities with difference in support keep the essence of the former rules in the sense that for an alternative to be declared winner a specific support fixed before the election is to be achieved. The challenge here is to formally generalise the rules to the case of being the preferences linguistic rather than numeric in nature. An additional challenge here is to relate the linguistic majorities with difference in support that can be obtained when the main two approaches to model and represent linguistic information are applied. On the one hand, linguistic preferences can be modelled using a cardinal approach by means of fuzzy sets and their associated membership functions (Zadeh [122]). On the other hand, an ordinal approach can be used to model and manage linguistic preferences using the 2–tuple symbolic representation (Herrera and Martínez [72]). Therefore, two new and different linguistic majorities with difference in support will be introduced: the *linguistic fuzzy majorities* ($LM_K$) and the 2*–tuple linguistic majorities* ($2TM_k$). Figure 4.1 illustrates the new linguistic majorities in relation with the corresponding ones developed for numerical preferences.

The remainder of the paper is structured as follows: The next section introduces concepts essential to the understanding of the rest of the paper. Following that, Section 4.3 introduces the concept of linguistic majorities with difference in support and their mathematical formulation for the main two approaches to model and represent linguistic information: fuzzy set representation (Subsection 4.3.1) and the 2–tuple symbolic representation (Subsection 4.3.2). Section 4.4 proves that both linguistic majorities are mathematically isomorphic when fuzzy sets are defuzzified into their centroid. In

Section 4.5, linguistic majorities based on difference in support are proved to verify the following relevant normative properties: anonymity, neutrality, monotonicity, weak Pareto and cancellativeness. Finally, in Section 4.6 conclusions are drawn and suggestions made for further work.

## 4.2 Preliminaries

Consider $m$ voters provide their preferences on pairs of alternatives of a set $X = \{x_1, \ldots, x_n\}$. The preferences of each voter can be represented using a matrix, $R^p = \left(r_{ij}^p\right)$, where $r_{ij}^p$ stands for the degree or intensity of preference of alternative $x_i$ over $x_j$ for voter $p$. The elements of $R^p$ can be numerical values or linguistic labels. In the following we focus on the former ones, leaving for Subsection 4.2.3 the second ones.

### 4.2.1 Numeric preferences

There are two main types of numeric preference relations: crisp preference relations and [0,1]–valued preference relations; with the second one being an extension of the first one, i.e. [0,1]–valued preference relations have crisp relations as a particular case.

1. A crisp preference relation is characterised for having elements $r_{ij}^p$ that belong to the discrete set of values $\{0, 0.5, 1\}$. In this context, when alternatives are pairwise compared, voters declare only their preference for one of the alternatives or their indifference between the two alternatives. Thus, if $r_{ij}^p = 1$ then voter $p$ prefers alternative $x_i$ to alternative $x_j$, while if $r_{ij}^p = 0.5$ the voter $p$ is indifferent between both alternatives. Moreover, it is always assumed that when $r_{ij}^p = 0.5$ it is also $r_{ji}^p = 0.5$; and when $r_{ij}^p = 1$ then $r_{ji}^p = 0$. This reciprocity property of preferences guarantees that preferences are represented by weak orderings, i.e. the asymmetric property is verified and 'inconsistent' situations where a voter could prefer two alternatives at the same time are avoided. Formally, a binary preference relation represented by $\succ_p$ is asymmetric if given two alternatives $x_i$ and $x_j$, $x_i \succ_p x_j$ implies that $x_j \not\succ_p x_i$ (Fishburn [41]).

2. The [0,1]–valued preference relation extends the crisp preference relation given that its elements $r_{ij}^p$ can take any value from the unit interval $[0,1]$, with the following interpretation: $r_{ij}^p > 0.5$ indicates that the individual $p$ prefers the alternative $x_i$ to the alternative $x_j$, with $r_{ij}^p = 1$ being the maximum degree of preference for $x_i$ over $x_j$; $r_{ij}^p = 0.5$ represents indifference between $x_i$ and $x_j$ for voter $p$. As in the previous case, the reciprocity property of preferences, $r_{ij}^p + r_{ji}^p = 1$, is usually assumed as an extension of the crisp asymmetry property described above. This type of preference relations will be referred to as reciprocal preference relations in this paper. In probabilistic choice theory, reciprocal preference relations are referred to as probabilistic binary preference relations. In fuzzy sets theory, reciprocal preference relations when used to represent intensities of preferences have usually been referred to as reciprocal fuzzy preference relations (Luce and Suppes [92]). Reciprocal preference relations can be seen as a particular case of (weakly) complete fuzzy preference relations, i.e. fuzzy preference relations satisfying $r_{ij} + r_{ji} \geq 1 \ \forall i, j$ (Fodor and Roubens [44]).

### 4.2.2 Majorities based on differences

In an attempt to overcome the support problems commonly attached to the simple majority rule in decision making contexts with crisp preferences, García-Lapresta and Llamazares [49] formalise the concept of majorities based on difference of votes or $M_k$ *majorities*, which was later axiomatically characterised in Llamazares [85] and Houy [74].

**Definition 4.1** ($M_k$ majorities)**.** Given $k \in \{0, \ldots, m-1\}$, and a profile of individual crisp preferences $(R^1, \ldots, R^m)$ on a set of alternatives $X = \{x_1, \ldots, x_n\}$, the $M_k$ *majority* is a mapping from $X \times X$ to $\{1, 0.5, 0\}$, with the following expression:

$$M_k(x_i, x_j) = \begin{cases} 1 & \text{if } m_{ij} > m_{ji} + k; \\ 0 & \text{if } m_{ji} > m_{ij} + k; \\ 0.5 & \text{otherwise;} \end{cases}$$

where $m_{ij}$ is the number of votes cast by the individuals for the alternative $x_i$ when compared with alternative $x_j$ and $m_{ji}$ is the number of votes cast for the alternative $x_j$.

Thus, under $M_k$ majorities, given a difference of votes $k$, an alternative, $x_i$, defeats another alternative, $x_j$, by $k$ votes ($M_k(x_i, x_j) = 1$) when the difference between the votes cast for the alternative $x_i$ and the votes cast for the alternative $x_j$ is greater than $k$. Compared with the simple majority rule, the main change introduced by the majorities based on difference of votes affects the indifference state. The indifference of preference between two alternatives happens when the difference between the votes cast for both alternatives in absolute value is lower than or equal to $k$, i.e. when the difference of votes belongs to $\{0, 1, \ldots, k\}$.

**Example 4.1.** Consider nine voters ($m = 9$) expressing their preferences between two alternatives, $(x_1, x_2)$, using the following profile of crisp preferences $(0, 0, 0, 0, 0, 0, 0.5, 1, 1, 1)$. In this case, we observe that five voters prefer alternative $x_2$ ($m_{21} = 5$), three voters prefer alternative $x_1$ ($m_{12} = 3$), while one voter is indifferent between both alternatives. Under simple majority rule $x_1$ would not be the winner alternative ($m_{12} < m_{21}$) but $x_2$. Simple majority rule corresponds by Definition 4.1 to $M_0$. We clearly see that under $M_k$ majorities $x_1$ is never declared the winner alternative (no matter the value of $k$), i.e. $M_k(x_1, x_2) \neq 1 \; \forall k$. Indeed, for this crisp preference profile we have the following:

$$M_k(x_1, x_2) = \begin{cases} 0 & \text{if } k \in \{0, 1\}; \\ 0.5 & \text{otherwise.} \end{cases}$$

Thus, when the difference of votes is set to be $k \in \{0, 1\}$ we have that $x_2$ collectively defeats $x_1$, otherwise there exists a collective indifference between both alternatives. Unanimous majority rule requires that all voters prefer one alternative to the other one. In our example this is only possible when $(m_{12}, m_{21}) \in \{(9, 0), (0, 9)\}$, which corresponds to the $M_k$ majority rule with $k = 8 (= m - 1)$.

$M_k$ majorities generalise other majority rules as the previous example has shown. Indeed, we have that $M_0$ majority is the simple majority rule, whereas $M_{(m-1)}$ majority is the unanimous majority rule. Moreover, $M_k$ majorities and *qualified majorities*, which are located between absolute majority and unanimity, are equivalent when individual indifference is ruled out from individual preferences (García-Lapresta and Llamazares [49]). These facts are summarised in Figure 4.2.

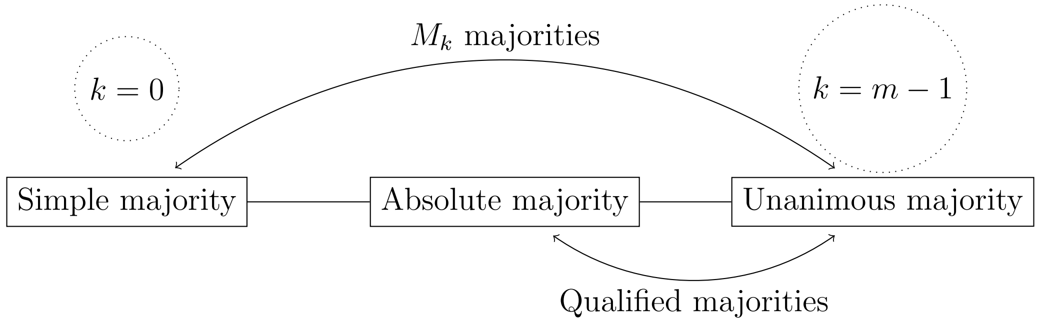


Figure 4.2: $M_k$ majorities *versus* other majorities.

García-Lapresta and Llamazares [50] extend $M_k$ majorities to the framework of $[0, 1]$–valued preferences. Majorities based on difference in support or $\widetilde{M}_k$ *majorities* allow voters to show their preferences between pairs of alternatives through reciprocal preference relations whilst still maintaining the requirement of a higher support to the winner alternative than with the simple majority rule. Under $\widetilde{M}_k$ majorities, an alternative, $x_i$, defeats another one, $x_j$, by a threshold of support $k$, when the sum of the intensities of preference of $x_i$ over $x_j$ for the $m$ voters exceeds the sum of the intensities of preference of $x_j$ over $x_i$ in a quantity greater than $k$.

**Definition 4.2** ($\widetilde{M}_k$ majorities)**.** Given a threshold $k \in [0, m)$ and a profile of individual reciprocal preference relations $(R^1, \ldots, R^m)$, the $\widetilde{M}_k$ *majority* is a mapping from $X \times X$ to $\{1, 0.5, 0\}$, with the following expression:

$$\widetilde{M}_k(x_i, x_j) = \begin{cases} 1 & \text{if } \sum_{p=1}^{m} r_{ij}^p > \sum_{p=1}^{m} r_{ji}^p + k; \\ 0 & \text{if } \sum_{p=1}^{m} r_{ji}^p > \sum_{p=1}^{m} r_{ij}^p + k; \\ 0.5 & \text{otherwise.} \end{cases}$$

With $\widetilde{M}_k$ majorities, indifference between two alternatives happens when the difference in support between the alternatives in absolute value is lower than or equal to $k$, i.e. it is a value in the closed interval $[0, k]$.

A direct consequence of the reciprocity property is that $\widetilde{M}_k$ majorities can be equivalently expressed in terms of the average of individual intensities of preference (García-Lapresta and Llamazares [50]):

$$\widetilde{M}_k(x_i, x_j) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} r_{ij}^p > \frac{m+k}{2m}; \\ 0 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} r_{ij}^p < \frac{m-k}{2m}; \\ 0.5 & \text{otherwise.} \end{cases} \tag{4.1}$$

The term $\frac{1}{m}\sum\limits_{p=1}^{m} r_{ij}^p$ can be interpreted as the collective preference (the average of all the votes) of the first alternative, $x_i$, over the second one, $x_j$. Under the $\widetilde{M}_k$ majorities, the indifference between two alternatives does not necessarily happen when the collective preference, expressed in terms of the arithmetic mean of the individual preferences, equals the value 0.5. $\widetilde{M}_k$ majorities declare indifference when the collective preference belongs to the closed interval $\left[0.5 - \frac{k}{2m}, 0.5 + \frac{k}{2m}\right]$, which we refer to as the indifference interval. When the collective preference is greater than the upper bound of the indifference interval, the first alternative is preferred to the second one. On the other hand, when the collective preference is lower than the lower bound of the indifference interval, the second alternative is preferred to the first one. In comparison with the simple majority rule, the $\widetilde{M}_k$ majorities promote an increase on the cases where the collective indifference is declared, which depends on the threshold of support required to define the strict preference state.

**Example 4.2.** Consider the previous nine voters of Example 4.1 express their preferences between two alternatives, $(x_1, x_2)$, using the following profile of $[0,1]$–valued preferences $(0.1, 0.25, 0.4, 0.4, 0.4, 0.5, 0.9, 0.9, 0.9)$. This profile of $[0,1]$–valued preferences represents the same preference states as that of the profile of crisp preferences of Example 4.1: five voters prefer alternative $x_2$, three voters prefer alternative $x_1$, while one voter is indifferent between both alternatives. The solution to the equation

$$\frac{1}{m}\sum_{p=1}^{m} r_{12}^p = \frac{m+k}{2m} \Leftrightarrow \frac{4.75}{9} = \frac{9+k}{18}$$

is $k = 0.5$, and we have

$$\widetilde{M}_k(x_1, x_2) = \begin{cases} 1 & \text{if } k \in [0, 0.5); \\ 0.5 & \text{otherwise.} \end{cases}$$

Notice that with the implementation of intensities of preference, $\widetilde{M}_k$ majorities produce different collective preference outputs than the ones under $M_k$ majorities in the corresponding crisp preference case. Indeed, in this case alternative $x_2$ never defeats alternative $x_1$ and, on the contrary, alternative $x_1$ can defeat alternative $x_2$ if the difference in support is set to be lower than 0.5. This means that alternative $x_1$ would be the winner under simple majority rule, which is not the case in the corresponding crisp preference case of Example 4.1. It is worth mentioning here that three of the five experts that prefer alternative $x_2$ to alternative $x_1$ are indicating a slightly preference that is close to indifference. In contrast, the three voters preferring alternative $x_1$ over $x_2$ are doing this with an intensity degree close to the maximum. We observe that the implementation of intensities of preference in deriving majority preferences with difference in support allows for a more precise discrimination than if it were not taken into account.

### 4.2.3 Linguistic preferences

As mentioned before, subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications, and individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences (Zadeh [125]). In these cases is still valid the following quotation by Zadeh [126]: 'Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximate characterisation of phenomena which are too complex or too ill-defined to be amenable to description in conventional quantitative terms.'

Let $\mathcal{L} = \{l_0, \ldots, l_s\}$ be a set of linguistic labels ($s \geq 2$), with semantic underlying a ranking relation that can be precisely captured with a linear order, i.e., $l_0 < l_1 < \cdots < l_s$. Table 4.1 provides an example with seven linguistic labels and their corresponding semantic meanings for the comparison of the ordered pair of alternatives $(x_i, x_j)$.

Assuming that the number of labels is odd and the central label $l_{s/2}$ stands for the indifference state when comparing two alternatives, the remaining labels are usually located symmetrically around that central assessment, which guarantees that a kind of reciprocity property holds as in the case of nu-

Table 4.1: Seven linguistic labels and their semantic meanings.

| Linguistic label | Semantic meaning |
|---|---|
| $l_0$ | $x_j$ is absolutely preferred to $x_i$ |
| $l_1$ | $x_j$ is highly preferred to $x_i$ |
| $l_2$ | $x_j$ is slightly preferred to $x_i$ |
| $l_3$ | $x_i$ and $x_j$ are equally preferred |
| $l_4$ | $x_i$ is slightly preferred to $x_j$ |
| $l_5$ | $x_i$ is highly preferred to $x_j$ |
| $l_6$ | $x_i$ is absolutely preferred to $x_j$ |

merical preferences previously discussed. Thus, if the linguistic assessment associated to the pair of alternatives $(x_i, x_j)$ is $l_{ij} = l_h \in \mathcal{L}$, then the linguistic assessment corresponding to the pair of alternatives $(x_j, x_i)$ would be $l_{ji} = l_{s-h}$. Therefore, the operator defined as $N(l_h) = l_g$ with $(g + h) = s$ is a negator operator because $N(N(l_h)) = N(l_g) = l_h$.

The corresponding matrix notation of linguistic individual preferences of voter $p$ is $R^p = (l^p_{ij})$ with $l^p_{ij} \in \mathcal{L}$. A profile of linguistic preferences for the pair of alternatives $(x_i, x_j)$ is the vector of its associated linguistic preferences given by a set of $m$ voters, $(l^1_{ij}, \ldots, l^m_{ij}) \in \mathcal{L}^m$. The main two representation formats of linguistic information are (Herrera *et al.* [70]): the cardinal, which is based on the use of fuzzy set characterised with membership functions and that are mathematically processed using Zadeh's *extension principle* [125]; and the ordinal, which is based on the use of *2-tuples symbolic methodology* (Herrera and Martínez [72]).

**Fuzzy set linguistic representation format**

Convex normal fuzzy subsets of the real line, also known as fuzzy numbers, are commonly used to represent linguistic terms. By doing this, each linguistic assessment is represented using a fuzzy number that is characterised by a membership function, with base variable the unit interval $[0, 1]$, describing its semantic meaning. The membership function maps each value in $[0, 1]$ to a degree of performance representing its compatibility with the linguistic assessment (Zadeh [125]). Figure 4.3 illustrates a fuzzy number with Gaussian membership function.

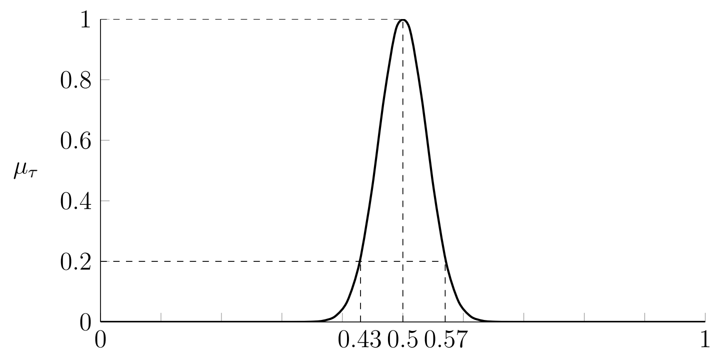


Figure 4.3: Representation of a fuzzy number with Gaussian membership function.

It is worth mentioning that some authors consider trapezoidal fuzzy numbers as the most appropriate to represent linguistic preferences (Delgado *et al.* [29], García-Lapresta *et al.* [51]) because they are more general than triangular and interval fuzzy numbers. Given four real numbers $t_1$, $t_2$, $t_3$, $t_4$, a trapezoidal fuzzy number (TFN) $\tau = (t_1, t_2, t_3, t_4)$ is characterised by the following membership function:

$$\mu_\tau(u) = \begin{cases} 0 & \text{if } u \leq t_1 \text{ or } u \geq t_4; \\ \dfrac{u - t_1}{t_2 - t_1} & \text{if } t_1 < u < t_2; \\ 1 & \text{if } t_2 \leq u \leq t_3; \\ \dfrac{t_4 - u}{t_4 - t_3} & \text{if } t_3 < u < t_4. \end{cases} \tag{4.2}$$

A representation of a set of seven balanced linguistic terms using trapezoidal fuzzy numbers is given in Figure 4.4. Alternative representations are possible. For instance in Herrera *et al.* [71], absolute preference of one alternative over another is represented using crisp values: $l_0 = (0, 0, 0, 0)$ and $l_6 = (1, 1, 1, 1)$.

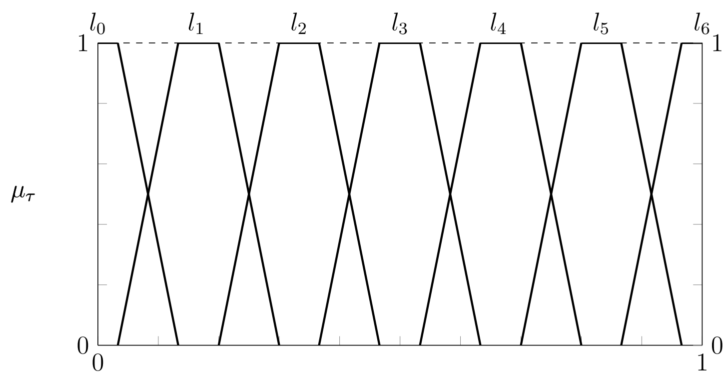


Figure 4.4: Representation of seven balanced linguistic terms with trapezoidal membership functions.

### 2–tuple linguistic representation format

Linguistic assessments can also be represented and aggregated using symbolic representation models based on an ordinal interpretation of the semantic meaning associated to the linguistic labels. Within this framework, the following different approaches have being developed: a linguistic symbolic computational model based on ordinal scales and max-min operators (Yager [120]), a linguistic symbolic computational model based on indexes (Delgado *et al.* [30] and Xu [119]).

In Herrera and Martínez [72], a more general approach was introduced: the 2–tuple linguistic model. This linguistic model takes as a basis the symbolic representation model based on indexes and in addition defines the concept of symbolic translation to represent the linguistic information by means of a pair of values called linguistic 2–tuple, $(l_b, \lambda_b)$, where $l_b \in \mathcal{L}$ is one of the original linguistic terms and $\lambda_b$ is a numeric value representing the symbolic translation. This representation structure allows, on the one hand, to obtain the same information than with the symbolic representation model based on indexes without losing information in the aggregation phase. On the other hand, the result of the aggregation is expressed on the same domain as the one of the initial linguistic labels and therefore, the well-known re-translation problem of the above methods is avoided.

**Definition 4.3** (Linguistic 2–tuple representation)**.** Let $a \in [0, s]$ be the result of a symbolic aggregation of the indexes of a set of labels assessed in a linguistic term set $\mathcal{L} = \{l_0, \ldots, l_s\}$. Let $b = round(a) \in \{0, \ldots, s\}$. The value $\lambda_b = a - b \in [-0.5, 0.5)$ is called a *symbolic translation*, and the pair of values $(l_b, \lambda_b)$ is called the 2*–tuple linguistic representation* of the symbolic aggregation $a$.

The 2–tuple linguistic representation of symbolic aggregation can be mathematically formalised with the following mapping:

$$\begin{aligned} \phi : [0, s] &\longrightarrow \mathcal{L} \times [-0.5, 0.5) \\ \phi(a) &= (l_b, \lambda_b). \end{aligned} \tag{4.3}$$

Based on the linear order of the linguistic term set and the complete ordering of the set $[-0.5, 0.5)$, it is easy to prove that $\phi$ is strictly increasing and continuous and, therefore its inverse function exists:

$$\begin{aligned} \phi^{-1} : \mathcal{L} \times [-0.5, 0.5) &\longrightarrow [0, s] \\ \phi^{-1}(l_b, \lambda_b) = b + \lambda_b &= a. \end{aligned} \tag{4.4}$$

The following negation operator is defined: $N(\phi(a)) = \phi(s - a)$. Figure 4.5 illustrates the application of the 2–tuple function $\phi$ and its inverse for a linguistic term set of cardinality seven. The value of the symbolic translation is assumed to be 3.7, which means that $round(3.7) = 4$ and therefore it can be represented with the 2–tuple $(l_4, -0, 3)$.

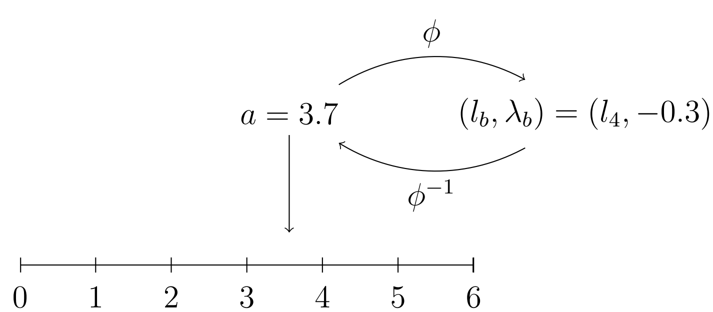


Figure 4.5: Ordinal linguistic representation: symbolic translation and 2–tuples.

## 4.3 Linguistic majorities with difference in support

Before majorities based on difference in support in the context of linguistic preferences are defined, we need to introduce the linguistic decision rule concept. Recall that a profile of linguistic preferences for a pair of alternatives $(x_i, x_j)$ is a vector of its associated linguistic preferences given by a set of $m$ voters $(l_{ij}^1, \ldots, l_{ij}^m) \in \mathcal{L}^m$.

**Definition 4.4.** Given a pair of alternatives $(x_i, x_j) \in X \times X$, a *linguistic decision rule* is a mapping

$$F : \mathcal{L}^m \longrightarrow \{0, 0.5, 1\},$$

such that:

$$F(l_{ij}^1, \ldots, l_{ij}^m) = \begin{cases} 1 & \text{if } x_i \text{ defeats } x_j; \\ 0 & \text{if } x_j \text{ defeats } x_i; \\ 0.5 & \text{if } x_i \text{ and } x_j \text{ tie.} \end{cases}$$

The generalisation of the majorities based on difference of votes from the context of numerical preferences to the context of linguistic preferences involves: (1) the computation of the voters average linguistic assessment for a pair of alternatives, and (2) the evaluation of the difference between two linguistic evaluations. In the following, we will formalise this in both linguistic representation methodologies.

### 4.3.1 Fuzzy linguistic majorities with difference in support

In what follows, $\widetilde{A}_{ij}^p$ denotes the normal and convex fuzzy set representing the linguistic preference of alternative $x_i$ over $x_j$ provided by voter $p$. As mentioned before, the formalisation of the fuzzy linguistic majorities with difference in support requires the computation of the average fuzzy linguistic preference, $\frac{1}{m} \sum_{p=1}^{m} \widetilde{A}_{ij}^p$, of a profile of linguistic preferences $\left(\widetilde{A}_{ij}^1, \ldots, \widetilde{A}_{ij}^m\right)$.

The extension principle allows the domain of a functional mapping to be extended from crisp elements to fuzzy sets as given below (Zadeh [125] and Hanss [69]).

**Definition 4.5** (Extension Principle)**.** Let $X_1 \times X_2 \times \cdots \times X_n$ be a universal product set and $F$ a functional mapping of the form

$$F\colon X_1 \times X_2 \times \cdots \times X_n \longrightarrow Y$$

that maps the element $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$ to the element $y = F(x_1, x_2, \ldots, x_n)$ of the universal set $Y$. Let $A_i$ be a fuzzy set over the universal set $X_i$ with membership function $\mu_{A_i}$ $(i = 1, 2, \ldots, n)$. The membership function $\mu_B$ of the fuzzy set $B = F(A_1, \ldots, A_n)$ over the universal set $Y$ is

$$\mu_B(y) = \begin{cases} \sup\limits_{y=F(x_1,x_2,\ldots,x_n)} [\mu_{A_1}(x_1) * \mu_{A_2}(x_2) * \cdots * \mu_{A_n}(x_n)] & \\ & \text{if } \exists\ x_1, \ldots, x_n :\ y = F(x_1, \ldots, x_n); \\ 0 & \text{otherwise}; \end{cases} \tag{4.5}$$

where $*$ is a t-norm.

For the work presented in this paper, the minimum t-norm ($\wedge$) is used.

In what follows we will first extend the real function $f\colon [0,1] \times [0,1] \longrightarrow [0,1]$,

$$f(u_1, u_2) = u_1 + u_2,$$

to $f(\widetilde{A}_1, \widetilde{A}_2)$ where $\widetilde{A}_1, \widetilde{A}_2$ are fuzzy sets over the set $[0,1]$ with associated membership functions $\mu_{\widetilde{A}_1}$, $\mu_{\widetilde{A}_2}$. The extension principle states that $\widetilde{B} = f(\widetilde{A}_1, \widetilde{A}_2)$ is a fuzzy set over the set $[0,1]$ with membership function $\mu_{\widetilde{B}}\colon [0,1] \to [0,1]$;

$$\mu_{\widetilde{B}}(u) = \sup_{\substack{u_1+u_2=u \\ u_1,u_2\in[0,1]}} \left[\mu_{\widetilde{A}_1}(u_1) \wedge \mu_{\widetilde{A}_2}(u_2)\right].$$

The representation theorem of fuzzy sets (Zadeh [125]) provides an alternative and convenient way to define a fuzzy set via its corresponding family of crisp $\alpha$–level sets. The $\alpha$–level set of a fuzzy set $\widetilde{A}$ over the universe $Z$ is defined as $\widetilde{A}^\alpha = \{z \in Z \mid \mu_{\widetilde{A}}(z) \geq \alpha\}$. The set of crisp sets $\{\widetilde{A}^\alpha \mid 0 < \alpha \leq 1\}$

is said to be a representation of the fuzzy set $\widetilde{A}$. Indeed, the fuzzy set $\widetilde{A}$ can be represented as

$$\widetilde{A} = \bigcup_{0<\alpha\leq 1} \alpha\widetilde{A}^{\alpha},$$

where $\alpha\widetilde{A}^{\alpha}$ is the scalar product of $\alpha$ with the set $\widetilde{A}^{\alpha}$. The membership function of $\widetilde{A}$ is as follows:

$$\mu_{\widetilde{A}}(z) = \sup_{\alpha:\, z\in\widetilde{A}_{\alpha}} \alpha.$$

More details about the representation theorem of fuzzy sets can be found in Zadeh [125].

Let $\widetilde{A}_1^{\alpha}$ and $\widetilde{A}_2^{\alpha}$ be the $\alpha$–level sets of fuzzy sets $\widetilde{A}_1$ and $\widetilde{A}_2$ described above. We have

$$f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right) = \left\{u_1 + u_2 \mid u_1 \in \widetilde{A}_1^{\alpha}, u_2 \in \widetilde{A}_2^{\alpha}\right\}.$$

Both $\widetilde{B}^{\alpha}$ and $f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right)$ are crisp sets. Furthermore, as we prove next, we have the following equality:

$$\widetilde{B}^{\alpha} = f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right). \tag{4.6}$$

I. Let $u \in \widetilde{B}^{\alpha}$. By definition, we have $\mu_{\widetilde{B}}(u) \geq \alpha$ and there exist at least two values $u_1, u_2 \in [0,1]$ such that $u_1 + u_2 = u$ and $\left[\mu_{\widetilde{A}_1}(u_1) \wedge \mu_{\widetilde{A}_2}(u_2)\right] \geq \alpha$. Therefore, it is true that $\mu_{\widetilde{A}_1}(u_1) \geq \alpha$ and $\mu_{\widetilde{A}_2}(u_2) \geq \alpha$, which means that $u_1 \in \widetilde{A}_1^{\alpha}$ and $u_2 \in \widetilde{A}_2^{\alpha}$. Consequently, $u \in f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right)$, i.e. $\widetilde{B}^{\alpha} \subseteq f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right)$.

II. Let $u \in f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right)$. There exist $u_1 \in \widetilde{A}_1^{\alpha}$ and $u_2 \in \widetilde{A}_2^{\alpha}$ such that $u_1 + u_2 = u$. We have that $\mu_{\widetilde{A}_1}(u_1) \geq \alpha$ and $\mu_{\widetilde{A}_2}(u_2) \geq \alpha$ and therefore it is:

$$\sup_{\substack{u_1+u_2=u \\ u_1\in\widetilde{A}_1^{\alpha}, u_2\in\widetilde{A}_2^{\alpha}}} \left[\mu_{\widetilde{A}_1}(u_1) \wedge \mu_{\widetilde{A}_2}(u_2)\right] \geq \alpha.$$

Because $\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha} \subseteq [0,1]$, we have:

$$\sup_{\substack{u_1+u_2=u \\ u_1,u_2\in[0,1]}} \left[\mu_{\widetilde{A}_1}(u_1) \wedge \mu_{\widetilde{A}_2}(u_2)\right] \quad \geq \sup_{\substack{u_1+u_2=u \\ u_1\in\widetilde{A}_1^{\alpha}, u_2\in\widetilde{A}_2^{\alpha}}} \left[\mu_{\widetilde{A}_1}(u_1) \wedge \mu_{\widetilde{A}_2}(u_2)\right].$$

We conclude that $u \in \widetilde{B}^{\alpha}$, i.e. $f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right) \subseteq \widetilde{B}^{\alpha}$.

A similar reasoning will lead us to conclude that the $\alpha$–level set of the average of fuzzy numbers is equal to the average of the $\alpha$–level set of fuzzy sets (Zhou *et al.* [129]). Denoting $f\left(\widetilde{A}_1^{\alpha}, \widetilde{A}_2^{\alpha}\right) = \widetilde{A}_1^{\alpha} + \widetilde{A}_2^{\alpha}$, it is safe to use the following notation

$$\widetilde{B} = \widetilde{A}_1 + \widetilde{A}_2 \Longleftrightarrow \left(\widetilde{A}_1 + \widetilde{A}_2\right)^{\alpha} = \widetilde{A}_1^{\alpha} + \widetilde{A}_2^{\alpha}.$$

The $\alpha$–level sets of fuzzy numbers are closed intervals, and therefore interval arithmetic yields:

$$\left(\widetilde{A}_1 + \widetilde{A}_2\right)^{\alpha} = \widetilde{A}_1^{\alpha} + \widetilde{A}_2^{\alpha} = [u_1^-, u_1^+] + [u_2^-, u_2^+] = [u_1^- + u_2^-, u_1^+ + u_2^+].$$

An example of the addition using the $\alpha$–level sets is shown in Figure 4.6. Given the fuzzy numbers $l_3$ and $l_4$ (Figure 4.4), $l_3 + l_4$ is constructed by applying (4.6) to compute the lower and upper bounds of its $\alpha$–level sets, followed by the application of the representation theorem of fuzzy sets. The computation of the lower bound of the 0.2–level set is given.

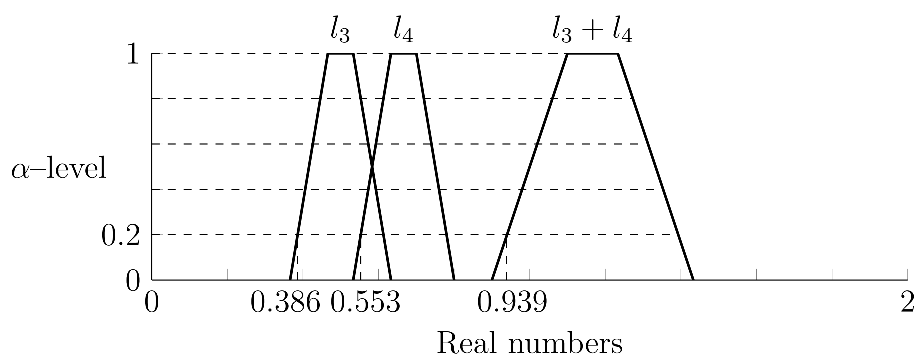


Figure 4.6: $\alpha$–level addition of linguistic terms.

Once we have solved the computation of the average linguistic preference of the profile of linguistic preferences associated to a pair of alternatives, the formalisation of fuzzy linguistic majorities with difference in support requires its classification regarding its containment in one of the intervals corresponding to the social preference or social indifference established by the value of $k$. In other words, we need to find out when the following inequality $\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{ij}^{p} > \frac{m+k}{2m}$ is true or when it is false. Because crisp numbers are particular types of fuzzy numbers, the above inequality involves the comparison of fuzzy numbers. Yager [121] pointed out that this problem has been extensively studied and that there is no unique best approach. Indeed, the set of fuzzy numbers is not totally ordered and therefore it is not possible to achieve a clear social decision in this case. This is clearly illustrated in Figure 4.7, where two different aggregated fuzzy set are displayed, namely $\widetilde{B}_1$ and $\widetilde{B}_2$. Because $\widetilde{B}_1$ completely belongs to one of the intervals of preference there is no doubt about the social decision, which in the illustrated case implies that alternative $x_j$ defeats alternative $x_i$ with a different in support $k$. A similar conclusion would derive if the fuzzy set complete belongs the interval of indifference between both alternatives. On the contrary, the case represented by $\widetilde{B}_2$ is ambiguous given that such set is located in between the interval of preference for $x_i$ and the indifference state. Thus, a different approach is needed if we are to provide a clear cut social choice as per Definition 4.4.

A widely used approach to rank fuzzy numbers consists in converting them into a representative crisp value, and perform the comparison on them, a methodology originally proposed by Zadeh in [124]. This approach has been proposed and used in the selection process of decision making problems under uncertainty where ranking of fuzzy or intuitionistic fuzzy sets is a must to arrive at a decision (Yager [121]). Recently, a study by Brunelli and Mezei [7] that compares different ranking methods for fuzzy numbers concludes that 'it is impossible to give a final answer to the question on what ranking method is the best. Most of the time choosing a method rather than another is a matter of preference or is context dependent.' Two defuzzification methods widely used in fuzzy set theory are the centre of area method (COA) and the mean of maximum method (MOM). The first one computes the centre of mass of the membership function of the fuzzy set (the centroid), whereas the second one computes the mid-point of the 1–level set of the fuzzy set. The COA method maintains the underlying semantic ranking relation within the set of linguistic labels as discussed before, i.e. given two linguistic labels $l_i, l_j \in \mathcal{L}$ such that $l_i < l_j$ then $\boldsymbol{u}_{COA}(l_i) < \boldsymbol{u}_{COA}(l_j)$, and

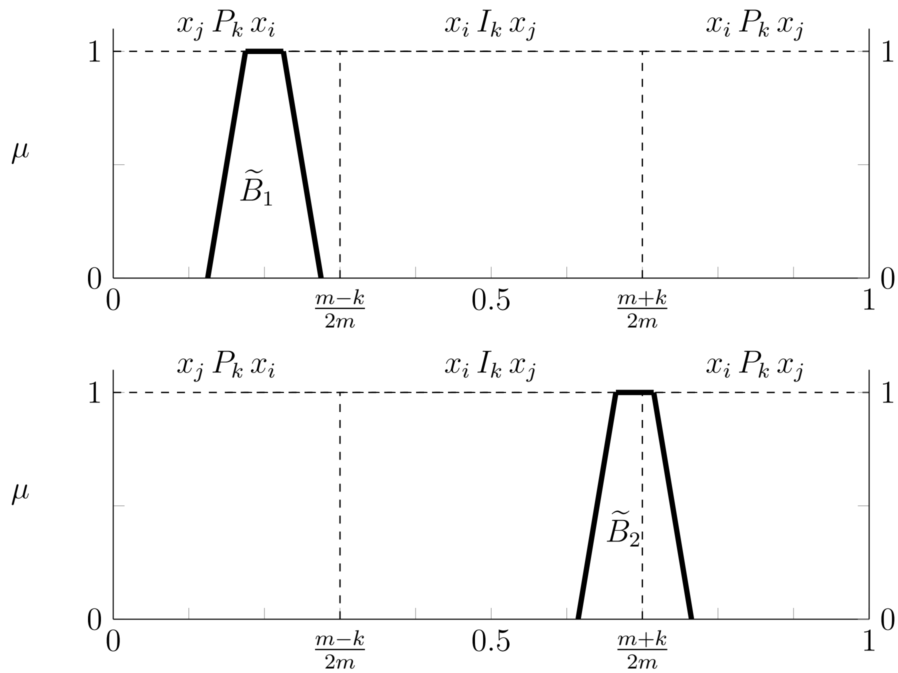


Figure 4.7: Comparison between aggregated fuzzy sets and preference–indifference states.

therefore its use is proposed in the approach presented here. It is worth mentioning Brunelli and Mezei's correlation study, and their centrality analysis associated to the corresponding correlation network representation, which shows the centre of area method as one of the highest central defuzzification methods. Furthermore, for a trapezoidal fuzzy number $\widetilde{A}$ with membership function (4.2), we have $\boldsymbol{u}_{COA}(\widetilde{A}) = (t_1 + t_4)/2$ and, under the assumed property of internal symmetry of the linguistic labels, it is $\boldsymbol{u}_{COA} \equiv \boldsymbol{u}_{MOM}$. From now on, we refer to the centre of area value of fuzzy number $\widetilde{A}$ simply as $\boldsymbol{u}(\widetilde{A})$. Given two trapezoidal fuzzy numbers, namely $\widetilde{A}_1$ and $\widetilde{A}_2$, it holds that $\boldsymbol{u}(\widetilde{A}_1 + \widetilde{A}_2) = \boldsymbol{u}(\widetilde{A}_1) + \boldsymbol{u}(\widetilde{A}_2)$. Hence, $\boldsymbol{u}$ is an additive function.

The range of function $\boldsymbol{u}$ is $[\boldsymbol{u}(l_0),\, \boldsymbol{u}(l_s)]$, while the range of $(m + k)/2m$ is $[0,\, 1]$. Thus, to carry out a fair comparison in the formalisation of the linguistic majority with difference in support, the following function $\boldsymbol{v}$ with range [0, 1] is used:

$$\boldsymbol{v}(\widetilde{A}) = \frac{\boldsymbol{u}(\widetilde{A}) \; - \; \boldsymbol{u}(l_0)}{\boldsymbol{u}(l_s) \; - \; \boldsymbol{u}(l_0)}.$$

Below, we formally define linguistic majorities with difference in support represented by fuzzy sets. Under these rules, an alternative, $x_i$, defeats another one, $x_j$, by a threshold of support $K$, if the defuzzified value of the average fuzzy set of the voters' linguistic valuations between $x_i$ and $x_j$ exceeds the value 0.5 in a quantity that depends on the threshold $K$, fixed before the election process.

**Definition 4.6** ($LM_K$ majorities with difference in support)**.** Given a set of alternatives $X$ and a profile of individual reciprocal fuzzy linguistic preference relations $R(X) = (R^1, \ldots, R^m)$, the $LM_K$ *majorities* with difference in support are the following linguistic decision rules:

$$LM_K(\widetilde{A}^1_{ij}, \, \ldots, \, \widetilde{A}^m_{ij}) = \begin{cases} 1 & \text{if } \boldsymbol{v}\left(\frac{1}{m}\sum\limits_{p=1}^{m} \widetilde{A}^p_{ij}\right) > \frac{m+K}{2m}; \\ 0 & \text{if } \boldsymbol{v}\left(\frac{1}{m}\sum\limits_{p=1}^{m} \widetilde{A}^p_{ij}\right) < \frac{m-K}{2m}; \\ 0.5 & \text{otherwise}; \end{cases} \tag{4.7}$$

where $\boldsymbol{v}\left(\frac{1}{m}\sum\limits_{p=1}^{m} \widetilde{A}^p_{ij}\right)$ is the defuzzified value of the fuzzy average linguistic preference of the profile of fuzzy linguistic preferences of the pair of alternatives $(x_i, x_j)$; and $K \in [0, \, m)$ represents the threshold of support required for an alternative to be the social winner.

In the following result we prove that function $\boldsymbol{v}$ is additive.

**Proposition 4.1.** *Function* $\boldsymbol{v}$ *verifies*

$$\boldsymbol{v}\left(\frac{1}{m}\sum_{p=1}^{m} \widetilde{A}^p_{ij}\right) \; = \; \frac{1}{m}\sum_{p=1}^{m} \boldsymbol{v}(\widetilde{A}^p_{ij}).$$

*Proof.* Because $\boldsymbol{u}$ is additive we have that

$$\boldsymbol{u}\left(\frac{1}{m}\sum_{p=1}^{m} \widetilde{A}^p_{ij}\right) \; = \; \frac{1}{m}\sum_{p=1}^{m} \boldsymbol{u}(\widetilde{A}^p_{ij}).$$

Also, we have that $\boldsymbol{u}$ and $\boldsymbol{v}$ are related in the form $\boldsymbol{u} = c \cdot \boldsymbol{v} + d$ where $c = \boldsymbol{u}(l_s) - \boldsymbol{u}(l_0)$ and $d = \boldsymbol{u}(l_0)$, it is:

$$\boldsymbol{u}\left(\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{ij}^{p}\right) = c \cdot \boldsymbol{v}\left(\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{ij}^{p}\right) + d,$$

$$\boldsymbol{u}\left(\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{ij}^{p}\right) = \frac{1}{m}\sum_{p=1}^{m}\boldsymbol{u}(\widetilde{A}_{ij}^{p}) = \frac{1}{m}\sum_{p=1}^{m}\left[c \cdot \boldsymbol{v}(\widetilde{A}_{ij}^{p}) + d\right] =$$

$$= c \cdot \frac{1}{m}\sum_{p=1}^{m}\boldsymbol{v}(\widetilde{A}_{ij}^{p}) + d.$$

Thus, we have:

$$\boldsymbol{v}\left(\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{ij}^{p}\right) = \frac{1}{m}\sum_{p=1}^{m}\boldsymbol{v}(\widetilde{A}_{ij}^{p}),$$

i.e. $\boldsymbol{v}$ is additive. □

Expression (4.7) can be rewritten as follows:

$$LM_K(\widetilde{A}_{ij}^{1}, \ldots, \widetilde{A}_{ij}^{m}) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m}\boldsymbol{v}(\widetilde{A}_{ij}^{p}) > \frac{m+K}{2m}; \\ 0 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m}\boldsymbol{v}(\widetilde{A}_{ij}^{p}) < \frac{m-K}{2m}; \\ 0.5 & \text{otherwise}; \end{cases} \tag{4.8}$$

where $K \in [0, m)$ and $\frac{1}{m}\sum\limits_{p=1}^{m}\boldsymbol{v}(\widetilde{A}_{ij}^{p})$ is the average of the defuzzified values associated with the profile of fuzzy linguistic preferences of the pair of alternatives $(x_i, x_j)$ as per the assessment of each individual voter.

It was mentioned before that the linguistic labels are located symmetrically around the central label $l_{s/2}$ and therefore it is also appropriate to assume that the membership functions defining the fuzzy linguistic labels result in centroids symmetrically distributed with respect to the centroid of the central label. Because the central fuzzy linguistic label $l_{s/2}$ stands for the indifference state when comparing two alternatives, it is also appropriate to have 0.5 as its centroid. Thus, denoting the centroid of fuzzy linguistic label

$l_h$ by $\boldsymbol{u}(l_h)$ we have that $\boldsymbol{u}(l_0) + \boldsymbol{u}(l_s) = 1$ and

$$\boldsymbol{u}(l_h) = \boldsymbol{u}(l_0) + h \cdot \frac{1 - 2 \cdot \boldsymbol{u}(l_0)}{s},$$

which guarantees an evenly distribution of the centroids of the fuzzy linguistic labels around the value 0.5. The application of function $\boldsymbol{v}$ results in

$$\boldsymbol{v}(l_h) = \frac{h}{s}. \tag{4.9}$$

In the following, we provide an example to illustrate the application of the $LM_K$ majorities with difference in support.

**Example 4.3.** Consider nine voters expressing their preferences between two alternatives, $(x_1, x_2)$, using the linguistic labels of Table 4.1 represented by a set of trapezoidal fuzzy numbers as illustrated in Figure 4.4 and verifying (4.9). Assume the voters provide the fuzzy linguistic profile of Table 4.2.

Table 4.2: Trapezoidal fuzzy numbers and centroids.

| Linguistic label | $\boldsymbol{v}(l_h)$ |
|---|---|
| $l_0$ | 0 |
| $l_1$ | 1/6 |
| $l_2$ | 1/3 |
| $l_3$ | 0.5 |
| $l_4$ | 2/3 |
| $l_5$ | 5/6 |
| $l_6$ | 1 |

In Table 4.3 two different $LM_K$ majorities with difference in support are computed: the simple linguistic majority $LM_0$, and $LM_3$.

Table 4.3: Aggregation and results for two different $LM_K$ majorities.

| $K$ | $\boldsymbol{v}\left(\frac{1}{m}\sum_{p=1}^{m}\widetilde{A}_{12}^{p}\right)=\frac{1}{m}\sum_{p=1}^{m}\boldsymbol{v}\left(\widetilde{A}_{12}^{p}\right)$ | $\frac{m+K}{2m}$ | $LM_K$ |
|---|---|---|---|
| 0 | $\boldsymbol{v}\left(\frac{l_0+l_1+l_2+l_2+l_2+l_3+l_6+l_6+l_6}{9}\right)=14/27$ | 0.5 | 1 |
| 3 | | 2/3 | 0.5 |

In the first case, it is enough to have an average centroid of the fuzzy linguistic profile greater than the centroid (0.5) of the central fuzzy linguistic assessment ($l_3$) to declare $x_1$ the social winner alternative. In the second case, the threshold required implies that the average centroid of the fuzzy linguistic profile is to be greater than the centroid of the fuzzy linguistic label $l_4$ for $x_1$ to be declared the social winner alternative. In the first case, we have that $x_1$ can indeed be declared the social $LM_0$ winner alternative, whilst there is social $LM_3$ indifference in the second case.

The $LM_K$ majorities with difference in support for this particular fuzzy linguistic profile have an expression similar to the one achieved for the case of $[0,1]$–valued preferences, being as follows:

$$LM_K(l_0,l_1,l_2,l_2,l_2,l_3,l_6,l_6,l_6)=\begin{cases}1 & \text{if } K\in[0,1/3);\\ 0.5 & \text{otherwise.}\end{cases}$$

The following observations are worth highlighting:

(i) $LM_K$ majorities with difference in support generalise the *simple linguistic majority* (García-Lapresta [47]). Indeed, $LM_0$ majority coincides with the simple majority based on linguistic labels. In this case, no difference of support between the alternatives is required.

(ii) *Linguistic unanimity* holds when all the voters involved in the election prefer the same alternative, even when their intensities of preference could differ from one to another. The following three linguistic profiles with nine voters and a set of seven linguistic terms will serve to illustrate this concept:

$$(l_0, l_0, l_0, l_1, l_1, l_1, l_2, l_2, l_2);$$
$$(l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6);$$
$$(l_0, l_0, l_0, l_0, l_0, l_0, l_0, l_0, l_3).$$

The first two profiles fulfill linguistic unanimity: in the first one all nine voters express a preference for the second alternative, whilst in the second one the first alternative is preferred by all nine voters. However, in the third profile there is no unanimity of preferences because voter 9 expresses indifference between both alternatives and therefore differs from the rest of voters, who strongly prefer the second alternative.

Given a profile of fuzzy linguistic preferences $(\widetilde{A}^1_{ij}, \ldots, \widetilde{A}^m_{ij})$, linguistic unanimity happens if $\boldsymbol{v}(\widetilde{A}^p_{ij}) \leq \boldsymbol{v}(l_{(s/2)-1}) \forall p$, or $\boldsymbol{v}(\widetilde{A}^p_{ij}) \geq \boldsymbol{v}(l_{(s/2)+1})\ \forall p$. In the first case, all voters prefer the second alternative over the first one, whilst the first alternative is preferred over the second one in the second case. Algebraic manipulation leads us to the following threshold values: $K < m - 2m \cdot \boldsymbol{v}(l_{(s/2)-1})$ for the social preference of the second alternative, and $K < 2m \cdot \boldsymbol{v}(l_{(s/2)+1}) - m$ for the social preference of the first alternative.

Because we are assuming that the linguistic labels are symmetrical and balanced around the central one, then if the fuzzy sets used to represent them are all of the same type and uniformly distributed in the domain $[0, 1]$, the normalised centroid function $\boldsymbol{v}$ would be $\boldsymbol{v}(l_h) = h/s\ \forall h$, and therefore the threshold value to assure linguistic unanimity would be $K < 2m/s$.

### 4.3.2 2–tuple linguistic majorities with difference in support

In order to extend the $M_k$ majorities to the framework of the 2–tuple, the addition as well as a rule to compare 2–tuples are needed.

**Definition 4.7** (2–tuple Addition (Herrera and Martinez [72])). The addition of 2–tuples, $\phi(a_1) = (l_{b_1}, \lambda_{b_1})$ and $\phi(a_2) = (l_{b_2}, \lambda_{b_2})$, with $b_1 = round(a_1)$, $b_2 = round(a_2)$, $\lambda_{b_1} = a_1 - b_1$ and $\lambda_{b_2} = a_2 - b_2$, is computed as follows:

$$\phi(a_1) + \phi(a_2) = (l_{b_{12}},\ \lambda_{b_{12}}),$$

with $b_{12} = round(a_1 + a_2)$, and $\lambda_{b_{12}} = (a_1 + a_2) - b_{12}$.

**Definition 4.8** (2–tuple Lexicographic Ordering (Herrera and Martinez [72]))**.** Given $\phi(a_1) = (l_{b_1}, \lambda_{b_1})$ and $\phi(a_2) = (l_{b_2}, \lambda_{b_2})$, we have that:

1. If $b_1$ is greater than $b_2$, then $\phi(a_1) > \phi(a_2)$.

2. If $b_1$ is equal to $b_2$ and $\lambda_{b_1}$ is greater than $\lambda_{b_1}$, then $\phi(a_1) > \phi(a_2)$.

3. If $b_1$ is equal to $b_2$ and $\lambda_{b_1}$ is equal to $\lambda_{b_1}$, then $\phi(a_1) = \phi(a_2)$.

Below, we formally define 2–tuple linguistic majorities with difference in support. Under these rules, an alternative, $x_i$, defeats another one, $x_j$, by a threshold of support $k$, if the 2–tuple linguistic representation of the average symbolic aggregation of the linguistic preferences of $x_i$ over $x_j$ exceeds the 2–tuple linguistic representation associated to the indifference state in a value that depends on the threshold $k$, fixed before the election process.

**Definition 4.9** ($2TM_k$ majorities with difference in support)**.** Given a set of alternatives $X$ and a profile of individual reciprocal 2–tuple linguistic preference relations $(R^1, \ldots, R^m)$, the $2TM_k$ *majority* with difference in support is the following linguistic decision rule:

$$2TM_k(a_{ij}^1, \ldots, a_{ij}^m) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} \phi\left(a_{ij}^p\right) > \phi\left(\frac{s \cdot m + k}{2m}\right); \\ 0 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} \phi\left(a_{ij}^p\right) < \phi\left(\frac{s \cdot m - k}{2m}\right); \\ 0.5 & \text{otherwise;} \end{cases} \tag{4.10}$$

where $\frac{1}{m}\sum\limits_{p=1}^{m} \phi\left(a_{ij}^p\right)$ is the average of the 2–tuple representation of the linguistic preferences provided by the voters for the pair of alternatives $(x_i, x_j)$, $\phi$ is the 2–tuple symbolic aggregation mapping (4.3); and $k \in [0,\ m \cdot s)$ represents the threshold of support required for an alternative to be the social winner.

In the context of the 2–tuple linguistic representation, the linguistic label $l_h$ is associated a valuation that coincides with its ordering position within $\mathcal{L}$, i.e. $h$, and therefore the maximum social preference value a set of voters can assign to an alternative when compared against another one is $m \cdot s$, which corresponds to the linguistic profile $(l_s, \ldots, l_s)$. This explains why $[0,\, m \cdot s)$ is the range of values for parameter $k$.

Given that in the ordinal representation of linguistic information the addition of linguistic labels is defined as $l_{a_1} + l_{a_2} = l_{a_1+a_2}$ [119], it is obvious that function $\phi$ is additive. Therefore expression (4.10) can be rewritten as follows:

$$2TM_k(a_{ij}^1, \ldots, a_{ij}^m) = \begin{cases} 1 & \text{if } \phi\left(\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p\right) > \phi\left(\frac{s \cdot m + k}{2m}\right); \\ 0 & \text{if } \phi\left(\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p\right) < \phi\left(\frac{s \cdot m - k}{2m}\right); \\ 0.5 & \text{otherwise;} \end{cases} \tag{4.11}$$

where $\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p$ is the symbolic aggregation, specifically the arithmetic mean, of the linguistic preferences provided by the voters for the pair of alternatives $(x_i, x_j)$.

The following example illustrates the use of the $2TM_k$ majorities with difference in support.

**Example 4.4** (Example 4.3 continuation)**.** Table 4.4 presents the results for two different $2TM_k$ majorities: $2TM_0$ and $2TM_{18}$. In the first one, the alternative $x_1$ is declared the winner when the 2–tuple representation of the symbolic arithmetic mean of the linguistic preferences provided by the voters for the pair of alternatives $(x_1, x_2)$ is greater than the indifference 2–tuple $(l_3,\, 0)$; while it has to be greater than the 2–tuple $(l_4,\, 0)$ in the second case.

The $2TM_k$ majorities with difference in support for this particular linguistic profile is as follows:

$$2TM_k(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6) = \begin{cases} 1 & \text{if } k \in [0, 2); \\ 0.5 & \text{otherwise.} \end{cases}$$

Table 4.4: Aggregation and results for two different $2TM_k$ majorities.

| $k$ | $\frac{1}{m}\sum_{p=1}^{m} a_{12}^p$ | $\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{12}^p\right)$ | $\phi\left(\frac{s\cdot m+k}{2m}\right)$ | $2TM_k$ |
|---|---|---|---|---|
| 0 | (0+1+2+2+2+3+6+6+6)/9 | $(l_3,\, 1/9)$ | $(l_3,\, 0)$ | 1 |
| 18 | = 28/9 | | $(l_4,\, 0)$ | 0.5 |

Examples 4.3 and 4.4 let us hypothesise that $LM_K$ majority and $2TM_k$ majority coincide when the following relationship holds $K = k/s$. This will be proved in the following section.

## 4.4 Equivalence between $LM_K$ and $2TM_k$ majorities with difference in support

So far, we have provided two apparently different extensions of $M_k$ majorities to the framework of the linguistic preferences. In this section, we prove that $LM_K$ and $2TM_k$ are equivalent.

Let $l_h \in \mathcal{L}$ be a linguistic label, $\boldsymbol{u}(l_h)$ its associated centroid following the fuzzy set linguistic representation introduced in Subsection 4.2.3, and $a_h = \phi^{-1}\left((l_h, 0)\right)$ the symbolic representation of the 2–tuple $(l_h, 0)$ as defined in (4.4). Let $\delta$ be the function that maps $a_h$ into $\boldsymbol{v}(l_h)$, i.e.

$$\delta\left(a_h\right) = \boldsymbol{v}(l_h). \tag{4.12}$$

Under these conditions, $\delta$ as defined in (4.12) is the restriction of a continuous and strictly increasing function with domain $[0, s]$:

$$\delta\colon [0, s] \longrightarrow [0, 1]$$

such that $\delta(0) = 0$, $\delta(s/2) = 0.5$ and $\delta(s) = 1$.

**Theorem 4.1** ($LM_K$ and $2TM_k$ Equivalence)**.** *If $\delta$ is additive then $LM_K$ majorities and $2TM_k$ majorities are equivalent.*

*Proof.* The following results is well known: if a continuous function verifies $F(x+y) = F(x) + F(y)\ \forall x, y \in \mathbb{R}$ then there exists a constant $a \in \mathbb{R}$ such that $F(x) = a \cdot x\ \forall x \in \mathbb{R}$ [2]. This result applied to function $\delta$ implies that $\delta(x) = x/s$. Therefore we have:

$$\frac{1}{m}\sum_{p=1}^{m} \boldsymbol{v}(\widetilde{A}_{ij}^{p}) > \frac{m+K}{2m} \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m} \delta\left(a_{ij}^{p}\right) > \frac{m+K}{2m},$$

i.e.

$$\frac{1}{m}\sum_{p=1}^{m} \boldsymbol{v}(\widetilde{A}_{ij}^{p}) > \frac{m+K}{2m} \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m} a_{ij}^{p} > \frac{s \cdot m + s \cdot K}{2m}.$$

We conclude that $LM_K$ majorities is equivalent to $2TM_k$ majorities when $k = s \cdot K$. □

Theorem 4.1 establishes the condition for $LM_K$ majorities and $2TM_k$ majorities to be mathematically isomorphic: $\delta(x) = x/s$. Notice that in Subsection 4.3.1 we proved that $\boldsymbol{v}$ is additive and that under the assumption of symmetric distribution of the linguistic labels with respect to the central assessment it is $\boldsymbol{v}(l_h) = h/s$, which allows us to conclude that $LM_K$ majorities and $2TM_k$ majorities are indeed equivalent when the membership functions of the fuzzy linguistic labels result in an evenly distribute of the centroids of the fuzzy linguistic labels. In the following section we prove a number of normative properties for the $2TM_k$ majorities with difference in support, which obviously apply to the $LM_K$ majorities using the proved equivalence.

## 4.5 Properties of linguistic majorities with difference in support

For convenience, we use expression (4.10) for $2TM_k$ majorities with difference in support:

$$2TM_k(a_{ij}^{1}, \ldots, a_{ij}^{p}) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} \phi\left(a_{ij}^{p}\right) > \phi\left(\frac{s \cdot m + k}{2m}\right); \\ 0 & \text{if } \frac{1}{m}\sum\limits_{p=1}^{m} \phi\left(a_{ij}^{p}\right) < \phi\left(\frac{s \cdot m - k}{2m}\right); \\ 0.5 & \text{otherwise;} \end{cases}$$

where $\frac{1}{m}\sum_{p=1}^{m}\phi\left(a_{ij}^{p}\right)$ is the average of the 2–tuple representation of the linguistic preferences provided by the voters for the pair of alternatives $(x_i, x_j)$, $\phi$ is the 2–tuple symbolic aggregation mapping (4.3); and $k \in [0,\, m \cdot s)$ represents the threshold of support required for an alternative to be the social winner.

The first normative property says that $2TM_k$ majorities fulfil *anonymity*, i.e. the order in which the linguistic valuations of the voters are given is irrelevant for the final social outcome. The proof is omitted because it is a direct consequence of the arithmetic mean being commutative.

**Proposition 4.2** (Anonimity)**.** *Given a profile of linguistic preferences* $(l_{ij}^1, \ldots, l_{ij}^m) \in \mathcal{L}^m$*, the following equality holds*

$$2TM_k(l_{ij}^1, \ldots, l_{ij}^m) = 2TM_k(l_{ij}^{\sigma(1)}, \ldots, l_{ij}^{\sigma(m)}),$$

*for any permutation* $\sigma: \{1, \ldots, m\} \to \{1, \ldots, m\}$.

**Example 4.5.** Given the profile of linguistic preferences $(l_6, l_2, l_6, l_2, l_3, l_1, l_0, l_6, l_2)$, then we have

$$\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p = (6+2+6+2+3+1+0+6+2)/9 = 28/9$$

and

$$\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p\right) = (l_3,\ 1/9)\,.$$

These values coincide with the result obtained in Example 4.4 and therefore it is

$$2TM_k(l_6, l_2, l_6, l_2, l_3, l_1, l_0, l_6, l_2) = 2TM_k(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6).$$

*Neutrality* means that the aggregation rule should treat alternatives equally, which is proved in the following proposition.

**Proposition 4.3** (Neutrality)**.** *Given a profile of linguistic preferences* $(l_{ij}^1, \ldots, l_{ij}^m) \in \mathcal{L}^m$*, the following equality holds*

$$2TM_k(N(l_{ij}^1), \ldots, N(l_{ij}^m)) = 1 -\ 2TM_k(l_{ij}^1, \ldots, l_{ij}^m).$$

*Proof.* We have to prove the following three statements:

1. If $2TM_k(N(a_{ij}^1), \ldots, N(a_{ij}^m)) = 1$, then $2TM_k(a_{ij}^1, \ldots, a_{ij}^m) = 0$.
2. If $2TM_k(N(a_{ij}^1), \ldots, N(a_{ij}^m)) = 0$, then $2TM_k(a_{ij}^1, \ldots, a_{ij}^m) = 1$.
3. If $2TM_k(N(a_{ij}^1), \ldots, N(a_{ij}^m)) = 0.5$, then $2TM_k(a_{ij}^1, \ldots, a_{ij}^m) = 0.5$.

Given a profile of linguistic preferences, $(l_{ij}^1, \ldots, l_{ij}^m)$, expressed in terms of its equivalent symbolic translation, i.e. $(a_{ij}^1, \ldots, a_{ij}^m)$, we have

$$2TM_k(N(a_{ij}^1), \ldots, N(a_{ij}^m)) = 1 \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m} \phi\left(N(a_{ij}^p)\right) \ > \ \phi\left(\frac{s \cdot m + k}{2m}\right).$$

Because $N(a_{ij}^p) = s - a_{ij}^p$ $(\forall\, p)$ then

$$\frac{1}{m}\sum_{p=1}^{m} \phi\left(N(a_{ij}^p)\right) \ > \ \phi\left(\frac{s \cdot m + k}{2m}\right) \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m} \phi(s - a_{ij}^p) \ > \ \phi\left(\frac{s \cdot m + k}{2m}\right).$$

Recall that function $\phi$ is additive, and therefore it is

$$\frac{1}{m}\sum_{p=1}^{m} \phi(s - a_{ij}^p) \ > \phi\left(\frac{s \cdot m + k}{2m}\right) \Leftrightarrow \phi\left(\frac{1}{m}\sum_{p=1}^{m}(s - a_{ij}^p)\right) > \phi\left(\frac{s \cdot m + k}{2m}\right).$$

Therefore we have:

$$2TM_k(N(a_{ij}^1), \ldots, N(a_{ij}^m)) = 1 \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m}(s - a_{ij}^p) \ > \ \frac{s \cdot m + k}{2m}.$$

Algebraic manipulation of the right hand side of this last equivalence yields:

$$\frac{1}{m}\sum_{p=1}^{m}(s - a_{ij}^p) \ > \ \frac{s \cdot m + k}{2m} \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m} a_{ij}^p \ < \ \frac{s}{2} - \frac{k}{2m}.$$

Because $\phi$ is strictly increasing we have

$$\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p \ < \ \frac{s}{2} - \frac{k}{2m} \Leftrightarrow \phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p\right) \ < \ \phi\left(\frac{s}{2} - \frac{k}{2m}\right).$$

Finally, applying again the additivity property of function $\phi$ we conclude that

$$2TM_k(N(a_{ij}^1),\dots,N(a_{ij}^m)) = 1 \Leftrightarrow \frac{1}{m}\sum_{p=1}^{m}\phi(a_{ij}^p) \,<\, \phi\left(\frac{s}{2}-\frac{k}{2m}\right)$$
$$\Leftrightarrow 2TM_k(a_{ij}^1,\dots,a_{ij}^m) = 0.$$

This proves item 1. The proofs of items 2 and 3 are similar. □

**Example 4.6.** Given the profile of linguistic preferences $(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6)$, then we have

$$\big(N(l_0), N(l_1), N(l_2), N(l_2), N(l_2), N(l_3), N(l_6), N(l_6), N(l_6)\big)$$
$$= (l_6, l_5, l_4, l_4, l_4, l_3, l_0, l_0, l_0).$$

Moreover,

$$\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p = (6+5+4+4+4+3+0+0+0)/9 = 26/9$$

and

$$\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p\right) = (l_3,\ -1/9)\,.$$

Notice that $\phi\left(\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p\right) > \phi\left(\frac{s\cdot m+k}{2m}\right)$ never happens, no matter the value of $k$. On the other hand, when $k < 2$ it is $\phi\left(\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p\right) < \phi\left(\frac{s\cdot m-k}{2m}\right)$. Consequently,

$$2TM_k(l_6, l_5, l_4, l_4, l_4, l_3, l_0, l_0, l_0) = \begin{cases} 0 & \text{if } k \in [0,2); \\ 0.5 & \text{otherwise.} \end{cases}$$

Recalling the result of Example 4.4, the following holds:

$$2TM_k(l_6, l_5, l_4, l_4, l_4, l_3, l_0, l_0, l_0) = 1 - 2TM_k(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6).$$

*Monotonicity* is proved next. Under this property, the majority value does not decrease when the individual linguistic preference evaluation of a profile increase.

**Proposition 4.4** (Monotonicity)**.** *Given two profiles of linguistic preferences,* $(l_{ij}^1, \ldots, l_{ij}^m)$ *and* $(l_{ij}'^1, \ldots, l_{ij}'^m)$*, such that it holds that* $l_{ij}^p \geq l_{ij}'^p$ $(\forall p)$ *then:*

$$2TM_k(l_{ij}^1, \ldots, l_{ij}^m) \geq 2TM_k(l_{ij}'^1, \ldots, l_{ij}'^m).$$

*Proof.* Recall that both function $\phi$ and the arithmetic mean are increasing, and therefore denoting $l_{ij}^p \equiv \phi(a_{ij}^p)$ and $l_{ij}'^p \equiv \phi(a_{ij}'^p)$ we have

$$l_{ij}^p \geq l_{ij}'^p \ (\forall p) \Rightarrow \frac{1}{m}\sum_{p=1}^{m} \phi(a_{ij}^p) \geq \frac{1}{m}\sum_{p=1}^{m} \phi(a_{ij}'^p),$$

which proves that

$$2TM_k(l_{ij}^1, \ldots, l_{ij}^m) \geq 2TM_k(l_{ij}'^1, \ldots, l_{ij}'^m). \qquad \square$$

**Example 4.7.** Given the profile of linguistic preferences $(l_3, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6)$, then we have

$$\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p = 31/9$$

and

$$\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p\right) = (l_3,\ 4/9)\,.$$

Because $\phi\left(\frac{1}{m}\sum\limits_{p=1}^{m} a_{ij}^p\right) > \phi\left(\frac{s\cdot m+k}{2m}\right)$ when $k < 8$, it is

$$2TM_k(l_3, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6) = \begin{cases} 1 & \text{if } k \in [0, 8); \\ 0.5 & \text{otherwise.} \end{cases}$$

Consequently,

$$2TM_k(l_3, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6) \geq 2TM_k(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6).$$

The *weak Pareto* property presented below, asserts that the result under the rule has to respect unanimous profiles.

**Proposition 4.5** (Weak Pareto)**.** *The following equalities hold:*

*1.* $2TM_k(l_s, \ldots, l_s) = 1$.

*2.* $2TM_k(l_0, \ldots, l_0) = 0$.

*Proof.*

1. We have

$$\frac{1}{m}\sum_{p=1}^{m}\phi(s) = \phi(s) > \phi\left(\frac{m \cdot s + k}{2m}\right) \ \forall k \in [0, m \cdot s),$$

   and therefore

$$2TM_k(l_s, \ldots, l_s) = 1.$$

2. We observe that $(l_0, \ldots, l_0) = \big(N(l_s), \ldots, N(l_s)\big)$. Thus, the proof of this case is obvious from case 1 above and Proposition 4.3. □

**Example 4.8.** Given the profile of linguistic preferences $(l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6)$, then we have $\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p = 6$ and $\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^p\right) = (l_6, 0)$. This is the highest collective preference valuation possible, and therefore it will be

$$2TM_k(l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6, l_6) = 1 \ \forall k.$$

Finally, the *cancellative* property is proved. Given two profiles with the same linguistic labels but two of them, then if the additions of the symbolic translations of the differing linguistic labels in each profile coincide, then the social majority is the same for the two profiles.

**Proposition 4.6** (Cancellative)**.** *Given two profiles of linguistic preferences,* $(l_{ij}^1, \ldots, l_{ij}^m)$ *and* $(l_{ij}'^1, \ldots, l_{ij}'^m)$*, such that*

$$l_{ij}^h = l_{ij}'^h \ \ \forall h \neq o, q; \ \ l_{ij}^o \neq l_{ij}'^o, \ l_{ij}^q \neq l_{ij}'^q \ \textit{with} \ l_{ij}^o + l_{ij}^q = l_{ij}'^o + l_{ij}'^q$$

*then*

$$2TM_k(l_{ij}'^1, \ldots, l_{ij}'^m) = 2TM_k(l_{ij}^1, \ldots, l_{ij}^m).$$

*Proof.* Notice that $l_{ij}^{h} = l_{ij}^{\prime h}\ \forall h \neq o, q;\ \ l_{ij}^{o} \neq l_{ij}^{\prime o},\ l_{ij}^{q} \neq l_{ij}^{\prime q}$ with $l_{ij}^{o} + l_{ij}^{q} = l_{ij}^{\prime o} + l_{ij}^{\prime q}$ implies $\frac{1}{m}\sum_{p=1}^{m}\phi(a_{ij}^{p}) = \frac{1}{m}\sum_{p=1}^{m}\phi(a_{ij}^{\prime p})$. □

**Example 4.9.** Given the following two profiles of linguistic preferences $(l_3, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_3)$ and $(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6)$, then we have $\frac{1}{m}\sum_{p=1}^{m} a_{ij}^{p} = 28/9$ and $\phi\left(\frac{1}{m}\sum_{p=1}^{m} a_{ij}^{p}\right) = (l_3,\ 1/9)$, for both of them.
Therefore,

$$2TM_k(l_3, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_3) = 2TM_k(l_0, l_1, l_2, l_2, l_2, l_3, l_6, l_6, l_6).$$

## 4.6 Conclusion

'Democratic theory is based on the premise that the resolution of a matter of social policy, group choice or collective action should be based on the desires or preferences of the individuals in the society, group or collective.' This quotation from Fishburn [40, page 3] fully justifies the use of preferences in social choice. However, democracy is recognised when decisions are made applying majority voting rule, which in its simple formulation means that the side with the most votes wins, whether it is an election, the select of the best candidate for a job when judged by a panel of experts, etc. This, however, is not the only majority rule available for social policy, group choice or collective action. Depending on the gravity or importance of the decision to make, other rules such as unanimous, absolute and qualified majorities may be more appropriate. These rules are easy to understand when each vote counts the same. However, there are many practical situations when this is not the case specially when experts or voters are allowed to indicate their degree of preference, which might be the case for example when selecting candidates for a job at a company. In these cases, it is possible to apply a new type of majority rules that are known as majorities based on difference in support, which allow to calibrate the amount of support required for the winner alternative by means of a difference of intensity of preference fixed before the election process. Majorities based on difference in support are defined when preferences are expressed quantitatively. This paper, however, deals with the comparison of two alternatives at a time and the experts or voters are allowed to express their preferences using qualitative ratings rather than quantitative ratings, a practice that is widely extended, for example, in market research for the introduction of new products by companies. This is because individuals, no mater their background, feel more comfortable using words to articulate their preferences than using numbers. In this context, the aim of this paper was to fill the gap between majorities based on difference of votes and majorities based on difference in support by providing new majority rules based on difference of support in the linguistic framework.

Linguistic majorities with difference in support extend majorities based on difference of votes from the context of crisp preferences to the framework of linguistic preferences. These linguistic majorities have been formalised for the two main representation methodologies of linguistic preferences: the cardinal, based on the use of fuzzy sets; and the ordinal, based on the use of the 2–tuples. It has been proved that both representations are mathematically isomorphic when fuzzy numbers are ranked using their respective centroids, and therefore it can be concluded that the cardinal approach constitutes a more general framework to model linguistic majorities with difference in support. Finally, a set of normative properties have been demonstrated to hold for the new linguistic majorities.

Some interesting extensions are left opened. Among them, the study of the collective consistency of linguistic majorities with difference in support when more than two alternatives are compared (Llamazares *et al.* [90]), and the development of a consistency based selection process seems to be worth further investigation. Also, it seems interesting to explore softer approaches to linguistic majorities with difference in support when the information is represented using fuzzy sets. Both research issues mentioned here could be addressed by developing an alternative methodology to the use of a representative crisp number more consistent with the fuzzy nature of the linguistic information. A potential avenue to investigate could be the construction of a collective fuzzy preference structure (Fodor and Roubens [44]) on the set of alternatives to represent the fuzzy linguistic majority with difference in support. Finally, the use of type–2 fuzzy sets also seems to be a challenging issue that deserves future research effort (Greenfield and Chiclana [67, 68]),which can be an appropriate representation model of preferences when different experts provide their information using sets of linguistic labels with different cardinality, a case that happens when experts have different levels of expertise. An alternative approach for this last scenario, which deserves further research, would be the application of aggregation operators able to aggregate fuzzy number not necessarily from the same set of linguistic labels, such as the type–1 OWA operator (Chiclana and Zhou[17] and Zhou *et al.* [128]).

# Conclusiones

En esta tesis se ha llevado a cabo un estudio detallado sobre la consistencia de las decisiones colectivas bajo las mayorías por diferencia de votos y las mayorías por diferencia de apoyo. Además, se ha presentado una aproximación lingüística de las mayorías por diferencia de votos.

Los dos primeros capítulos se centran en las mayorías por diferencia de apoyo. En concreto, se analizan las condiciones sobre los umbrales de apoyo que permiten garantizar que las relaciones de preferencia fuerte que se derivan de la aplicación de dichas mayorías estén libres de ciclos y de intransitividades.

En el Capítulo 1 se obtienen las siguientes conclusiones respecto a la transitividad de las relaciones de preferencia fuerte obtenidas bajo las mayorías por diferencia de apoyo:

- Siempre que el umbral de apoyo sea menor que el número total de votantes menos uno, se pueden encontrar ejemplos de relaciones de preferencia recíprocas para los que las relaciones de preferencia fuerte obtenidas bajo las mayorías por diferencia de apoyo no son transitivas.
- Si el umbral de apoyo es mayor o igual que el número total de votantes menos uno y las relaciones de preferencia recíprocas son am–transitivas o menos racionales que éstas, se pueden encontrar ejemplos en los que las relaciones de preferencia fuerte que se derivan de la aplicación de las mayorías por diferencia de apoyo no son transitivas.
- Las relaciones de preferencia fuerte resultado de la aplicación de las mayorías por diferencia de apoyo son transitivas si el umbral de apoyo es mayor o igual que el número total de votantes menos uno y las

relaciones de preferencia recíprocas son am–transitivas o cumplen una condición de transitividad más exigente que dicha transitividad.

Los resultados anteriores subrayan la importancia de la consideración de relaciones de preferencia recíprocas altamente racionales para obtener relaciones de preferencia fuerte transitivas. En concreto, aquéllas que cumplan como mínimo la condición de am–transitividad. Lo anterior puede estar relacionado con el hecho de que las mayorías por diferencia de apoyo se pueden formalizar por medio del operador de agregación media aritmética. Además, el requisito sobre el umbral que permite garantizar la transitividad de las relaciones de preferencia fuerte es muy elevado.

En el Capítulo 2 se obtienen las siguientes conclusiones sobre la triple-aciclicidad de las relaciones de preferencia fuerte que se derivan del uso de las mayorías por diferencia de apoyo:

- Las relaciones de preferencia fuerte que resultan de la aplicación de las mayorías por diferencia de apoyo son triple-acíclicas, teniendo en cuenta relaciones de preferencia recíprocas 0,5–transitivas, si el umbral de apoyo es mayor o igual a la parte entera de dos tercios del número total de votantes.

- Las relaciones de preferencia fuerte obtenidas bajo las mayorías por diferencia de apoyo son triple-acíclicas, tomando en consideración relaciones de preferencia recíprocas mín–transitivas o máx–transitivas, si el umbral de apoyo es mayor o igual a un tercio del número total de votantes.

- Para el caso de dos individuos con relaciones de preferencia recíprocas máx–transitivas, las relaciones de preferencia fuerte originadas por el uso de las mayorías por diferencia de apoyo son triple-acíclicas si el umbral de apoyo es mayor o igual que la mitad de la parte entera de dos tercios del número total de votantes. Se conjetura que dicho resultado se debe producir para cualquier número de votantes.

Los resultados sobre la triple-aciclicidad de las relaciones de preferencia fuerte evidencian que el hecho de requerir una condición de consistencia colectiva más débil que la transitividad origina resultados positivos menos exigentes. Por un lado, se demuestra que se pueden obtener umbrales para garantizar la triple-aciclicidad de las relaciones de preferencia fuerte teniendo en cuenta relaciones de preferencia recíprocas débilmente racionales como las 0,5–transitivas y las mín-transitivas, si bien el umbral necesario para las primeras resulta bastante elevado. Por otro lado, se demuestra que el umbral que asegura la triple-aciclicidad de las relaciones de preferencia fuerte para relaciones de preferencia recíprocas máx–transitivas es menos severo que el que se requiere para garantizar la transitividad de las relaciones de preferencia fuerte.

En el Capítulo 3 se estudian las probabilidades asociadas a que las relaciones de preferencia fuerte sean transitivas, las correspondientes a que las relaciones de preferencia fuerte sean triple-acíclicas y las referidas a que las relaciones de preferencia débil sean transitivas, tanto para las mayorías por diferencia de votos como para las mayorías por diferencia de apoyo. Dichas probabilidades se calculan para diferentes números de votantes.

En el caso de las mayorías por diferencia de votos, las principales conclusiones que se pueden extraer del análisis de las probabilidades obtenidas son las siguientes:

- Se obtienen resultados similares tanto si las preferencias individuales son órdenes lineales como si son órdenes débiles:
  - Las diferencias de votos necesarias para alcanzar probabilidades de relaciones de preferencia fuerte triple-acíclicas iguales a uno son razonables. Su valor supone alrededor de un tercio del número total de votantes.
  - Las diferencias de votos necesarias para alcanzar probabilidades de relaciones de preferencia fuerte transitivas iguales a uno son muy elevadas excepto en el caso de tres votantes en el que toma el valor de uno.
  - En el caso de relaciones de preferencia débil transitivas, no es posible alcanzar probabilidades iguales a uno.
- Las probabilidades de relaciones de preferencia fuerte triple-acíclicas y transitivas son algo mayores si las preferencias individuales son órdenes débiles que si son órdenes lineales.

En el caso de las mayorías por diferencia de apoyo, destacan los siguientes resultados al analizar las probabilidades asociadas a que las relaciones de preferencia fuerte sean transitivas, a que sean triple-acíclicas y a que las relaciones de preferencia débil sean transitivas:

- Para las relaciones de preferencia débil transitivas, las relaciones de preferencia fuerte transitivas y las fuerte triple-acíclicas, se obtienen probabilidades iguales a uno para umbrales de apoyo diferentes de cero.
- En el caso de las relaciones de preferencia débil, en general, la mayor exigencia en cuanto a la transitividad de las relaciones de preferencia recíprocas no se refleja en que menores umbrales sean suficientes para alcanzar probabilidades de relaciones de preferencia débil transitivas iguales a uno. Ese comportamiento solo se produce en el caso de 100 000 individuos y en el caso de que la diferencia de apoyo sea nula.
- En el caso de las relaciones de preferencia fuerte conforme aumenta el número de votantes, el peso del umbral necesario para obtener probabilidades de relaciones de preferencia fuerte transitivas y triple-acíclicas iguales a uno, respecto al número total de votantes, es proporcionalente menor. Además, excepto en el caso de 100 individuos, a medida que las condiciones de transitividad sobre las relaciones de preferencia recíprocas son más exigentes, los umbrales necesarios para que las probabilidades de las relaciones de preferencia fuerte transitivas y triple-acíclicas alcancen el valor de uno son proporcionalmente menores.

Los resultados anteriores complementan los presentados en los Capítulos 1 y 2. Por una parte, se obtienen umbrales para la transitividad de las relaciones de preferencia fuerte con probabilidad igual a uno en aquellos casos en los que, bajo el análisis teórico, no se podía garantizar la transitividad de dichas relaciones de preferencia. Por otra parte, se obtienen umbrales razonables para garantizar la transitividad y la triple-aciclicidad de las relaciones de preferencia fuerte con probabilidad igual a uno, en comparación con los obtenidos en los desarrollos teóricos. Además, en el caso de relaciones de preferencia recíprocas máx–transitivas, se observa que las probabilidades asociadas a la triple-aciclicidad de las relaciones de preferencia fuerte para umbrales de apoyo iguales a la mitad de la parte entera de dos tercios del número de votantes alcanzan el valor de uno. Estos resultados apoyan la conjetura que se realizaba en el Capítulo 2 sobre los umbrales de apoyo necesarios para garantizar la triple-aciclicidad de las relaciones de preferencia fuerte cuando se tienen en cuenta relaciones de preferencia recíprocas máx–transitivas.

En el Capítulo 4 se presenta una aproximación lingüística de las mayorías por diferencia de votos con el objetivo de llenar el espacio entre dichas mayorías, donde se tienen en cuenta preferencias individuales *crisp* y las mayorías por diferencia de apoyo, donde se consideran relaciones de preferencia recíprocas.

Para ello, se utilizan preferencias lingüísticas modelizadas bajo la metodología cardinal, basada en conjuntos difusos y en sus funciones de pertenencia, y bajo la ordinal, basada en el modelo de las 2-tuplas.

En el primer caso, la regla se construye a través de los centroides de los números difusos trapezoidales que representan las preferencias colectivas entre pares de alternativas. En el segundo caso, la regla se construye sobre la 2-tupla colectiva, que no es sino la media de las 2-tuplas que representan las preferencias lingüísticas individuales.

Se demuestra que, bajo ciertas condiciones de regularidad sobre los números difusos trapezoidales, ambas reglas son equivalentes. Por último se demuestra que estas reglas cumplen propiedades atractivas como son el anonimato, la neutralidad, la monotonía, la condición débil de Pareto y la cancelación.

Los inconvenientes apuntados en la literatura a la hora de ordenar números difusos quedan patentes en este capítulo. Para poder ordenar dichos números se opta por utilizar los centroides, una solución que da lugar a la comparación de escalares. Por tanto, la vaguedad que se pretendía recoger en el modelo pasa a ser en cierto modo testimonial.

Queda pendiente una modelización alternativa de este tipo de reglas que permita una comparación de los números difusos consistente con la vaguedad de su naturaleza.

# Concluding remarks

This thesis presents a detailed study of the consistency of collective decisions under majorities based on difference of votes and majorities based on difference in support. In addition, a linguistic approach to majorities based on difference of votes is introduced.

The first two chapters focus on majorities based on difference in support. Specifically, the conditions under thresholds of support to guarantee non cyclic and transitive strict preference relations under such majorities are analysed.

In Chapter 1, the following conclusions about the transitivity of strict preference relations under majorities based on difference in support are obtained:

- Whenever the threshold of support is lower than the total number of voters minus one, examples can be found of reciprocal preference relations for which the strict preference relations obtained under majorities based on difference in support are not transitive.

- If the threshold of support is equal to or greater than the total number of voters minus one and the reciprocal preference relations are am–transitive or lower in terms of rationality, examples can be found for which the strict preference relations derived from majorities based on difference in support are not transitive.

- The strict preference relations derived from the application of majorities based on difference in support are transitive if the threshold of support is equal to or greater than the total number of voters minus one and the reciprocal preference relations are am-transitive or fulfil a more stringent transitivity condition.

The results above highlight the importance of considering highly rational reciprocal preference relations in order to obtain transitive strict preference relations, specifically those that at least fulfil the condition of am–transitivity. This could be related to the fact that majorities based on difference in support can be formalised by means of the arithmetic mean aggregation operator. Moreover, the requirement made of the threshold that enables the transitivity of the strict preference relation to be guaranteed is very high.

In Chapter 2 the following conclusions about the triple-acyclicity of the strict preference relations derived from majorities based on difference in support are obtained:

- The strict preference relations obtained under majorities based on difference in support are triple-acyclic, taking into account 0.5–transitive reciprocal preference relations if the threshold of support is equal to or greater than the integer part of two thirds the total number of voters.

- The strict preference relations obtained under majorities based on difference in support are triple-acyclic, taking into account min–transitive or max–transitive reciprocal preference relations, if the threshold of support is equal to or greater than one third the number of voters.

- In the case of two individuals with max–transitive reciprocal preference relations, the strict reciprocal preference relations derived from the use of majorities based on difference in support are triple-acyclic if the threshold of support is equal to or greater than half of the integer part of two thirds the total number of voters. It is conjectured that this result must be true for any number of voters.

Results on triple-acyclicity of strict preference relations show that requiring a collective consistency condition weaker than transitivity causes less stringent positive results. On the one hand, it is proven that thresholds can be obtained to guarantee the triple-acyclicity of the strict preference relation bearing in mind weakly rational reciprocal preference relations such as 0.5–transitive and min–transitive ones, although the threshold needed for the first ones is quite high. On the other hand, it is proven that the threshold that guarantees the triple-acyclicity of the strict preference relations for max–transitive reciprocal preference relations is less severe than that required to guarantee the transitivity of the strict preference relation.

Chapter 3 examines the probabilities associated with strict preference relations being transitive, with strict preference relations being triple-acyclic and with weak preference relations being transitive both for majorities based on difference of votes and for majorities based on difference in support. These probabilities are calculated for different numbers of voters.

In the case of majorities based on difference of votes, the main conclusions drawn from the analysis of the probabilities are the following:

- The results are similar regardless of whether the individual preferences are linear orderings or weak orderings.
- The differences in votes needed to reach probabilities of triple-acyclic strict preference relations of one are reasonable, at around one third of the total number of voters.
- The differences in votes needed to reach probabilities of triple-acyclic strict preference relations of one are very high except in the case of three voters, where the difference is one.
- In the case of transitive weak preference relations, it is not possible to reach probabilities of one.
- The probabilities of triple-acyclic and transitive strict preference relations are slightly higher when individual preferences are weak orderings than when they are linear orderings.

In the case of majorities based on difference in support, the following results stand out when the probabilities associated with transitive strict preference relations, with triple-acyclic strict preference relations and with transitive weak preference relations are analysed:

- For transitive weak preference relations, transitive strict preference relations and triple-acyclic strict preference relations, probabilities of one are reached for thresholds of support other than zero.

- In the case of weak preference relations, in general, the greater requirement for the transitivity of the reciprocal preference relation is not reflected in a smaller threshold of support for reaching probabilities of transitive preference relations of one. Such behaviour is only found in the case of 100 000 individuals and in the case of zero difference in support.

- In the case of strict preference relations, as the number of voters increases the proportion of the total number of voters represented by the threshold required to reach probabilities of transitive and triple-acyclic strict preference relations of one becomes proportionally lower. In addition, with the exception of the case of 100 individuals, as the transitivity conditions applicable to reciprocal preference relations become more strict, the thresholds required become lower in proportion to the probabilities of the transitive and triple-acyclic preference relations reaching a value of one.

The results indicated above complement those presented in Chapters 1 and 2. On the one hand, thresholds with a probability of transitive strict preference relations of one are obtained in cases in which the transitivity of such preference relations cannot be guaranteed under the theoretical analysis. On the other hand, reasonable thresholds are obtained to guarantee the transitivity and triple-acyclicity of the strict preference relation with a probability of one, in comparison with the theoretical figures. Furthermore, in the case of max–transitive reciprocal preference relations, the probabilities associated with the triple-acyclicity of the strict preference relations reach a value of one for thresholds of support equivalent to half of the integer part of two thirds of the total number of voters. These results support the conjecture made in Chapter 2 concerning the thresholds of support required to guarantee the triple-acyclicity of strict preference relations when the reciprocal preference relations considered are max–transitive.

In Chapter 4 a linguistic approach to majorities based on difference of votes is introduced with a view to filling the gap between these majorities, where crisp individual preferences are considered and majorities based on difference in support, where reciprocal preference relations are considered.

To that end, linguistic preferences are used, modelled under the cardinal method, based on fuzzy sets and on their membership functions, and under the ordinal method, based on the 2-tuples model.

In the first case the rule is built up by means of the centroids of the trapezoidal fuzzy numbers that represent the collective preferences in pairs of alternatives. In the second case, the rule is built up by means of the collective 2-tuple which is the arithmetic mean of the 2-tuples that represent the individual linguistic preferences.

It is proven that the two rules are equivalent under certain regularity conditions for trapezoidal fuzzy numbers, Finally, it is proven that these rules have attractive properties such as anonymity, neutrality, monotonicity, a weak Pareto condition and cancelativeness.

The drawbacks pointed out in the relevant literature concerning the ranking of fuzzy numbers appear in this Chapter. To rank such numbers, centroids are used. This solution leads to the comparison of scalars. Therefore, the vagueness that the model seeks ends up being symbolic.

An alternative modelling of rules of this type is opened up that allows a comparison of fuzzy numbers consistent with their vague nature.

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