Equilibrium strategies in a multiregional transboundary pollution differential game with spatially distributed controls*

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Abstract

We analyze a differential game with spatially distributed controls to study a multiregional transboundary pollution problem. The dynamics of the state variable (pollution stock) is defined by a two dimensional parabolic partial differential equation. The control variables (emissions) are spatially distributed variables. The model allows for a, possibly large, number of agents with predetermined geographical relationships. For a special functional form previously used in the literature of transboundary pollution dynamic games we analytically characterize the feedback Nash equilibrium. We show that at the equilibrium both the level and the location of emissions of each region depend on the particular geographical relationship among agents. We prove that, even in a simplified model, the geographical considerations can modify the players' optimal strategies and therefore, the spatial aspects of the model should not be overlooked.

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1 Introduction

Most of the literature on dynamic models settled for the analysis of different economic and environmental problems takes into account the temporal aspect but disregards the spatial aspect. The addition of the spatial dimension although enriches the model and consequently its prescriptions comes at the cost of making the analysis harder. Over the last decade the spatial dimension has been introduced in different economic and environmental contexts, such as, for example, allocation of economic activity or technological diffusion (Brito (2004), Camacho et al. (2008), Brock & Xepapadeas (2008a), Boucekkine et al. (2009, 2013a, 2013b, 2019a), Desmet & Rossi-Hansberg (2010), Brock et al. (2014a) and Fabbri (2016)) or environmental and climate problems (Brock & Xepapadeas (2008b, 2010), Brock et al. (2014b), Camacho & Pérez-Barahona (2015), Xepapadeas (2010), Anita et al. (2013, 2015), Desmet & Rossi-Hansberg (2015), La Torre et al. (2015), Augeraud-Véron et al. (2017, 2019a, 2019b), Boucekkine et al. (2019b), and De Frutos & Martín-Herrán (2019a, 2019b)). Except De Frutos & Martín-Herrán (2019a, 2019b) all the previously cited papers study finite or infinite time horizon optimal control problems extended to infinite dimensional state space. All these papers analyze problems where there is only one decision-maker.

One of the main differences between De Frutos & Martín-Herrán (2019a, 2019b) and the other contributions to this literature is the introduction of strategic interactions between the decision-makers. The addition of the strategic behavior of the agents implies a methodological change, moving from an optimal control problem to a dynamic game. More specifically, De Frutos & Martín-Herrán (2019a, 2019b) study an intertemporal transboundary pollution dynamic game where there is a continuum of spatial sites and the pollution stock diffuses over these sites. In both papers the J-player (region or country) model is formulated in continuous space and continuous time with two spatial dimensions and one temporal dimension, the spatio-temporal evolution of the stock of a pollutant is described by a diffusion partial differential equation (PDE) and general boundary conditions are assumed. In De Frutos & Martín-Herrán (2019a) each player decides the emission level so as to maximize the present value of benefits net of environmental damages due to the concentration of a pollutant over his spatial domain, taking into account the PDE that describes the spatio-temporal evolution of the stock of a pollutant. In De Frutos & Martín-Herrán (2019b) in addition to choosing the level of emissions, each player also decides the investment in clean technology, taking into account the temporal evolution of the stock of clean technology and that the greater this stock the lower the emission-output ratio.

De Frutos & Martín-Herrán (2019a, 2019b) follow the same spatial discretization approach to characterize the equilibrium emission strategies in a multiregional setting with spatial effects. In the space-discretized model there are J state variables, the average pollution in each one of the J regions or countries, and the temporal dynamics of these variables is described by a system of J ordinary differential equations. Graß & Uecker (2017) uses a similar spatial discretization approach to analyze spatially distributed optimal control problems. Fabbri et al. (2020) recently analyze a spatially discrete model with two agents that strategically exploit a natural resource in a two region setup. In De Frutos & Martín-Herrán (2019a) the feedback Nash equilibrium of the two-player space-discretized model is analytically characterized, while for more than two players, the model is numerically solved by adapting a numerical algorithm presented in De Frutos & Martín-Herrán (2015). The linear-quadratic specification of the model is inspired in the literature of transboundary pollution dynamic games (Jørgensen et al. (2010) surveyed this literature), specifically in the seminal papers by Dockner & Long (1993) and Van der Ploeg and De Zeeuw (1992). The numerical examples show that once the spatial dimension and the strategic behavior of the decision-makers are introduced the environmental policies greatly differ from those characterized ignoring either the spatial dimension or the strategic interactions among the agents. In De Frutos & Martín-Herrán (2019b) particular functional forms borrowed from Jørgensen & Zaccour (2001) are considered in such a way that the dynamic game belongs to the class of linear-state differential games. For this formulation the space-discretized model is exactly solved and is proved to be a clear generalization of the model which ignores the spatial transport phenomena. The equilibrium environmental policy of the spatial model coincides with the equilibrium policy of the non-spatial model when the diffusion parameter, that describes how pollution diffuses among regions, tends to infinity.

The main contribution of this paper is twofold. First, we contribute to the literature on optimal control of PDEs adding the strategic behavior of the decision makers, and then, moving from optimal control to differential game methodology. Second, we contribute to the literature on differential games incorporating in the modelling the spatial dimension of the problem at hand. As far as we know, De Frutos & Martín-Herrán (2019a, 2019b) are the first papers dealing simultaneously with these two aspects. In the present paper instead of using a space discretized approach of the problem as in the two previously cited works, we follow a direct approach analytically studying the problem.

More specifically, in the present paper we revisit the study of the equilibrium emission strategies in a multiregional dynamic game in a spatial context. The main objective of the paper is to analytically characterize the feedback Nash equilibrium emission strategies of the continuous space and continuous time J-player (region or country) model where the spatio-temporal evolution of the stock of a pollutant is described by a diffusion partial differential equation. We depart from the spatial discretization approach presented in De Frutos & Martín-Herrán (2019a, 2019b) and for a linear-state specification inspired in Jørgensen & Zaccour (2001) and De Frutos & Martín-Herrán (2019b) we explicitly solve a functional Hamilton-Jacobi-Bellman system in an infinite dimensional Hilbert space. Thanks to the linear-state framework, guessing an affine form of the players' value functions we can explicitly compute these functions and as a consequence also the feedback Nash equilibrium emission strategies. The resolution method involves solving several elliptical problems, one for each player. Although the equilibrium strategies are constant in time as a direct consequence of the linear-state structure of the dynamic game, interestingly, these strategies are not constant in space. Solving the original model formulation enables to strategically determine the average total emission in each region (as was the case in the space-discretized formulation of the model in De Frutos & Martín-Herrán (2019a, 2019b)), but also to characterize the particular point in the space where each region is emitting pollutants. Using the new approach proposed in this paper we can analyze the optimal intraregional distribution of emissions of the pollutant, a question that has been previously neglected. Furthermore, the new approach allows the study not only of the diffusive effect, but also the case where advection is important and particles are transported due to some external convective field, as for example, wind or water flow.

Our analytical results show that the geographical aspects are essential ingredients when determining the equilibrium emission strategies. Through several examples with different geographical configurations we show that the type and behavior of the neighbors of one region have an influence in its own equilibrium environmental policies, not only by determining the optimal amount of emissions but also establishing the optimal spatial location. The results corroborate those obtained in the analysis of the space-discretized model which allows us to conclude that this simplified model correctly captures the spatial essence of the setting. However, the continuous-space and continuous-time model analyzed in the present paper is richer which in turn allows us to have an accurate view of how the spatial ingredients affect the equilibrium emission strategies and hence the pollution stock.

The rest of the paper is organized as follows. Section 2 presents the multiregional transboundary pollution differential game with spatially distributed controls and introduces some technical hypotheses and definitions. Section 3 analytically studies the Hamilton-Jacobi-Bellman system of equations and derives the feedback Nash equilibrium of the differential game. Section 4 presents some examples to highlight the properties of the Nash equilibria. The paper finishes with some concluding remarks.

2 The model

Let us denote by Ω a bounded planar domain endowed with a partition Ω_j, $j = 1, \ldots, J$, such that

$$\overline{\Omega} = \bigcup_{j=1}^{J} \overline{\Omega}_j, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j,$$

where $\overline{\Omega}$ is the closure of Ω. We denote by ∂_{ij} the common boundary between subdomains Ω_i and Ω_j, that is $\partial_{ij} := \partial\Omega_i \cap \partial\Omega_j = \overline{\Omega}_i \cap \overline{\Omega}_j$, $i \neq j$.

The model is a J-player differential game in which the control variable of player i is the pollutant emissions in region Ω_i. The game is played non-cooperatively. In what follows we identify player i with the region Ω_i and use the word country to distinguish Ω_i from the whole region Ω. We assume that each of the countries, Ω_i, $i = 1, \ldots, J$, can exclusively emit pollution in its own territory at a rate represented by a function $u_i : \Omega_i \times [0, +\infty) \to \mathbb{R}_+$, $i = 1, \ldots, J$. Pollutant emissions accumulate into a stock of pollution denoted by $P(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Omega$ and $t \geq 0$. The pollution stock diffuses over the whole region Ω. The stock of pollution, P, is the state variable of the differential game, and each player has a unique control variable, the emission rate, u_i.

Utility is assumed to be isoelastic with constant elasticity of intertemporal substitution

$1/\sigma$, $\sigma > 0$. More precisely, utility U is defined by

$$U(u_i) = \begin{cases} \frac{u_i^{1-\sigma}-1}{1-\sigma}, & \sigma \neq 1, \\ \log(u_i), & \sigma = 1. \end{cases}$$

The objective of player i, $i = 1, \ldots, J$, is to maximize the following functional with respect to u_i,

$$J_i(u_1, \ldots, u_J, P_0) = \int_0^{+\infty} \int_{\Omega_i} e^{-\rho t} \Big(U(u_i) - \varphi_i P \Big) \, d\boldsymbol{x} \, dt \tag{1}$$

subject to

$$\begin{aligned} &\frac{\partial P}{\partial t} = \nabla \cdot (k \nabla P) + \boldsymbol{b} \cdot \nabla P - cP + F(u_1, \ldots, u_J), \quad \boldsymbol{x} \in \Omega, \\ &P(\boldsymbol{x}, 0) = P_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ &\alpha P(\boldsymbol{x}, t) + k \nabla P \cdot \boldsymbol{n} = \alpha P_b(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega. \end{aligned} \tag{2}$$

In (1) we follow the common approach in the literature (see, for example, Jørgensen et al. (2010)) and assume that the instantaneous welfare is given by the average over Ω_i of the utility derived from production, and hence, from emissions, minus the damage caused by the pollution stock. This damage is assumed to be proportional to the pollution stock and measured by parameter $\varphi_i > 0$. Therefore, each player maximizes over an infinite time horizon the instantaneous welfare discounted at a constant time discount rate $\rho > 0$.

In (2) F is assumed to be a real smooth function of its arguments, P_0 and P_b are known functions representing respectively, the initial distribution of pollution and the external pollution that enters into Ω through its boundary. In what follows we assume without loss of generality that function P_b is identically zero. The diffusion coefficient $k(\boldsymbol{x})$ is assumed to be a smooth function of the spatial variables satisfying $k_1 \leq k(\boldsymbol{x}) \leq k_2$ for some positive constants k_1 and k_2. The coefficient $c(\boldsymbol{x})$ represents the natural decay of pollution and it is supposed to be a non-negative smooth function. Function $\boldsymbol{b}(\boldsymbol{x})$ represents an external convective field. We assume that $\boldsymbol{b}(\boldsymbol{x})$ is smooth and divergence free $\nabla \cdot \boldsymbol{b} = 0$. Finally, α is a positive real number.

We assume that $F(u_1, \ldots, u_J) = \sum_{j=1}^{J} u_j 1_{\Omega_j}$ where 1_{Ω_j} is the characteristic function of the set Ω_j, $j = 1, \ldots, J$. We implicitly consider that function u_i can be arbitrarily extended outside of the domain Ω_i.

Let us consider $\mathbb{X} = L^2(\Omega)$ the Hilbert space of square integrable functions defined over domain Ω. As usual we identify $\mathbb{X}$ with its dual $\mathbb{X}^*$. Here and in the rest of the paper, we denote by $H^s(\Omega)$ the Sobolev space of functions with s derivatives, in the distributional sense, in $L^2(\Omega)$.

We define the set $\mathcal{U}_i$ of admissible controls for player i, $i = 1, \ldots, J$, as the set of measurable functions u_i defined in $\Omega_i \times \mathbb{R}_+ \to \mathbb{R}_+$. Although it is not essential here, we also assume that functions $u_i \in \mathcal{U}_i$ are essentially bounded. We also assume that functions $u_i \in \mathcal{U}_i$ satisfy that there exist non-negative constants $\underline{u}$ and $\overline{u}$ such that $\underline{u} \leq \|u\|_\infty \leq \overline{u}$. Note that choosing $\underline{u}$ small enough and $\overline{u}$ large enough, the constraints never become active, see Section 3.

The hypotheses above guarantee that given an initial condition $P_0 \in \mathbb{X}$, the state equation (2) has a unique weak solution for each choice of the controls $u_i \in \mathcal{U}_i$, $i = 1, \ldots, J$, see Barbu (1993), Li & Yong (1995), Tröltzsch (2009).

With this model we adopt the simplest version of the economic and environmental model that still presents two important features allowing us to answer our main research question. First, the strategic behaviour of the players, emissions by one player affects the environment of all; and second, the spatial aspect that allows us to show that at the equilibrium both the level and the location of emissions of each region depend on the particular geographical relationship among agents.

We are interested in stationary Markov-perfect Nash equilibria (MPNE) of the game. Then, we look for controls of the form $u_i(\boldsymbol{x}, t) = \Lambda_i(P(\boldsymbol{x}, t))$, $i = 1, \ldots, J$. Here the admissible strategies Λ_i are functionals $\Lambda_i : \mathbb{X} \to \mathcal{U}_i$ such that the controlled dynamics

$$\begin{aligned} &\frac{\partial P}{\partial t} = \nabla \cdot (k \nabla P) + \boldsymbol{b} \cdot \nabla P - cP + \sum_{j=1}^{J} \Lambda_j(P) 1_{\Omega_j}, \quad \boldsymbol{x} \in \Omega, \\ &P(\boldsymbol{x}, \tau) = P_\tau(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \\ &\alpha P(\boldsymbol{x}, t) + k \nabla P \cdot \boldsymbol{n} = \alpha P_b(\boldsymbol{x}, t), \quad \boldsymbol{x} \in \partial\Omega, \end{aligned} \tag{3}$$

has a unique solution defined in $[\tau, \infty)$ for every $\tau \geq 0$ and $P_\tau \in \mathbb{X}$.

Definition 1. A vector $\boldsymbol{\Lambda}^* = [\Lambda_1^*, \ldots, \Lambda_J^*]$ of admissible strategies is a Markov Perfect Nash Equilibrium if $J_i(\boldsymbol{u}^*, P_0) \geq J_i([u_i, \boldsymbol{u}_{-i}^*], P_0)$, for all $u_i = \Lambda_i(P)$ with Λ_i an admissible strategy, $i = 1, \ldots, J$. Here, $\boldsymbol{u}^* = [u_1^*, \ldots, u_J^*]$, $u_j^* = \Lambda_j^*(P^*)$ and P^* is the solution of (3) with $\Lambda_i = \Lambda_i^*$, $i = 1, \ldots, J$. We use $[u_i, \boldsymbol{u}_{-i}^*]$ to denote $[u_1^*, \ldots, u_i, \ldots, u_J^*]$.

Given a stationary MPNE $\boldsymbol{\Lambda}^* = [\Lambda_1^*, \ldots, \Lambda_J^*]$, $V_i(P) = J_i(\boldsymbol{u}^*, P)$ is called the value function of Player i.

The objective of the next section is to characterize the MPNE of the differential game (1)-(2) through the study of the value function.

3 The Hamilton-Jacobi-Bellman equation

For simplicity in the exposition we assume in this section that $\boldsymbol{b}\cdot\boldsymbol{n}=0$ in $\partial\Omega$. We define the linear operator $\boldsymbol{A}: D(\boldsymbol{A})\to X$ by

$$\boldsymbol{A}P = \nabla\cdot(k\nabla P) + \boldsymbol{b}\cdot\nabla P - cP, \quad \forall P\in\mathbb{X},$$

where

$$D(\boldsymbol{A}) = \{P\in H^2(\Omega) | \alpha P(\boldsymbol{x}) + k\nabla P\cdot\boldsymbol{n} = 0\}, \tag{4}$$

is the domain of $\boldsymbol{A}$. The linear operator $\boldsymbol{A}$ is continuous from $D(\boldsymbol{A})$ in $\mathbb{X}$ and is the infinitesimal generator of a contraction semigroup e^{At} in $\mathbb{X}$, see Li & Yong (1995). In what follows $\nabla W(P)$ denotes the Fréchet derivative of a functional W and $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{X}$.

The following proposition is a consequence of Proposition 1.2 (Ch. 6, p.225) in Li & Yong (1995), see also Başar & Olsder (1999), Haurie et al. (2012).

Proposition 1. *Let $\boldsymbol{\Lambda}^* = [\Lambda_1^*,\dots,\Lambda_J^*]$ be a MPNE. Let us assume that $V^i(P)$ is of class $\mathcal{C}^1(\mathbb{X})$, $i=1,\dots,J$. The value functions V^i, $i=1,\dots,J$, satisfy the functional Hamilton-Jacobi-Bellman system*

$$\rho V^i(P) = \sup_{u_i}\Big\{\mathcal{G}^i(P,u_i) + \big\langle \boldsymbol{A}P + u_i 1_{\Omega_i} + \sum_{j\neq i}\Lambda_j^*(P)1_{\Omega_j}, \nabla V^i(P)\big\rangle\Big\}, \quad i=1,\dots,J, \tag{5}$$

where

$$\mathcal{G}^i(P,u_i) = \int_{\Omega_i}\Big(U(u_i) - \varphi_i P\Big)\,d\boldsymbol{x}.$$

Furthermore, $\Lambda_i^*(P)$, $i=1,\dots,J$, is a maximizer of the right hand side of (5).

The regularity hypothesis in Proposition 1 is very demanding. It is well known that even in the finite dimensional case Hamilton-Jacobi-Bellman equations can fail to have enough regularity and one has to resort to weaker concepts as viscosity solutions, see, for example, Barbu (1993), Cannarsa & Da Prato (1990), Li & Yon (1995).

Following Dockner et al. (2000) (Th. 3.3, page 63, and Th. 4.4, page 103), transversality conditions of the form

$$\lim_{t\mapsto\infty} e^{-\rho t} V_i(P^*(\boldsymbol{x},t)) = 0, \quad i = 1,\dots,J, \tag{6}$$

have to be satisfied. The equilibrium we are explicitly computing leads to controlled dynamics that possesses a (unique) stable stationary state, so that the transversality conditions will be automatically satisfied.

We remark that, in general, Hamilton-Jacobi-Bellman systems of the form (5) with boundary conditions as (6) have multiple solutions (see De Frutos & Martín-Herrán (2018)), which correspond with possible multiple MPNE. Because the differential game fits the linear-state class, we concentrate in the rest of the paper on value functions that are affine in the state variable. More precisely, we look for affine value functions of the form

$$V^i(P) = w_i + \int_\Omega v_i(\boldsymbol{x})P(\boldsymbol{x})\,d\boldsymbol{x}, \quad i = 1,\dots,J, \tag{7}$$

for some unknowns $w_i \in \mathbb{R}$ and $v_i \in \mathbb{X}$.

We remark that affine functions of the form (7) are obviously of class $\mathcal{C}^1(\mathbb{X})$. Then, we can apply Th. 4.4 in Dockner et al. (2000) (page 103) to guarantee that if tranversality conditions as (6) are satisfied, affine solutions of Hamilton-Jacobi-Bellman system (5) are in fact value functions of the problem. Here, as usual, we are using the catching-up optimality criterion for infinite horizon.

We observe that the Fréchet derivative of the affine functional $V^i(P)$ in (7) is the linear operator defined by

$$\left\langle \nabla V^i(P), h \right\rangle = \int_\Omega v_i(\boldsymbol{x})h(\boldsymbol{x})\,d\boldsymbol{x}, \quad h \in \mathbb{X}. \tag{8}$$

Using (8) and integrating by parts we have

$$\left\langle \nabla V^i(P), \boldsymbol{A}P \right\rangle = \int_\Omega v_i \boldsymbol{A}P d\boldsymbol{x} = \int_\Omega \boldsymbol{A}^* v_i P d\boldsymbol{x}, \tag{9}$$

where $\boldsymbol{A}^*$ denotes the adjoint of $\boldsymbol{A}$ defined for $v_i \in D(\boldsymbol{A}^*) = D(\boldsymbol{A})$ by

$$\boldsymbol{A}^* v_i = \nabla \cdot (k \nabla v_i) - \boldsymbol{b} \cdot \nabla v_i - c v_i.$$

Note that in deriving (9) we use the homogeneous Robin boundary conditions defined in (4), together with the simplifying hypothesis $\boldsymbol{b} \cdot \boldsymbol{n} = 0$ in $\partial\Omega$. Using again (8) we have

$$\left\langle \nabla V^i(P), u_j 1_{\Omega_j} \right\rangle = \int_\Omega v_i u_j 1_{\Omega_j} d\boldsymbol{x}, \quad j = 1,\dots,J. \tag{10}$$

Finally, the derivatives with respect to u_i of the functionals $\mathcal{G}^i(P, u_i)$ and

$$\mathcal{F}^i(u_i) := \int_\Omega v_i u_i 1_{\Omega_i} d\boldsymbol{x}$$

are given by

$$\left\langle \nabla_{u_i} \mathcal{G}^i(P, u_i), h \right\rangle = \int_\Omega u_i^{-\sigma} h 1_{\Omega_i} d\boldsymbol{x}, \quad h \in \mathbb{X}, \tag{11}$$

$$\left\langle \nabla_{u_i} \mathcal{F}^i(u_i), h \right\rangle = \int_\Omega v_i h 1_{\Omega_i} d\boldsymbol{x}, \quad h \in \mathbb{X}. \tag{12}$$

We can write now the first-order condition for the maximization of the right hand side in (5) using (8)-(12). We have

$$\int_\Omega u_i^{-\sigma} h 1_{\Omega_i} d\boldsymbol{x} + \int_\Omega v_i h 1_{\Omega_i} d\boldsymbol{x} = 0, \quad \forall h \in \mathbb{X} = L^2(\Omega). \tag{13}$$

So that

$$u_i = (-v_i)^{-1/\sigma} 1_{\Omega_i}, \quad i = 1, \ldots J. \tag{14}$$

Substituting the guess (7) in the HJB equation (5) and using the previous results we have

$$\begin{aligned} &\rho w_i + \rho \int_\Omega v_i P \, d\boldsymbol{x} = \\ &\quad \int_\Omega \big(U((-v_i)^{-1/\sigma}) - \varphi_i P\big) 1_{\Omega_i} d\boldsymbol{x} + \int_\Omega \boldsymbol{A}^* v_i P d\boldsymbol{x} + \int_\Omega \sum_{j=1}^J (-v_j)^{-1/\sigma} v_i 1_{\Omega_j} d\boldsymbol{x}. \end{aligned} \tag{15}$$

From (15) it is clear that function v_i satisfies

$$\boldsymbol{A}^* v_i = \rho v_i + \varphi_i 1_{\Omega_i}, \quad i = 1 \ldots, J. \tag{16}$$

The scalar w_i can be computed from

$$w_i = \frac{1}{\rho} \int_\Omega U((-v_i)^{-1/\sigma}) 1_{\Omega_i} d\boldsymbol{x} + \frac{1}{\rho} \int_\Omega \sum_{j=1}^J (-v_j)^{-1/\sigma} v_i 1_{\Omega_j} d\boldsymbol{x}. \tag{17}$$

Summarizing we have proved the following proposition

Proposition 2. *There exists a stationary Markov Perfect Nash Equilibrium of the differential game (1)- (2), such that the value function of player i has the form (7) where v_i is the unique solution of the elliptic problem*

$$\begin{aligned} \nabla \cdot (k \nabla v_i) - \boldsymbol{b} \cdot \nabla v_i - c v_i - \rho v_i = \varphi_i 1_{\Omega_i}, &\quad \textit{in } \Omega \\ \alpha v_i + k \nabla v_i \cdot \boldsymbol{n} = 0, &\quad \textit{on } \partial\Omega \end{aligned} \tag{18}$$

and $w_i \in \mathbb{R}$ is given by (17). The strategy of player i, u_i, is positive and can be computed by (14).

Furthermore, the stationary steady state of the pollution stock is given by the solution of the elliptic problem

$$\begin{aligned} \nabla \cdot (k \nabla P) + \boldsymbol{b} \cdot \nabla P - cP + \sum_{j=1}^{J} u_j 1_{\Omega_j} = 0, \quad &\textit{in } \Omega \\ \alpha P + k \nabla P \cdot \boldsymbol{n} = 0, \quad &\textit{on } \partial\Omega. \end{aligned} \tag{19}$$

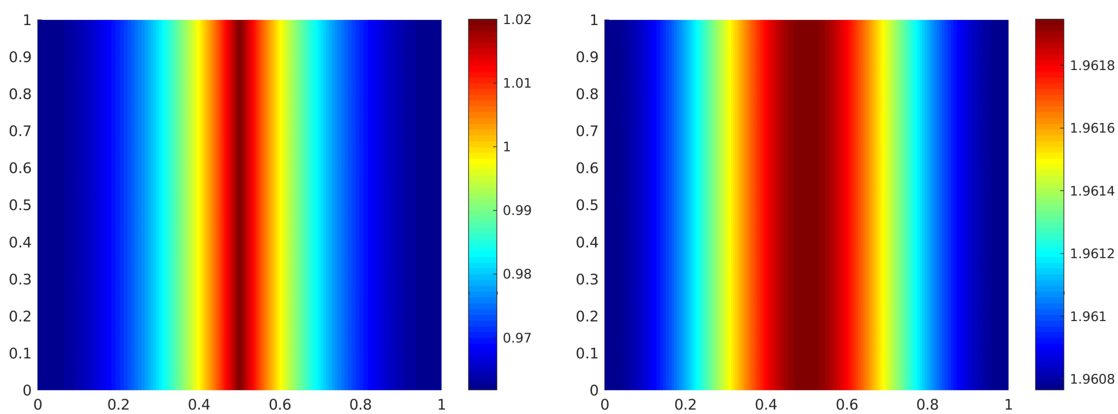
4 Numerical examples

In this section we present some examples in order to illustrate the effects of the spatial configurations on the strategic behaviour of the agents. We remark that even with this simple formulation of the problem it is evident that on the one hand, the model is able to capture the differences that can be expected in the equilibrium policies. On the other hand, the examples show that, as expected, the optimal location of emissions as well as its size depend on the geographical position among the players.

We fixed the following values of the parameters: $c = 0.5$, $\varphi_i = 1$, $\rho = 0.01$, $k = 1$. Also and without any loss of generality we choose $P_b = 0$. We remark that with this choice of the parameters all the agents are symmetric except, perhaps, for the geographic relative position. It should be apparent that the qualitative results do not depend on the particular values chosen if we restrict our study to constant coefficients (isotropic diffusion). The case of anisotropic diffusion, although interesting, is out of the scope of this paper.

For brevity in what follows we only present the results for a logarithm utility function ($\sigma = 1$). For values of σ different from one, the results are qualitatively similar. See Remark 1.

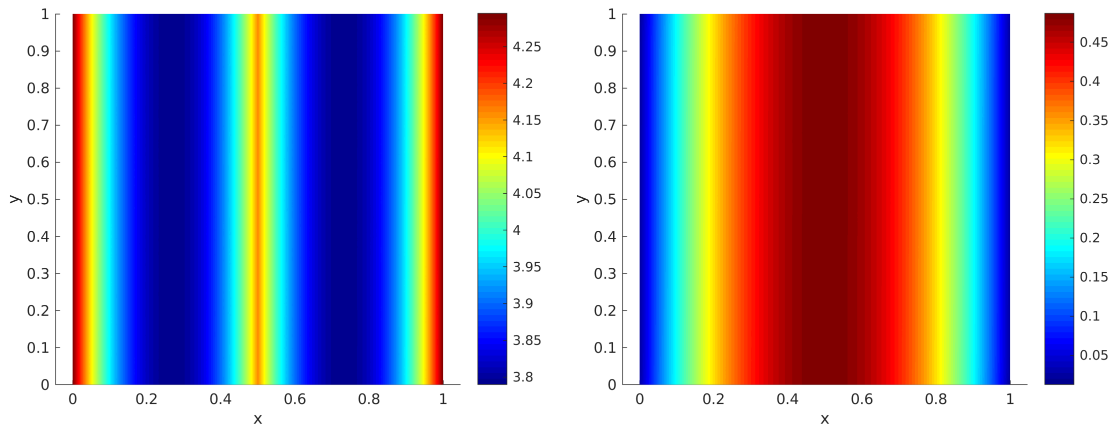
Example 1. In this example we consider a rectangular region Ω subdivided in two identical countries Ω_i, $i = 1, 2$. More explicitly $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 = [0, 0.5] \times [0, 1]$ and $\Omega_2 = [0.5, 1] \times [0, 1]$, respectively. We consider that region Ω is completely isolated from the exterior, that is we put $\alpha = 0$ in (2). We consider $\boldsymbol{b} = \boldsymbol{0}$ (no convection) in this example. The left picture in Figure 1 represents the emissions and the right picture represents the stock of pollution at steady state. We fix this convention for the rest of examples in this paper. We can observe that both countries behave in a symmetric manner as corresponds to the completely symmetric geometry. On the one hand, each country chooses the emission rate symmetrically with respect to the horizontal axis. On the other hand, both countries emit more near the common boundary between Ω_1 and Ω_2. The emission rate decreases as the distance from the common boundary increases. Because both regions are isolated from outside both strategically decide to reduce as much as possible the emission rates at the points in space where the exchange of pollution stock with its neighbour is more difficult. The steady-state levels of the pollution stock compare as the equilibrium emission rates.



Emissions **Pollution Stock**

Figure 1: Two isolated symmetric countries.

Example 2. The geometry of this example is identical to the previous one with the difference that we consider that $\alpha = 1$ in the part of the boundary defined by $x = 0$ and $x = 1$ and $\alpha = 0$ in the lines defined by $y = 0$ and $y = 1$. This choice models a situation in which the region is isolated from the outside in the top and bottom boundaries defined by $y = 0$ and $y = 1$, and can freely exchange pollution with the clean ($P_b = 0$) exterior through the vertical lines $x = 0$ and $x = 1$. The behaviour is similar to the previous example except for the effect of the open vertical boundaries. The two countries are able to detect that they can get rid of part of the pollution stock at no cost and, consequently, they increase the emission near this open boundary, see Figure 2. Note that the size of the emissions is larger than in Example 1 whereas the stock of pollution is smaller, clearly reflecting the effect of the open boundaries.

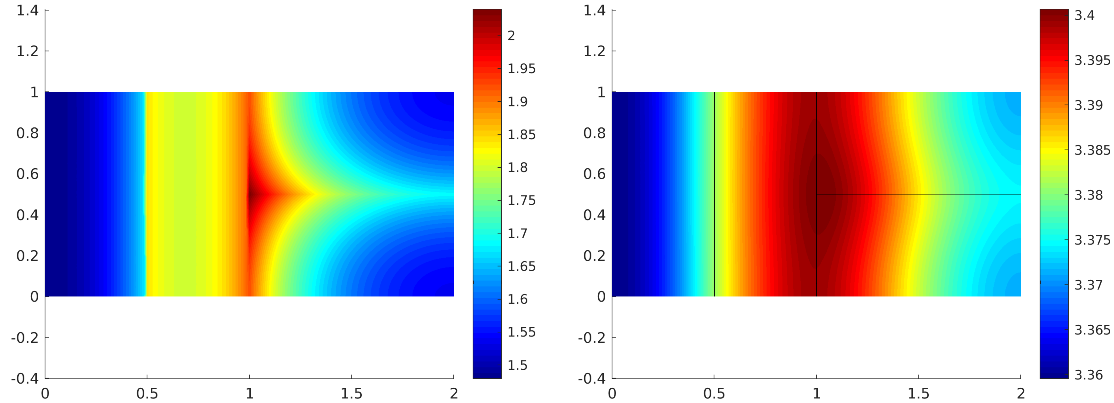


Emissions **Pollution Stock**

Figure 2: Two symmetric countries. Open boundaries

Example 3. In this example we consider again a rectangular region $\Omega = [0,2] \times [0,1]$, but this time subdivided in four identical countries with different relative positions. More explicitly the countries are defined by $\Omega_1 = [0,0.5] \times [0,1]$, $\Omega_2 = [0.5,1] \times [0,1]$, $\Omega_3 = [1,2] \times [0,0.5]$ and $\Omega_4 = [1,2] \times [0.5,1]$. As in Example 1 Ω is isolated from the exterior ($\alpha = 0$ in (2)) and there is no convection ($\boldsymbol{b} = \mathbf{0}$). We can observe in Figure 3 that the smaller the number of neighbouring countries, the lower the equilibrium emissions level. This result is consistent with the two previous examples: Each country knows that its own emissions are being diffused away so they increase the emissions if they have a larger number of neighbours. This comes from the fact that in this model, the positive effect of the emissions is not shared with other countries, whereas the negative effect of the concentration of pollution is shared through the diffusive state equation. We can also observe the effect of the relative position on the equilibrium strategies. The emissions in Ω_3 and Ω_4, which are completely symmetric, are much higher than expected in the proximity of Ω_2, because both countries recognize the increase of emissions in Ω_2. On the contrary, the emissions along the common boundary of Ω_3 and Ω_4 decrease as the distance to Ω_2 increases. As in Example 1, the steady-state levels of the pollution stock compare as the equilibrium emission rates.

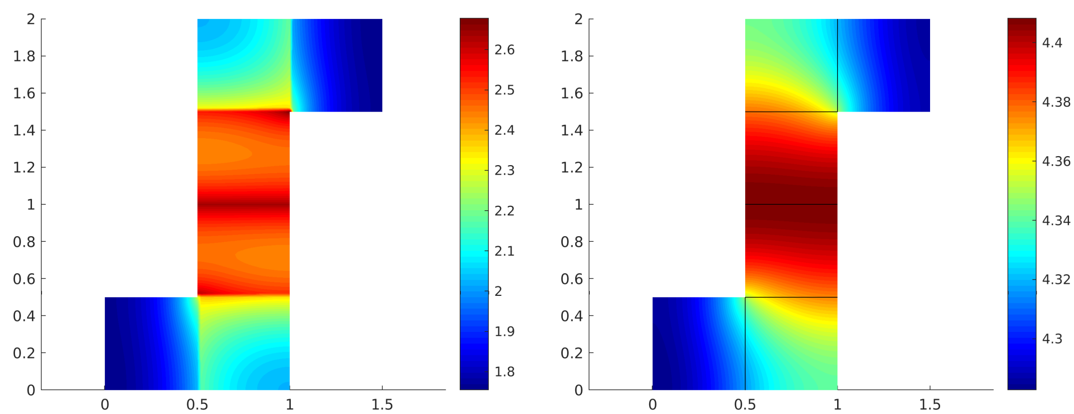
Example 4. Next three examples model a situation in which six, otherwise identical countries, are consecutively positioned along a channel of, say, groundwater forming a (double) L-shaped geometry. The geometry is defined by $\Omega_1 = [0,0.5] \times [0,0.5]$, $\Omega_2 = [0.5,1] \times [0,0.5]$, $\Omega_3 = [0.5,1] \times [0.5,1]$, $\Omega_4 = [0.5,1] \times [1,1.5]$, $\Omega_5 = [0.5,1] \times [1.5,2]$ and $\Omega_6 = [1,1.5] \times [1.5,2]$.



Emissions **Pollution Stock**

Figure 3: Four countries. Different number of neighbours

In this example region $\Omega = \bigcup_{j=1}^{6} \Omega_j$ is isolated from the exterior ($\alpha = 0$ in (2)) and there is no convection ($\boldsymbol{b} = \mathbf{0}$).

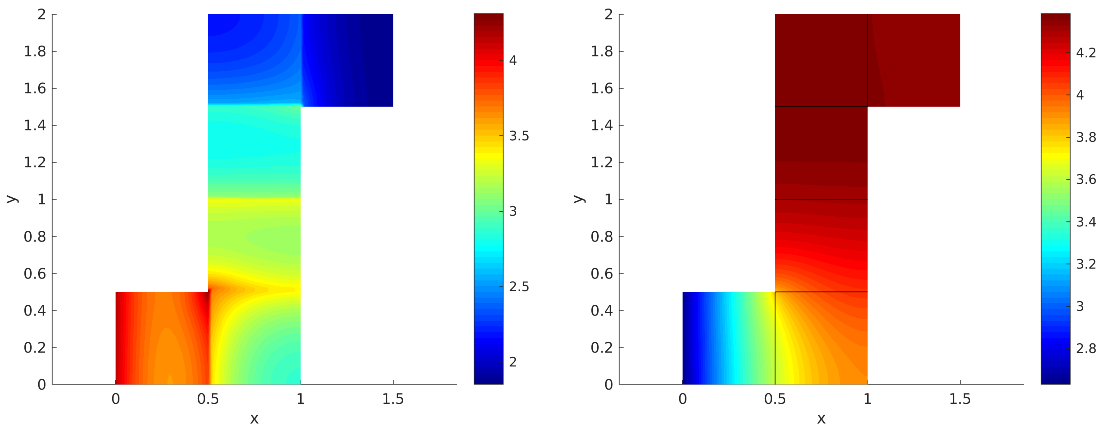


Emissions **Pollution Stock**

Figure 4: Six countries in an isolated L-shaped domain.

In Figure 4 we observe again the dependence of each country emissions on the number of its neighbours. In this picture we can observe a new feature not presented in the previous examples: The dependence on the distance from the boundary. The emissions in countries Ω_3 and Ω_4 are the highest among the six countries because their neighbours Ω_2 and Ω_5 have themselves another farther neighbour and the diffusion can spread the stock of pollution to a larger and more distant area. We shall use this example to compare with the next and last two examples.

Example 5. The geometry of this example is the same as in Example 4 except that now country Ω_1 has an open boundary defined by the line $x = 0$. That is, $\alpha = 0$ in $\partial\Omega$ except in $\partial\Omega \cap \{x = 0\}$ where $\alpha = 1$. Figure 5 shows how a small change in the geographical setting can cause a dramatic change in the equilibrium emissions which is reflected in the change of the steady state of the stock of pollution. Country Ω_1 that has the open boundary takes advantage of this fact emitting at the highest level (see left picture in Figure 5). The emissions of the rest of the countries are lower the longer the distance to the open boundary. However, the stock of pollution follows the opposite pattern: the larger the distance to the open boundary, the greater the steady-state stock of pollution.



Emissions **Pollution Stock**

Figure 5: Six countries in an L-shaped domain. Open boundary

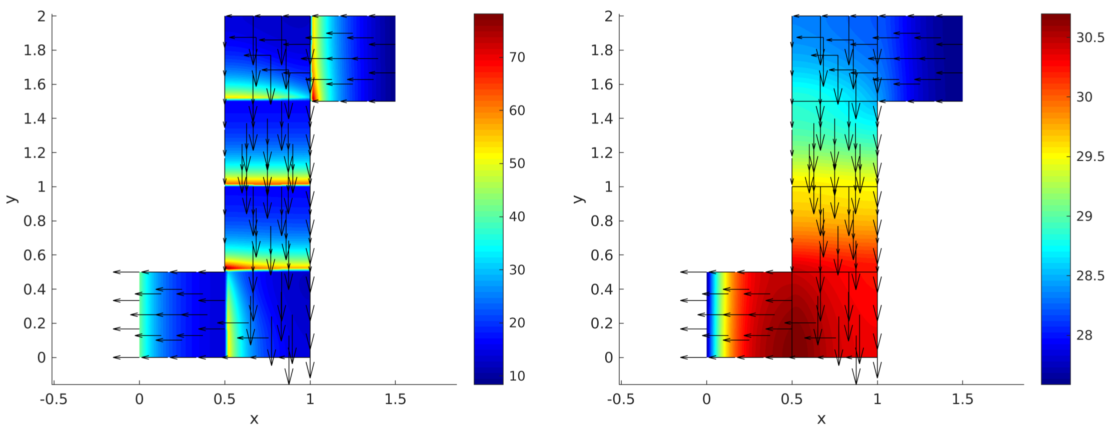
It is also interesting to compare the size of emissions in Examples 4 and 5. The following facts are apparent from Figure 4 and Figure 5. First, the emissions are larger in average if Ω has some part of the boundary open to exchange pollution with the exterior. Of course for this to be true, we have to consider that the exterior is clean ($P_b = 0$). However, the size of the steady-state stock of pollution is about the same size in Figure 4 (right picture) and Figure 5 (right picture). Then although the effect of the open boundary is positive for the overall region Ω, because higher emissions with the same stock of pollution lead to higher welfare, this is not the case if we individually analyze each one of the countries Ω_i. Clearly, Ω_6 suffers the consequences of the relocation of the stock of pollution which can give, depending on the parameters value, a lower welfare for this country.

Example 6. In the last example we consider again the L-shaped domain of Example 5.

However, this time we consider a convective flow given by

$$\boldsymbol{b}(\boldsymbol{x}) = \begin{cases} (4,0), & \boldsymbol{x} \in \Omega_1 \cup \Omega_6 \cup \big(\Omega_2 \cap \{(x,y), y < x-1\}\big) \cup \big(\Omega_5 \cap \{(x,y), y \geq 5/2 - x\}\big) \\ (0,4), & \boldsymbol{x} \in \Omega_3 \cup \Omega_4 \cup \big(\Omega_2 \cap \{(x,y), y \geq x-1\}\big) \cup \big(\Omega_5 \cap \{(x,y), y < 5/2 - x\}\big). \end{cases}$$

This means that in this example we are considering an external convective field in the direction of $-\boldsymbol{b}(\boldsymbol{x})$ which is represented in Figure 6. The rest of the parameters and boundary conditions are the same as in Example 5.



Emissions **Pollution Stock**

Figure 6: Six countries in an L-shaped domain with convection.

We note that the convective field $\boldsymbol{b}$ is not parallel to the boundary in $\Omega_1 \cap \{(x,y), x = 0\}$, where an outward flow is prescribed and in $\Omega_6 \cap \{(x,y), x = 1.5\}$, where an inward flow is prescribed. The adjoint problem to be solved in each subdomain Ω_i, $i = 1, \ldots, 6$, in this particular problem is

$$\begin{aligned} \nabla \cdot (k \nabla v_i) - \boldsymbol{b} \cdot \nabla v_i - c v_i - \rho v_i = \varphi_i 1_{\Omega_i}, &\quad \text{in } \Omega \\ \nabla v_i \cdot \boldsymbol{n} - (\boldsymbol{b} \cdot \boldsymbol{n}) v_i = 0, &\quad \text{on } \Gamma_N \\ \nabla v_i \cdot \boldsymbol{n} + (1 - \boldsymbol{b} \cdot \boldsymbol{n}) v_i = 0, &\quad \text{on } \Gamma_R \end{aligned}$$

where $\Gamma_R = \partial\Omega \cap \{(x,y), x = 0\}$ and $\Gamma_N = \partial\Omega \backslash \Gamma_R$.

In Figure 6 we observe that there is a change in scale due to the size of the convective field $\boldsymbol{b}$. Qualitatively the effect of the convective flow is clear. The countries concentrate their emissions near the boundary of the downstream neighbour. That is, the countries profit from the fact that pollution is transported away with the flow whereas the positive effect of the emissions in the welfare remains in the country. The steady state of the stock of pollution also presents some interesting and new characteristics. We can clearly see in the right picture of Figure 6 that most part of the stock of pollution is concentrated downstream, even if the size of emissions is similar in the six countries. Interestingly, we also observe that near the boundary Γ_R where the outward flow drains the pollution out of Ω the stock of pollution is remarkably low due to the open boundary.

Remark 1. It is clear from (14) that the results concerning emission rates and pollution stock are qualitatively similar regardless the value of parameter σ. The difference being in the scale of the emission rates, and consequently, of the pollution stock. As the elasticity of intertemporal substitution increases (σ decreases), the countries augment their emissions. This more aggressive behavior reproduces what has been called the voracity effect in the literature (Tornell & Lane (1999)).

5 Concluding remarks

This paper studies a transboundary pollution dynamic game that takes into account both the temporal and the spatial dimension of the environmental-economic problem. The spatial-temporal evolution of the stock of a pollutant is described by a diffusion partial differential equation and general boundary conditions are assumed. The main difference with respect to most of the recent literature that adds the spatial aspect in the study of different economic and environmental problems is that instead of a single decision maker, several decision makers and strategic interactions between them are considered. As far as we know only these two papers De Frutos & Martín-Herrán (2019a, 2019b)) have introduced the strategic interactions between the decision-makers. These papers follow a spatial discretization approach to characterize the equilibrium emission strategies of transboundary pollution dynamic games with spatial effects. In this paper we depart from the spatial discretization approach and characterize the feedback Nash equilibrium emission strategies of the J-player model formulated in continuous space and continuous time with two spatial dimensions and one temporal dimension. For a linear-state specification of transboundary pollution dynamic game inspired in Jørgensen & Zaccour (2001) and De Frutos & Martín-Herrán (2019b) we can explicitly compute the players' value functions and hence, the feedback Nash equilibrium emission strategies. These strategies are constant in time but remarkably they are space-dependent. To the best of our knowledge, we are the first in the literature to characterize the optimal intraregional distribution of the emissions. The present approach allows us, first, to determine the average total emission in each region (as in De Frutos & Martín-Herrán (2019a, b) using the space-discretized model); and second, the optimal spatial location of the emissions of each region. Our analytical results and examples with different geographical configurations show that the spatial aspects play an important role in the determination of the equilibrium emission strategies.

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