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Optimal policy for multi-item systems with stochastic demands, backlogged shortages and limited storage capacity

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ABSTRACT

In this paper, an inventory model for multiple products with stochastic demands is developed. The scheduling period or inventory cycle is known and prescribed. Demands are independent random variables and they follow power patterns throughout the inventory cycle. For each product, an aggregate cycle demand is realized first and then the demand is released to the inventory system gradually according to power patterns within a cycle. These demand patterns express different ways of drawing units from inventory and can be a good approach to modelling customer demands in inventory systems. Shortages are allowed and they are fully backlogged. It is assumed that the warehouse where the items are stored has a limited capacity. For this inventory system, we determine the inventory policy that maximizes the expected profit per unit time. An efficient algorithmic approach is proposed to calculate the optimal inventory levels at the beginning of the inventory cycle and to obtain the maximum expected profit per unit time. This inventory model is applicable to on-line sales of a wide variety of products. In this type of sales, customers do not receive the products at the time of purchase, but sellers deliver goods a few days later. Also, this model can be used to represent inventories of products for in-shop sales when the withdrawal of items from the inventory is not at the purchasing time, but occurs in a period after the sale of the products. This inventory model extends various inventory systems studied by other authors. Numerical examples are introduced to illustrate the theoretical results presented in this work.

1. Introduction

Inventory management refers to decision making on the process of ordering, storing, and holding items in any company. This process includes the management of raw materials, components, and finished products, as well as warehousing and processing such items. Companies can use different inventory management methods according to the type of business or product being analysed. One of these management methods takes the economic order quantity into account. This model

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is used in inventory management by calculating the number of units a company should order or produce to replenish the inventory, minimizing the total cost of its inventory, while it is assumed that consumer demand is constant, and no shortages are allowed. Another of the management methods considers a prescribed scheduling period and seeks to determine the optimal inventory level of every item at the beginning of the inventory cycle. This model is called an order-level model and allows shortages in the inventory. Shortages occur when there is not enough stock to satisfy customer demand. In this case, shortages may be backordered or be lost sales. We assume in this work that shortages are completely backordered. This means that customers would be willing to wait for the delivery of new merchandise to meet their demands. The costs of inventory in the model include holding, backlogging, and ordering costs. The order-level model seeks to ensure that the right amount of inventory is ordered per batch, so a company does not have to make orders very frequently, does not have too many shortages and there is not an excess of inventory sitting on hand.

1.1. Problem statement

This article considers a multi-item inventory system with a known scheduling period. The planning horizon does not need to be finite, nor do the demands need to be integers. Demands are independent random variables, and they follow power patterns throughout the inventory cycle. For each product, an aggregate cycle demand is realized first and then the demand is released to the inventory system gradually according to power patterns within a cycle. Thus, if the power demand pattern indexes of products tend to infinite, then we get the multi-item newsboy problem. Therefore, this paper extends the newsvendor model. The inventory system is developed assuming full backlogging, that is, the backorders are satisfied with the arrival of the next replenishment. According to the assumptions of the inventory system, a mathematical model is presented to determine the optimal inventory policy that maximizes the expected profit per unit time. Thus, the optimal inventory levels at the beginning of the inventory cycle, the economic lot sizes, the reorder points, the minimum expected inventory cost, and the maximum expected profit per unit time are established. In addition, when the warehouse space required for the storage of products has a limited capacity, a new optimal policy for the inventory system is proposed. Hence, this last policy extends the optimal solution of the newsvendor problem with a single linear resource constraint. The assumption of limited storage allows realistic inventory models to be analysed and this assumption provides a more complete approach to the management decision-making.

The model studied in this article can be adapted and applied to several real situations of the management and commercialization of products. For example, the on-line sale of clothing, accessories or footwear products requires that the distribution of the products to customers does not take place at the time of purchase, but rather it takes place a few days later. The same occurs with the on-line sale of products with high-tech components such as cell phones, computers, tablets, etc., where an aggregate demand is first requested for each product and then the demand is released from the inventory gradually within the inventory cycle. In addition, other real examples, where this stock management model is suitable, arise in the sale of household appliances such as refrigerators, washing machines, dryers, dishwashers, etc. In these cases, after customers confirm the purchase of any of these products, the supply of the same and, consequently, their extraction from the inventory, are performed in days after the sale. Therefore, the inventory model studied in this article can be adapted and applied to the management and commercialization of these products.

1.2. Literature review

1.2.1. Stochastic demand

The classical inventory models assume that customer demand is known and constant throughout time. However, in a lot of real systems, demand for an item may be uncertain and must be considered a random variable. One of the most well-known inventory problems where random demand for the product is assumed is the so-called newsboy problem or newsvendor problem. This inventory model considers a single decision period and represents the situation faced by a newspaper vendor who must decide how many newspapers to stock in the face of uncertain demand and knowing that unsold newspapers will be worthless at the end of the day. The optimal stocking quantity or inventory level of the newsvendor which minimizes the expected cost is the value of the generalized inverse cumulative distribution function at a critical point. This critical point represents the ratio or proportion between the unit backlogging cost and the sum of the unit holding cost and the unit backlogging cost. Several works have studied the newsboy problem considering various additional characteristics that condition the search for the optimal solution. Qin et al. [1] examined specific extensions of the newsvendor problem in the context of modelling customer demand, supplier costs, and the buyer risk profile. Choi [2] presented in a handbook different models, extensions, and applications of the newsvendor problem. Watt and Vázquez [3] studied the standard newsboy problem under two assumptions: the wholesaler is an expected profit maximiser who sets the wholesale price optimally and then considers the salvage value at which the newsboy can return unsold items to the wholesaler, and that the salvage value is a choice variable of the newsboy, and thus, it acts as a standard insurance device. Mitra [4] studied the newsvendor problem assuming that the salvage/clearance price is a decision variable. He presented three stochastic inventory models considering that the seasonal demand and end-of-season demand can be endogenous or exogenous. Surti et al. [5] studied the cause of the deviation of the newsvendor ordering from the optimal risk-neutral behaviour. They explored the prospect theory as possible explanation to this deviation. Our work extends the inventory model of the newsboy problem, allowing that the extraction of items from the inventory is not done instantly, but items are extracted from the inventory throughout the inventory cycle. In addition, the inventory model is analysed considering a storage capacity constraint.

Many researchers have also analysed other models where demands are stochastic and follow certain probability distributions during the inventory cycle. Girlich [6] considered an EOQ model with non-linear demand patterns and proved that, under certain conditions, Harris' formula for optimal lot size was correct even in the stochastic case. Eynan and Kropp [7] analysed a periodic review system under stochastic demand with variable stock-out costs. Sana [8] developed an economic order quantity model for stochastic demand when the capacity of own warehouse is limited and a rented warehouse can be used, if needed. Chuang and Chiang [9] empirically tested the classic EOQ model, with respect to its original form and assumptions, in the U.S. automobile industry. They proved that demand uncertainty has a highly significant impact on order quantity. Maddah and Noueihed [10] considered a variant of the EOQ model, assuming that demand was generated from a renewal stochastic process. They showed that the optimal order quantity that minimizes the expected inventory cost follows the classic EOQ formula. More recently, Liao and Deng [11] developed an evolved environmental sustainability EOQ model to optimize the environmental indexes in remanufacturing systems under uncertainties in both acquisition quantity and market demand. Dos Santos and Oliveira [12] developed an inventory management model with periodic review policy under demand uncertainty with the possibility of considering partial backorder. Zhang et al. [13] analysed the problem of the inventory decision of a single type of building material under the dual constraints of stochastic demand and the carbon cap-and-trade mechanism. Sarkar et al. [14] developed a two-echelon supply chain model with shortages under stochastic lead time which is assumed to be a function of order quantity and production rate.

1.2.2. Power demand pattern

To meet customer demand, items are usually withdrawn from inventory at a constant rate per unit time. However, there exist other ways or alternatives to remove quantities from the inventory. Naddor [15] proposed a function dependent on time to represent diverse ways to take out stock units. This function, which describes the behaviour of demand during the inventory cycle, is known as the power demand pattern. It is a time-dependent power function, which is appropriate to describe the variation of the inventory level over time. This demand pattern represents different ways to take out units from inventory and extends the classical constant demand rate. Therefore, the power demand pattern may represent other situations where customer demands are initially low and then increase during the inventory cycle, or when demands are high at the beginning of the inventory cycle and then decrease gradually over time. Thus, the power demand pattern can be a good approach to represent customer demand in the modelling of inventory systems.

Several inventory models that assume a power demand pattern during the inventory cycle have been analysed by different researchers. Thus, Lee and Wu [16] proposed an economic order quantity model for items with deterioration, shortages, and a power demand pattern. Singh et al. [17] analysed an inventory model for perishable items with power demand and considering partial backordering. Rajeswari and Vanjikkodi [18] developed a deteriorating inventory model with power demand and partial backlogging. Sicilia et al. [19] studied several deterministic inventory systems with power demand pattern. Mishra and Singh [20] presented a computational approach on a partial backlogging inventory model for queued customers with power demand and quadratic deterioration. Sicilia et al. [21] studied an economic order quantity model for a power demand pattern system with deteriorating items. Sicilia et al. [22] developed an inventory model for deteriorating items with shortages and time-varying demand. Sicilia et al. [23] presented the optimal policy for an inventory system with power demand, backlogged shortages and production rate proportional to demand rate. Sicilia et al. [24] studied optimal inventory policies for uniform replenishment systems with time-dependent demand. Later, Tripathi et al. [25] proposed inventory models with power demand and inventory-induced demand with holding cost functions. San-José et al. [26] studied the optimal inventory policy under power demand pattern and partial backlogging. More recently, San-José et al. [27] developed an inventory system with demand dependent on both time and price, assuming backlogged shortages. Keshavarzfard et al. [28] analysed the optimization of imperfect economic manufacturing models with a power demand rate dependent production rate. San-José et al. [29] analysed the optimal price and quantity under power demand pattern and non-linear holding cost. San-José et al. [30] studied an inventory system with discrete scheduling period, time-dependent demand and backlogged shortages. Adaraniwon and Omar [31] presented an inventory model for delayed deteriorating items with power demand considering shortages and lost sales. San-José et al. [32] developed the best pricing and optimal policy for an inventory system under time-and-price-dependent demand and backordering. In these above papers, many of the inventory systems refer to deterministic systems with a single product. However, in this work, we study a probabilistic multi-item inventory system and develop the optimal inventory policy for this system.

1.2.3. Multi-item inventory system

Several researchers have analysed multi-product inventory systems. Nahmias and Schmidt [33] developed various efficient heuristic solution procedures for the multi-item newsboy problem subject to a single constraint, which expresses limited storage capacity or bounded budget. Lau and Lau [34] presented formulations and solution procedures for handling multiple newsboy-type products under one resource-capacity constraint and when multiple resource constraints are considered. Zhang [35] analysed the multi-product newsboy problem with supplier quantity discounts and a budget constraint. Nia et al. [36] developed a multi-item multi-constraint EOQ model with shortage for a single-supplier single-buyer supply chain under green vendor managed inventory policy. Cárdenas-Barrón and Sana [37] presented a multi-item EOQ model in a two-layer supply chain where demand is sensitive to the promotional effort from a collaborative perspective. Dordevic

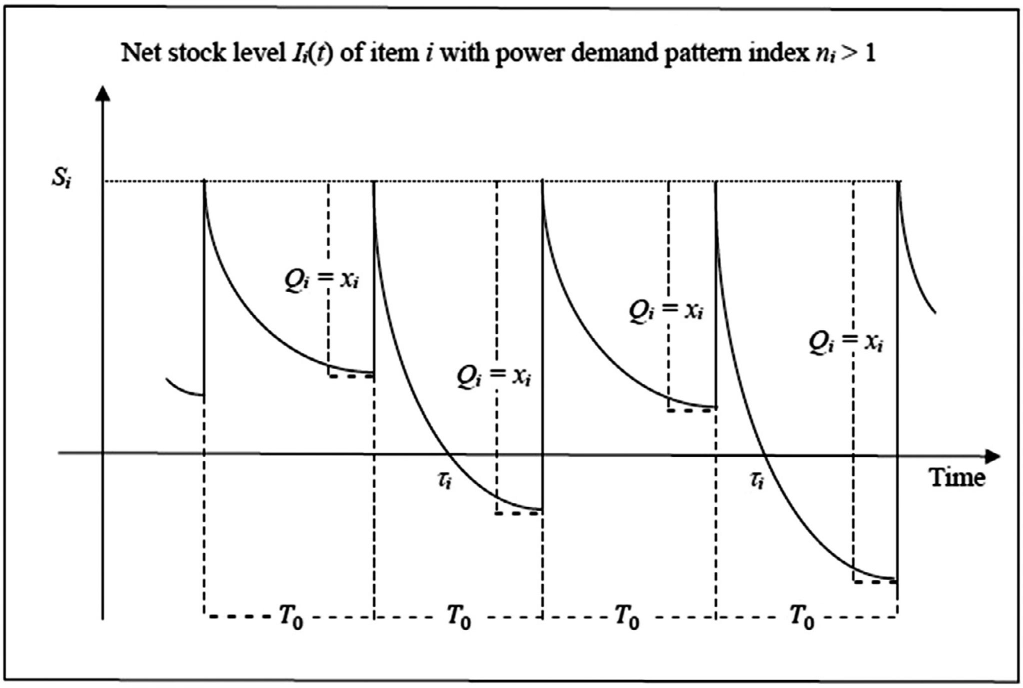

Fig. 1. Graphic of the net inventory level for the i th item considering stochastic demand and a power demand pattern with index $n_i > 1$ throughout the inventory cycle.

et al. [38] developed the static time-continuous multiproduct EOQ-based inventory problem with storage space constraints, modelled as a combinatorial optimization problem and approximately solved using two heuristics. More recently, Malik and Sarkar [39] considered a continuous-review inventory model for multi-items, assuming stochastic lead time and service level and storage space constraints. Hormozzadefighalati et al. [40] developed a spatial model for a multi-product supplier selection model with service level and budget constraints. Khouja et al. [41] developed two heuristics for optimizing the newsvendor's response to new in-season demand information for a multi-product single-period problem.

1.3. Outline of the article

The paper is structured as follows. Section 2 introduces the properties of the inventory system to be studied, presents the notation used in the article and describes the evolution of the inventory level of each product. Section 3 analyses the probabilistic inventory model for multiple items with power demand patterns and fixed scheduling period. Next, the equivalence of the problems on the maximization of the expected profit per unit time and the minimization of the expected inventory cost per unit time is established. Thus, the problem of minimizing the expected total inventory cost per unit time for the full backlogging probabilistic inventory model is formulated and the optimal solution for this problem is developed. Section 4 determines the optimal inventory policy for the probabilistic multi-item system with power demand patterns when there exists limited storage capacity. An efficient algorithm is presented to solve this inventory problem. Section 5 describes the optimal inventory levels when demands follow Pareto distributions. Numerical examples to illustrate the theoretical results previously obtained are introduced in Section 6. A sensitivity analysis of the optimal inventory policies is presented in Section 7. Some managerial insights are commented in Section 8. Finally, in Section 9, we discuss some aspects of the optimal inventory policy proposed in this paper and present some future research lines in inventory management.

The flow chart of Fig. 1 shows the steps that have been taken in doing this paper.

2. Hypotheses and notation of the inventory system

The paper deals with a multi-item inventory system in which the demand for products is not known with certainty. This system works under the following hypothesis or characteristics:

- The inventory system considers several different items.
- Items are jointly replenished every certain time-period or inventory cycle. This scheduling period is known and prescribed.

- The replenishment is instantaneous for all the products.
- The lead time of each item is insignificant or null.
- Shortages are allowed and fully backlogged.
- Demands for items are random variables that follow known probability distributions.
- Items are withdrawn throughout the period following power demand patterns.
- A probabilistic order-level multi-item system is analysed, in which the optimal inventory level for each item at the beginning of the inventory cycle must be determined.
- Three costs are considered in the inventory system: the expected holding cost, the expected shortage cost and the expected replenishing cost.
- The aim is to maximize the expected profit per unit time.

2.1. Notation

Next, we present the notation of the inventory system that is developed throughout the article:

- Let N be the number of items considered in the inventory system.
- Let T_0 represent the length of the scheduling period or inventory cycle. Items are jointly replenished every T_0 time units.
- Let Q_i denote the replenishment quantity or lot size of the ith item, with $i = 1, 2, \ldots, N$. This amount is equal to the demanded quantity during the inventory cycle.
- Let S_i represent the inventory level of the item i at the beginning of each inventory cycle, with $i = 1, 2, \ldots, N$. For each item i, the stock of this product goes up to the level S_i after the replenishment size Q_i is included in the inventory. These quantities S_i are unknown values and should be determined so that the expected profit per unit time is maximized.
- Let s_i be the replacement level or reorder point of the product i, with $i = 1, 2, \ldots, N$; that is, $s_i = S_i - Q_i$.
- Let h_i be the holding cost per unit time of a unit in stock of the ith item, with $i = 1, 2, \ldots, N$.
- Let A represent the replenishing or ordering cost. It is a known constant and independent of the quantity ordered for each item.
- Let ω_i denote the unit backlogging cost per unit time of the ith article, for $i = 1, 2, \ldots, N$.
- Let p_i be the unit selling price of the item i, with $i = 1, 2, \ldots, N$.
- Let c_i be the unit purchasing price of the item i, for $i = 1, 2, \ldots, N$. Obviously, it is assumed that $c_i < p_i$, for $i = 1, 2, \ldots, N$.
- Let v_i designate the storage space required per unit of the item i, with $i = 1, 2, \ldots, N$.
- Let W denote the total space available in the warehouse to store the items.
- Let X_i be the total demand along the inventory cycle T_0 of the ith item ($i = 1, 2, \ldots, N$). It is a nonnegative random variable with probability density function $f_i(x_i)$ defined on the interval $[a_i, b_i)$, with $a_i = \inf\{x_i \geq 0 : f_i(x_i) > 0\}$ and $b_i = \sup\{x_i \geq 0 : f_i(x_i) > 0\}$. The cumulative distribution function of X_i is denoted by $F_i(x_i)$. We assume that X_i (with $i = 1, 2, \ldots, N$) are independent random variables. Let μ_i denote the average demand of the item i during the period T_0.
- Demand accumulated up to time t for the ith item is denoted by $D_i(t)$, with $0 \leq t < T_0$. It is assumed that this demand is time-dependent and is given by

$$D_i(t) = X_i\left(\frac{t}{T_0}\right)^{1/n_i}, \quad i = 1, 2, \ldots, N \tag{1}$$

  For the ith article, the value n_i represents the index of demand pattern, with $n_i > 0$. If the demand pattern index of the ith item is $n_i > 1$, then a greater part of the demand occurs at the beginning of the period. Thus, if we consider that n_i tends to infinity, then the demand is totally concentrated at beginning of the inventory cycle (newsvendor case). If $n_i = 1$, the demand rate is constant throughout the scheduling period. When $n_i < 1$, a larger portion of the demand occurs toward the end of the inventory cycle. Hence, if we suppose $n_i \to 0$, then the demand is totally concentrated at the end of the inventory cycle.
  From (1), the demand rate at time t (with $0 < t < T_0$) for the $i-$ th item ($i = 1, 2, \ldots, N$) is

$$D_i'(t) = \frac{X_i}{T_0 n_i}\left(\frac{t}{T_0}\right)^{\frac{1-n_i}{n_i}} \tag{2}$$

  This demand rate is known as the power demand pattern. Several authors, such as Naddor [15], Lee and Wu [16], Tripathi et al. [25], Keshavarzfard et al. [28], and San-José et al. [32], have developed inventory systems considering this type of demand pattern when demand is deterministic. However, in this paper, power demand patterns with stochastic demands are assumed.
- Let $AQ_i(X_i, S_i)$ designate the average quantity held in stock of the item i during the period T_0.
- Let $AB_i(X_i, S_i)$ denote the average backlogging of the item i during the period T_0.
- Let RC be the replenishing cost per unit time.

- Let $EQ_i(S_i)$ be the expected average quantity held in stock of the item i during the period T_0.
- Let $EB_i(S_i)$ be the expected average backlogging of the item i during the period T_0.
- Let $EHC(S_1, S_2, \dots, S_N)$ represent the expected total holding cost per unit time.
- Let $EBC(S_1, S_2, \dots, S_N)$ denote the expected total backlogging cost per unit time.
- Let ERC be the expected replenishing cost per unit time.
- Let $C(S_1, S_2, \dots, S_N)$ denote the expected total inventory cost per unit time.
- Let $EP(S_1, S_2, \dots, S_N)$ represent the expected profit per unit time.

2.2. Evolution and behaviour of the inventory system over time

In this section, we analyse the fluctuations in the stock of items throughout the inventory cycle. The net inventory level $I_i(t)$ for the ith item at time t, with $0 \le t < T_0$, is understood as the difference between the stock level at the beginning of the inventory cycle and the demand accumulated up to time t, that is,

$$I_i(t) = S_i - X_i\left(\frac{t}{T_0}\right)^{1/n_i}, \quad 0 \le t < T_0 \tag{3}$$

The net inventory level $I_i(t)$ is a continuous and differentiable function on the time interval $[0, T_0)$. This function starts with $I_i(0) = S_i$ units and is decreasing due to demand throughout the inventory cycle.

Since b_i represents the superior extreme of the demand X_i, the inventory level S_i must be less than or equal to b_i. If b_i is a finite value and S_i is greater than b_i, then there would be an unnecessary excessive amount in stock of item i, and the holding cost could be reduced by considering an inventory level S_i less than or equal to b_i in the system.

If the demand X_i is less than or equal to the inventory level S_i, then no shortages occur in the system. However, if the demand X_i is greater than the inventory level S_i, there exists a time τ_i in which the inventory level of the ith product drops to zero. Next, the inventory level continues to decrease, and shortages occur. If x_i is a realization of demand during the period T_0, then the net inventory level $I_i(t)$ of the item i decreases until it reaches the value s_i at the end of the inventory cycle, and the inventory is replenished with a lot size of $Q_i = x_i$ units. Thus, if the demanded amount x_i is greater than the inventory level S_i, the quantity $s_i = S_i - x_i$ is negative and $-s_i$ represents the demand for the ith item which is not met (backorders). The replenishment quantity Q_i increases the stock up to level S_i and the behaviour of the inventory level is repeated on the next scheduling period.

Fig. 2 displays the fluctuations of the inventory level for the ith product during the inventory cycle, when demand is stochastic and follows a power demand pattern with index $n_i > 1$. That is, the way by which quantities are taken out of the inventory follows the time-dependent power function given by (2) with an index greater than one.

In the following paragraphs, we analyse the probabilistic inventory model for the system with multiple products and present the optimal inventory policy when shortages are fully backordered. We then develop the optimal policy assuming that a constraint on the storage capacity is also incorporated into the model.

3. Inventory model for multiple items with stochastic demands, power demand patterns, fixed scheduling period and fully backlogged shortages

From Fig. 2, it can be seen that, for any item i, the average quantity held in stock $AQ_i(X_i, S_i)$ and the average backlogging $AB_i(X_i, S_i)$ depend on the inventory level S_i and on the demand X_i. Thus, considering the relative values of the demand X_i and the inventory level S_i at the beginning of the period T_0, two possible situations arise in this system: (i) $X_i \le S_i$ and (ii) $X_i > S_i$. Both situations are shown in Fig. 2. In the first case, when $X_i \le S_i$, the average quantity in inventory $I_1(X_i, S_i)$ of the item i for a demand X_i is

$$AQ_i(X_i, S_i) = \frac{1}{T_0}\int_0^{T_0} I_i(t)dt = \frac{1}{T_0}\int_0^{T_0}\left(S_i - X_i\left(\frac{t}{T_0}\right)^{1/n_i}\right)dt = S_i - \frac{X_i n_i}{n_i + 1} \tag{4}$$

In this case, there are no shortages in the system. Thus, the average shortage $AB_i(X_i, S_i)$ is zero.

In the second case, when $X_i > S_i$, we have to calculate the time in which the inventory level $I_i(t)$ is zero for each item i. Thus, if τ_i denotes the point in time when the ith item inventory level drops to zero, then $I_i(\tau_i) = 0$ and, from (3), we have

$$\tau_i = \frac{S_i^{n_i} T_0}{X_i^{n_i}}, \quad i = 1, 2, \dots, N \tag{5}$$

Hence, from (5), the average quantity held in stock is

$$AQ_i(X_i, S_i) = \frac{1}{T_0}\int_0^{\tau_i} I_i(t)dt = \frac{1}{T_0}\int_0^{\tau_i}\left(S_i - X_i\left(\frac{t}{T_0}\right)^{1/n_i}\right)dt = \frac{S_i}{n_i + 1}\left(\frac{S_i}{X_i}\right)^{n_i} \tag{6}$$

The average backlogging is

$$AB_i(X_i, S_i) = \frac{1}{T_0}\int_{\tau_i}^{T_0} (-I_i(t))dt = \frac{-1}{T_0}\int_{\tau_i}^{T_0}\left(S_i - X_i\left(\frac{t}{T_0}\right)^{1/n_i}\right)dt = \frac{n_i X_i}{n_i + 1} + \frac{S_i}{n_i + 1}\left(\frac{S_i}{X_i}\right)^{n_i} - S_i \tag{7}$$

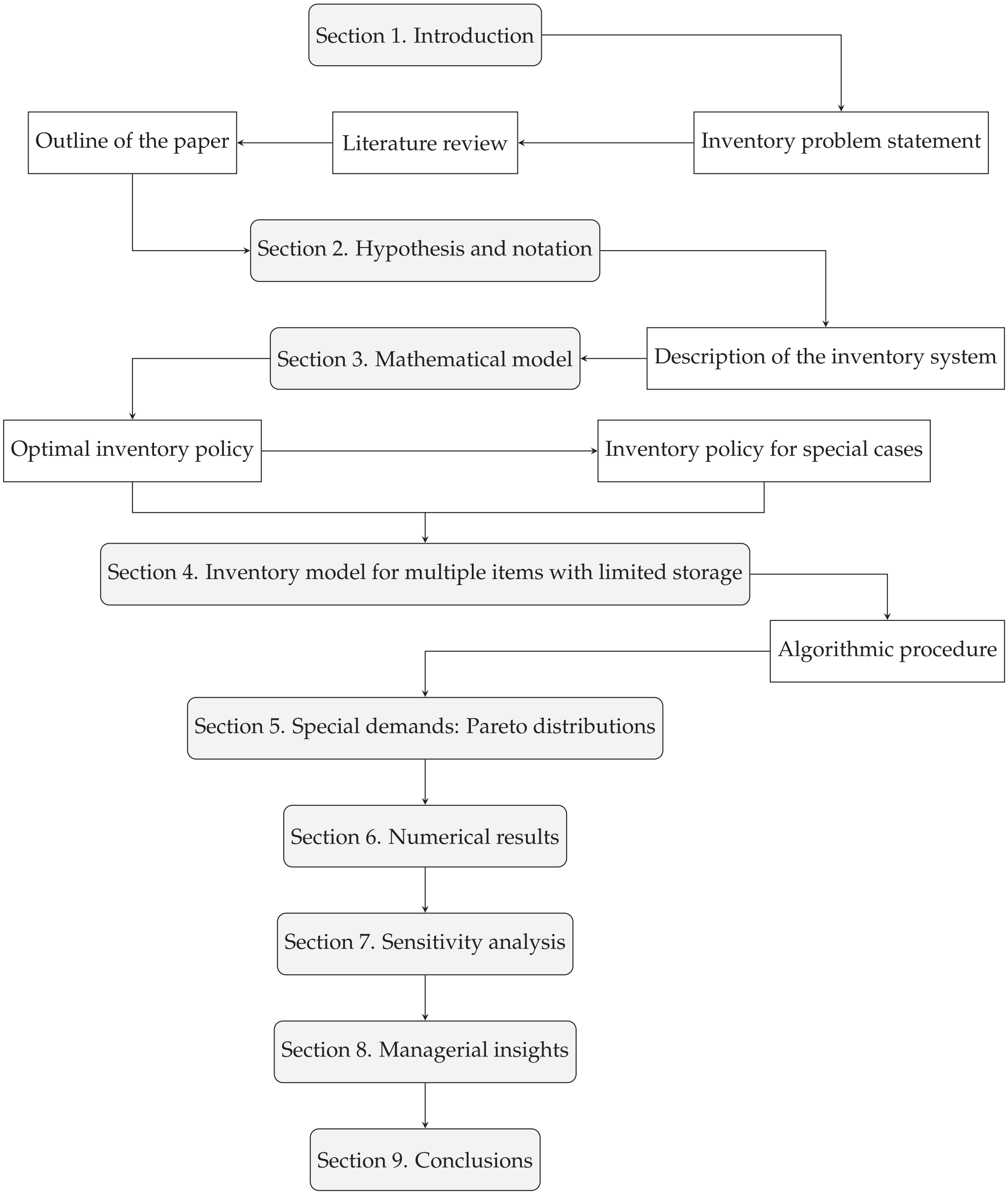

Fig. 2. Research framework flow chart.

Summarizing, from (4) and (6), the average amount held in the inventory for item i is

$$AQ_i(X_i, S_i) = \begin{cases} S_i - \frac{X_i n_i}{n_i+1}, & X_i \le S_i \\ \frac{S_i}{n_i+1}\left(\frac{S_i}{X_i}\right)^{n_i}, & X_i > S_i \end{cases} \tag{8}$$

and, from (7), the average shortage is

$$AB_i(X_i, S_i) = \begin{cases} 0, & X_i \le S_i \\ \frac{n_i X_i}{n_i+1} + \frac{S_i}{n_i+1}\left(\frac{S_i}{X_i}\right)^{n_i} - S_i, & X_i > S_i \end{cases} \tag{9}$$

Therefore, from (8), the expected average amount in the inventory $EQ_i(S_i)$ is

$$EQ_i(S_i) = E[AQ_i(S_i)] = \int_0^{S_i} \left(S_i - \frac{x_i n_i}{n_i+1}\right) f_i(x_i) dx_i + \int_{S_i}^{\infty} \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i \tag{10}$$

and, from (9), the expected average shortage $EB_i(S_i)$ is

$$EB_i(S_i) = E[AB_i(S_i)] = \int_{S_i}^{\infty} \left(\frac{n_i x_i}{n_i+1} + \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} - S_i \right) f_i(x_i) dx_i \tag{11}$$

For each item $i = 1, 2, \ldots, N$, the holding cost per unit time is $h_i EQ_i(S_i)$ and the backlogging cost per unit time is $\omega_i EB_i(S_i)$. Thus, the expected total holding cost is given by

$$\begin{aligned} EHC(S_1, \ldots, S_N) &= \sum_{i=1}^{N} h_i EQ_i(S_i) \\ &= \sum_{i=1}^{N} h_i \left[\int_0^{S_i} \left(S_i - \frac{x_i n_i}{n_i+1} \right) f_i(x_i) dx_i + \int_{S_i}^{\infty} \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i \right] \end{aligned} \tag{12}$$

and the expected total backlogging cost is

$$EBC(S_1, \ldots, S_N) = \sum_{i=1}^{N} \omega_i EB_i(S_i) = \sum_{i=1}^{N} \omega_i \left[\int_{S_i}^{\infty} \left(\frac{n_i x_i}{n_i+1} + \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} - S_i \right) f_i(x_i) dx_i \right] \tag{13}$$

The replenishing cost per unit time is

$$RC = \begin{cases} 0, & \textit{if } X_i = 0 \textit{ for all } i = 1, 2, \ldots, N \\ A/T_0, & \textit{otherwise} \end{cases}$$

Then, the expected replenishing cost is

$$ERC = \theta \frac{A}{T_0}$$

where θ is the probability given by

$$\theta = 1 - \Pr\{X_1 = 0, X_2 = 0, \ldots, X_N = 0\} = 1 - \prod_{i=1}^{N} \Pr\{X_i = 0\}$$

Notice that if the demands are continuous random variables, the probabilities of such demands to be zero are always null. Therefore, in this case, the expected replenishing cost is A/T_0.

The expected total inventory cost per unit time of the system is the sum of the above costs, that is

$$\begin{aligned} C(S_1, \ldots, S_N) &= EHC(S_1, \ldots, S_N) + EBC(S_1, \ldots, S_N) + ERC = \\ &= \sum_{i=1}^{N} h_i \int_0^{S_i} \left(S_i - \frac{x_i n_i}{n_i+1} \right) f_i(x_i) dx_i + \sum_{i=1}^{N} (h_i + \omega_i) \int_{S_i}^{\infty} \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i \\ &\quad + \sum_{i=1}^{N} \omega_i \int_{S_i}^{\infty} \left(\frac{n_i x_i}{n_i+1} - S_i \right) f_i(x_i) dx_i + \theta \frac{A}{T_0} \end{aligned} \tag{14}$$

Now, as the ordered quantity for item i or lot size Q_i is equal to the demanded quantity X_i, then $E[Q_i] = E[X_i]$. Thus, the expected benefit per unit time obtained from the sale of items in each inventory cycle is given by

$$E\left[\sum_{i=1}^{N} \frac{(p_i - c_i)Q_i}{T_0} \right] = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} E(Q_i) = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} \mu_i$$

The expected profit $EP(S_1, \ldots, S_N)$ per unit time during each inventory cycle is the difference between the expected benefit per unit time due to the sale of items and the expected total inventory cost per unit time given by (14), that is

$$EP(S_1, \ldots, S_N) = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} \mu_i - C(S_1, \ldots, S_N) \tag{15}$$

The aim is to maximize the above profit per unit time, which is equivalent to minimizing the expected inventory cost per unit time. Thus, the objective consists in determining the optimal values of the initial inventory levels S_i $(i = 1, 2, \ldots, N)$ such that the cost function $C(S_1, \ldots, S_N)$ given in (14) is minimized on the region $\Omega = \{(S_1, \ldots, S_N)/b_i \geq S_i \geq 0, \text{ with } i = 1, 2, \ldots, N\}$.

3.1. Optimal policy

To find the optimal policy that minimizes the expected inventory cost per unit time shown in (14), we first need to prove the following result.

Proposition 1. *For each item i, with $i = 1, 2, \dots, N$, the function*

$$Z_i(S_i) = \int_{S_i}^{\infty} \left(1 - \left(\frac{S_i}{x_i}\right)^{n_i}\right) f_i(x_i) dx_i \tag{16}$$

defined on the interval $[0, \infty)$ is a continuous function, which takes values in the interval (0,1]. In addition, this function is strictly decreasing on $[0, b_i)$.

Proof. See the Appendix. □

Notice that the function $Z_i(S_i)$ given by (16) can be rewritten as

$$Z_i(S_i) = 1 - \left(F_i(S_i) + \int_{S_i}^{\infty} \left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i\right)$$

Next, we prove the convexity of the expected inventory cost function and present the conditions that must satisfy the optimal inventory levels when there is no limitation on the storage space.

Theorem 1.

(i) *The expected inventory cost $C(S_1, \dots, S_N)$ proposed in (14) is a strictly convex function on the set $\{(S_1, \dots, S_N)/b_i > S_i > 0$, with $i = 1, 2, \dots, N\}$.*
(ii) *The optimal inventory levels S_i^0, with $i = 1, 2, \dots, N$, that minimize the inventory cost function $C(S_1, \dots, S_N)$ must satisfy the constraints*

$$\int_{S_i}^{\infty} \left(1 - \left(\frac{S_i}{x_i}\right)^{n_i}\right) f_i(x_i) dx_i = \frac{h_i}{h_i + \omega_i}, \quad i = 1, 2, \dots, N \tag{17}$$

Moreover, the point $(S_1^0, \dots, S_N^0) \in Int(\Omega)$.

Proof. See the Appendix. □

From Proposition 1, the Eq. (17) is equivalent to

$$Z_i(S_i) = \frac{h_i}{h_i + \omega_i}, \quad i = 1, 2, \dots, N \tag{18}$$

and, from Theorem 1, Eq. (18) has a unique solution S_i^0, for each $i = 1, 2, \dots, N$.

Thus, the optimal inventory levels are given by

$$S_i^0 = Z_i^{-1}\left(\tfrac{h_i}{h_i+\omega_i}\right), \quad i = 1, 2, \dots, N \tag{19}$$

The Eq. (18) is equivalent to

$$F_i(S_i) + \int_{S_i}^{\infty} \left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i = \frac{\omega_i}{h_i + \omega_i}, \quad i = 1, 2, \dots, N$$

That is, the optimality condition for the inventory level S_i of the item i is that the ratio $\omega_i/(h_i + \omega_i)$ is equal to the expected value of $m(X_i, S_i)$, where $m(X_i, S_i)$ is the function defined as $m(X_i, S_i) = \min\left\{1, \left(\frac{S_i}{X_i}\right)^{n_i}\right\}$.

From (12) and (19), the minimum expected holding cost per unit time is

$$\begin{aligned} EHC(S_1^0, \dots, S_N^0) &= \sum_{i=1}^{N} h_i EQ_i(S_i^0) \\ &= \sum_{i=1}^{N} h_i \left[\frac{n_i}{n_i + 1} \int_0^{S_i^0} \left(S_i^0 - x_i\right) f_i(x_i) dx_i + \frac{S_i^0 \omega_i}{(n_i + 1)(h_i + \omega_i)}\right] \end{aligned} \tag{20}$$

From (13) and (19), and considering that the average demand for the product i is μ_i, the minimum expected backlogging cost per unit time is

$$EBC(S_1^0,\dots,S_N^0)=\sum_{i=1}^{N}\omega_i EB_i(S_i^0)$$
$$=\sum_{i=1}^{N}\omega_i\left[\frac{n_i}{n_i+1}\int_0^{S_i^0}\left(S_i^0-x_i\right)f_i(x_i)dx_i+\frac{n_i}{n_i+1}\left(\mu_i-S_i^0\right)-\frac{S_i^0h_i}{(n_i+1)(h_i+\omega_i)}\right] \tag{21}$$

From (14) and (19), the minimum expected inventory cost per unit time is given by

$$C_0=C(S_1^0,\dots,S_N^0)=\sum_{i=1}^{N}(h_i+\omega_i)\frac{n_i}{n_i+1}\int_0^{S_i^0}\left(S_i^0-x_i\right)f_i(x_i)dx_i$$
$$+\sum_{i=1}^{N}\omega_i\frac{n_i}{n_i+1}\left(\mu_i-S_i^0\right)+\theta\frac{A}{T_0} \tag{22}$$

From (15) and (22), the maximum expected profit EP^0 per unit time is given by

$$EP^0=EP(S_1^0,\dots,S_N^0)=\sum_{i=1}^{N}\frac{(p_i-c_i)}{T_0}\mu_i-C(S_1^0,\dots,S_N^0)$$

Therefore, the inventory policy given by (19) and (22) is the optimal solution for the inventory problem when either there is no limit to the storage capacity, or when the optimal inventory levels do not fill the available warehouse capacity.

3.2. Specific cases

(i) If the power demand pattern indices are $n_i=1$, for all $i=1,2,\dots,N$, then the inventory policy coincides with the optimal policy for a probabilistic multi-item inventory system with constant demands, fixed scheduling period, backlogged shortages, and assuming that backlogging costs are equal for all the items (Chikán [42, pages 320 – 321]).

(ii) (ii) If we have only a single article, that is, if we assume $N=1$, then the optimal policy is equivalent to the optimal policy for a probabilistic inventory system with fixed inventory cycle, fully backlogged shortages and power demand pattern (Chikán [42, pages 175 – 176]).

(iii) Considering both conditions simultaneously, that is, if $N=1$ and $n_1=1$, then the optimal policy obtained coincides with the efficient policy for the inventory system with a single article, stochastic demand, instantaneous replenishment, fixed scheduling period, fully backlogged shortages, and uniform demand pattern (see Naddor [15, pages 130 – 135] and Chikán [42, pages 173 – 174]).

(iv) If, for each product i, we consider that the power demand pattern index n_i tends to infinite, then the expected total inventory cost per unit time given in (14) is reduced to

$$C(S_1,\dots,S_N)=\sum_{i=1}^{N}h_i\int_0^{S_i}(S_i-x_i)f_i(x_i)dx_i+\sum_{i=1}^{N}\omega_i\int_{S_i}^{\infty}(x_i-S_i)f_i(x_i)dx_i+\theta\frac{A}{T_0}$$

For each item i, with $i=1,2,\dots,N$, the function (16) becomes

$$Z_i(S_i)=\int_{S_i}^{\infty}f_i(x_i)dx_i=1-F_i(S_i)$$

Thus, in this case, the optimal inventory policy given by (19) coincides with the well-known solution of the newsvendor problem.

4. Inventory model for multiple items with stochastic demands, power demand patterns, backlogged shortages, and limited storage

In this section, we study the optimal policy for the probabilistic multi-item inventory system with power demands and full backlogging, assuming that the inventory cycle is fixed and the total storage space available in the warehouse is limited.

Let W be the total available space in the warehouse for all N items. Considering that v_i represents the unit space of item i, with $i=1,2,\dots,N$, and S_i is the initial inventory level of the ith item, then the total volume of the N items must be less than or equal to the available storage space W. Thus, the following condition must be satisfied

$$\sum_{i=1}^{N}v_iS_i\le W \tag{23}$$

The objective function $C(S_1,\dots,S_N)$ for this inventory system is proposed in (14), but now we additionally consider the constraint of limited storage given in (23). Thus, the new inventory problem is

$$\begin{array}{rcl} \min C(S_1,\ldots,S_N) & = & \sum_{i=1}^N h_i \int_0^{S_i}\left(S_i - \frac{x_i n_i}{n_i+1}\right) f_i(x_i)dx_i + \sum_{i=1}^N (h_i+\omega_i)\int_{S_i}^\infty \frac{S_i}{n_i+1}\left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i)dx_i \\ & & + \sum_{i=1}^N \omega_i \int_{S_i}^\infty \left(\frac{n_i x_i}{n_i+1} - S_i\right) f_i(x_i)dx_i + \theta\frac{A}{T_0} \\ \textit{subject to} & & \sum_{i=1}^N v_i S_i \le W \\ & & 0 \le S_i, \quad \text{for } i = 1,2,\ldots,N \end{array} \tag{24}$$

4.1. Optimal policy for the multi-item system with limited storage

If the inventory levels S_i^0, with $i = 1,2,\ldots,N$, determined by (19), satisfy the constraint (23), then they will be the optimal inventory levels.

Otherwise, as the cost function $C(S_1,\ldots,S_N)$ is a convex function on the region Ω, if the levels S_i^0, with $i = 1,2,\ldots,N$, do not satisfy the limited storage constraint, it is clear that the optimal inventory levels for this system with limited storage must hold the equality in (23). Hence, we can now use the Lagrangian multipliers technique to obtain the optimal inventory levels S_i^*, for $i = 1,2,\ldots,N$. From the problem (24), the Lagrangian function L is given by the following expression

$$L(S_1,\ldots,S_N,\lambda) = C(S_1,\ldots,S_N) + \lambda\left(\sum_{i=1}^N v_i S_i - W\right),$$

where λ is the Lagrangian multiplier. Hence, for an optimal solution, we have to calculate the partial derivatives of $L(S_1,\ldots,S_N,\lambda)$ with respect to the variables S_i (for $i = 1,2,\ldots,N$), and the multiplier λ. This leads to

$$\frac{\partial L}{\partial S_i} = (h_i+\omega_i)\left(\int_0^{S_i} f_i(x_i)dx_i + \int_{S_i}^\infty \left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i)dx_i\right) - \omega_i + \lambda v_i, \quad i = 1,2,\ldots,N \tag{25}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^N v_i S_i - W \tag{26}$$

Equating (25) to zero, we obtain

$$\int_0^{S_i} f_i(x_i)dx_i + \int_{S_i}^\infty \left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i)dx_i = \frac{\omega_i - \lambda v_i}{h_i+\omega_i}, \quad i = 1,2,\ldots,N$$

This condition is equivalent to

$$\int_{S_i}^\infty \left(1 - \left(\frac{S_i}{x_i}\right)^{n_i}\right) f_i(x_i)dx_i = \frac{h_i + \lambda v_i}{h_i+\omega_i}, \quad i = 1,2,\ldots,N \tag{27}$$

Taking into account the function $Z_i(S_i)$ defined in (16), the above Eq. (27) leads to

$$Z_i(S_i) = \frac{h_i + \lambda v_i}{h_i+\omega_i}, \quad i = 1,2,\ldots,N \tag{28}$$

From Proposition 1, the function $Z_i(S_i)$ is strictly positive and decreasing on $[0, b_i)$ and takes values in the interval $(0, 1]$. Thus, to find a solution to Eq. (28) for each item, the right side of this equation must be less than or equal to one. Hence the multiplier λ has to be less than or equal to ω_i/v_i, for any item i. Moreover, if λ were greater than ω_i/v_i for some item i, then the level S_i for this item must be zero. Given a value of λ, let $S_i(\lambda)$ be the inventory level that solves the Eq. (28) for each item i, that is

$$S_i(\lambda) = \begin{cases} Z_i^{-1}\left(\frac{h_i+\lambda v_i}{h_i+\omega_i}\right), & \textit{if } \lambda \le \omega_i/v_i \\ 0, & \textit{if } \lambda > \omega_i/v_i \end{cases} \tag{29}$$

The inventory level $S_i(\lambda)$ for each item i must be greater than or equal to zero. Note that if $\lambda = 0$, then Eq. (28) is reduced to Eq. (18). Therefore, the inventory levels $S_i(0)$ obtained from (29) are equal to levels S_i^0, given in (19), with $i = 1,2,\ldots,N$.

From (26), the inventory levels $S_i(\lambda)$ given in (29) must meet the condition

$$\sum_{i=1}^N v_i S_i(\lambda) = W$$

Let $g(\lambda)$ be the function defined by

$$g(\lambda) = \sum_{i \in I(\lambda)} v_i S_i(\lambda) - W \tag{30}$$

on the interval $[0,\infty)$, with $I(\lambda) = \{i/\omega_i - \lambda v_i \ge 0 \text{ and } 1 \le i \le N\}$.

The next result characterizes the optimal inventory policy when the inventory levels S_i^0 do not satisfy the constraint of limited storage.

Theorem 2. *The function $g(\lambda)$ given by (30) is a continuous and strictly decreasing function. Moreover, if $g(0) > 0$, then the function $g(\lambda)$ has a unique positive root λ^* and, from (29), the optimal inventory level of item i is given by $S_i(\lambda^*)$.*

Proof. See the Appendix. □

As, in this section, we have assumed that the inventory levels S_i^0 given in (19), with $i = 1, 2, \dots, N$, do not meet the constraint (23), then $g(0) > 0$. Therefore, in this case, from Theorem 2, the equation $g(\lambda) = 0$ always has a unique positive solution λ^* on the interval $\left[0, \max_{1\le i\le N}\{\omega_i/\nu_i\}\right]$. Once the value of λ^* has been determined, the optimal inventory levels $S_i^* = S_i(\lambda^*)$, for $i = 1, 2, \dots, N$ can be calculated from Eq. (29).

Suppose that the optimal inventory level of item i is $S_i^* = S_i(\lambda^*)$, the minimum expected holding cost per unit time is

$$\begin{aligned} EHC^* &= EHC(S_1^*, \dots, S_N^*) \\ &= \sum_{i=1}^{N} \frac{h_i n_i}{n_i+1} \int_0^{S_i^*} \left(S_i^* - x_i\right) f_i(x_i) dx_i + \sum_{i=1}^{N} h_i \frac{(\omega_i - \lambda^* \nu_i) S_i^*}{(n_i+1)(h_i+\omega_i)} \end{aligned} \tag{31}$$

Considering that the average demand of the product i is μ_i, the minimum expected backlogging cost per unit time is

$$\begin{aligned} EBC^* &= EBC(S_1^*, \dots, S_N^*) \\ &= \sum_{i=1}^{N} \frac{\omega_i n_i}{n_i+1} \int_0^{S_i^*} \left(S_i^* - x_i\right) f_i(x_i) dx_i + \sum_{i=1}^{N} \frac{\omega_i n_i}{n_i+1}\left(\mu_i - S_i^*\right) - \sum_{i=1}^{N} \omega_i \frac{(h_i + \lambda^* \nu_i) S_i^*}{(n_i+1)(h_i+\omega_i)} \end{aligned} \tag{32}$$

Therefore, the minimum expected inventory cost per unit time is given by

$$\begin{aligned} C^* &= C(S_1^*, \dots, S_N^*) \\ &= \sum_{i=1}^{N} \frac{(h_i+\omega_i) n_i}{n_i+1} \int_0^{S_i^*} \left(S_i^* - x_i\right) f_i(x_i) dx_i + \sum_{i=1}^{N} \frac{\omega_i n_i}{n_i+1}\left(\mu_i - S_i^*\right) - \sum_{i=1}^{N} \frac{\lambda^* \nu_i}{n_i+1} S_i^* + \theta \frac{A}{T_0} \end{aligned} \tag{33}$$

From (15) and (33), the maximum expected profit EP^* per unit time is given by

$$EP^* = EP(S_1^*, \dots, S_N^*) = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} \mu_i - C(S_1^*, \dots, S_N^*)$$

4.2. Algorithmic procedure

Considering the previous results, we present an efficient procedure (Algorithm 1) to solve the inventory problem formulated in (24) for the probabilistic multi-item system with power demand patterns, backlogged shortages, and limited storage. The minimum of the expected inventory cost $C(S_1, \dots, S_N)$ subject to the constraints is also provided by this procedure.

Algorithm 1 gives the optimal inventory levels S_i^*, for $i = 1, 2, \dots, N$. Moreover, in each inventory cycle, the replenishing quantity or lot size Q_i for the ith item must be equal to the demanded quantity on the period T_0. Thus, assuming that x_i is a realization of the demand X_i of the item i during the period T_0, then the lot sizes are $Q_i^* = x_i$, for $i = 1, 2, \dots, N$; and the reorder points are $s_i^* = S_i^* - Q_i^* = S_i^* - x_i$, for $i = 1, 2, \dots, N$.

Note that the non-linear equation $g(\lambda) = 0$, with λ in the interval $(\lambda_{j-1}, \lambda_j]$, can be solved using some numerical procedure such as the secant method or the bisection method (see Stoer and Bulirsch [43]).

In the following section, we illustrate the previous algorithmic procedure considering that demands follow certain probability distributions.

Algorithm 1.

Step 1.	From (19), calculate $S_i^0 = S_i(0) = Z_i^{-1}\left(\frac{h_i}{h_i+\omega_i}\right)$, for every $i = 1, \ldots, N$.
	From (30), calculate $g(0)$.
Step 2.	If $g(0) \le 0$, then $\lambda^* = 0$ and $S_i^* = S_i^0$, for $i = 1, 2, \ldots, N$. Go to Step 8.
	Else, go to Step 3.
Step 3.	Let $j = 0$, $\lambda_0 = 0$, $I_1 = \{1, \ldots, N\}$.
Step 4.	Calculate
	$j = j + 1$
	$\lambda_j = \min_{i \in I_j}\{\frac{\omega_i}{\nu_i}\}$
	$I_{j+1} = I_j - \{i \in I_j / \lambda_j = \frac{\omega_i}{\nu_i}\}$
	From (29), calculate $S_i(\lambda_j) = Z_i^{-1}\left(\frac{h_i+\lambda_j\nu_i}{h_i+\omega_i}\right)$, for every $i \in I_j$.
	From (30), get $g(\lambda_j)$.
Step 5.	If $g(\lambda_j) \le 0$, then go to Step 6.
	Otherwise, come back to the beginning of Step 4.
Step 6.	Obtain $\lambda^* = \arg_{\lambda_{j-1} < \lambda \le \lambda_j}\{g(\lambda) = 0\}$.
Step 7.	For $i = 1$ to N do
	$S_i^* = S_i(\lambda^*)$ for $i \in I_j$
	$S_i^* = 0$ for $i \notin I_j$
Step 8.	From (33), calculate the minimum expected inventory cost $C^* = C(S_1^*, \ldots, S_N^*)$.
	From (15), obtain the maximum expected profit $EP^* = EP(S_1^*, \ldots, S_N^*)$.

5. Special demands: Pareto distributions

In the next paragraphs, we apply the methodology proposed in the above sections to stochastic customer demands, which follow Pareto probability distributions.

Let us consider an inventory system with N items where the replacement of the products is carried out jointly every T_0 time units. Shortages are allowed and completely backlogged. Item i is taken out of the inventory following a power demand pattern given by (2). The demand X_i of each item i follows a Pareto distribution on the period T_0. Thus, for the item i, the probability density of the demand $f_i(x_i)$ is

$$f_i(x_i) = \frac{\alpha_i \eta_i^{\alpha_i}}{x_i^{\alpha_{i+1}}}, \quad \text{for all} \quad x_i \ge \eta_i \quad \text{with} \quad \alpha_i > 2, i = 1, 2, \ldots, N \tag{34}$$

where η_i is the scale parameter and α_i is the shape parameter. Let μ_i be the average demand of the item i during the period T_0. These average demands are

$$\mu_i = \frac{\alpha_i \eta_i}{\alpha_i - 1}, \qquad i = 1, 2, \ldots, N \tag{35}$$

For each item $i = 1, 2, \ldots, N$, the function $Z_i(S_i)$ given in (16) depends on the relative values of S_i, and η_i. Thus, we have

$$Z_i(S_i) = \begin{cases} 1 - \frac{\alpha_i}{\alpha_i+n_i}\left(\frac{S_i}{\eta_i}\right)^{n_i}, & S_i \le \eta_i \\ \frac{n_i}{\alpha_i+n_i}\left(\frac{\eta_i}{S_i}\right)^{\alpha_i}, & S_i > \eta_i \end{cases} \tag{36}$$

Note that

$$Z_i(\eta_i) = \frac{n_i}{\alpha_i + n_i}, \qquad i = 1, 2, \ldots, N \tag{37}$$

Therefore, from (19), the optimal inventory levels are

$$S_i{}^0 = Z_i{}^{-1}\left(\frac{h_i}{h_i + \omega_i}\right) = \begin{cases} \eta_i\left(\frac{(\alpha_i+n_i)\omega_i}{\alpha_i(h_i+\omega_i)}\right)^{1/n_i}, & \frac{n_i}{\alpha_i+n_i} \le \frac{h_i}{h_i+\omega_i} \\ \eta_i\left(\frac{n_i(h_i+\omega_i)}{h_i(\alpha_i+n_i)}\right)^{1/\alpha_i}, & \frac{n_i}{\alpha_i+n_i} > \frac{h_i}{h_i+\omega_i} \end{cases}, \quad i = 1, 2, \ldots, N \tag{38}$$

The minimum expected inventory cost per unit time C_0 is given by (22). Before calculating this expected inventory cost, we first need to evaluate the integral

$$\int_0^{S_i^0} \left(S_i^0 - x_i\right) f_i(x_i) dx_i = \begin{cases} 0, & S_i^0 \le \eta_i \\ \frac{\eta_i^{\alpha_i}}{\alpha_i - 1}\left(S_i^0\right)^{-\alpha_i+1} + S_i^0 - \frac{\alpha_i \eta_i}{\alpha_i - 1}, & S_i^0 > \eta_i \end{cases} \tag{39}$$

Note that the value of this integral depends on the relative values of S_i^0 and η_i. Now, to calculate the minimum cost, we need to partition the set of item indices $I = \{1, 2, \dots, N\}$ into two sets, which are denoted by $J_1 = \{i \in I / S_i^0 \le \eta_i\}$ and $J_2 = \{i \in I / S_i^0 > \eta_i\}$. From (22) and (39), we have

$$C_0 = \sum_{i \in J_2} (h_i + \omega_i) \frac{n_i}{n_i + 1} \left(\frac{\eta_i^{\alpha_i}}{\alpha_i - 1} \left(S_i^0\right)^{-\alpha_i + 1} + S_i^0 - \frac{\alpha_i \eta_i}{\alpha_i - 1} \right) + \sum_{i=1}^{N} \omega_i \frac{n_i \mu_i}{n_i + 1} - \sum_{i \in J_1} \omega_i \frac{n_i S_i^0}{n_i + 1} - \sum_{i \in J_2} \omega_i \frac{n_i S_i^0}{n_i + 1} + \theta \frac{A}{T_0} \tag{40}$$

Substituting the optimal inventory levels given by (38) in the above equation, we obtain the minimum expected cost per unit time

$$C_0 = \sum_{i \in J_2} \frac{\alpha_i h_i \eta_i}{\alpha_i - 1} \left(\frac{n_i (h_i + \omega_i)}{h_i (\alpha_i + n_i)} \right)^{1/\alpha_i} - \sum_{i \in J_1} \frac{\omega_i n_i \eta_i}{n_i + 1} \left(\frac{(\alpha_i + n_i) \omega_i}{\alpha_i (h_i + \omega_i)} \right)^{1/n_i} + \sum_{i=1}^{N} \frac{\omega_i n_i \mu_i}{n_i + 1} - \sum_{i \in J_2} \frac{(h_i + \omega_i) n_i \alpha_i \eta_i}{(n_i + 1)(\alpha_i - 1)} + \theta \frac{A}{T_0} \tag{41}$$

From (15) and (41), the maximum expected profit per unit time is

$$EP^0 = EP(S_1^0, \dots, S_N^0) = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} \mu_i - C_0$$

We now study the optimal policy for the probabilistic multi-item inventory system with power demand patterns, fixed inventory cycle, full backlogging, demands following Pareto distributions and assuming that the total storage space available in the warehouse is limited.

From Algorithm 1 presented in Section 4, in steps 1 and 2, we check if the inventory levels S_i^0 given by (38) satisfy the storage constraint (23).

If the levels S_i^0 meet this constraint, then they are the optimal inventory levels and the minimum inventory cost per unit time is given by (41).

However, if the levels S_i^0 do not satisfy the storage constraint, then, in Step 3, we set $j = 0$, $\lambda_0 = 0$, $I_1 = \{1, \dots, N\}$. Next, we must obtain the new values $S_i(\lambda)$ according to (29). Given a value of λ, the inventory level $S_i(\lambda)$ that solves the Eq. (28) for each item i is

$$S_i(\lambda) = Z_i^{-1}\left(\tfrac{h_i + \lambda v_i}{h_i + \omega_i}\right) = \begin{cases} \eta_i \left(\frac{(\alpha_i + n_i)(\omega_i - \lambda v_i)}{\alpha_i (h_i + \omega_i)} \right)^{1/n_i}, & \frac{n_i}{\alpha_i + n_i} \le \frac{h_i + \lambda v_i}{h_i + \omega_i} \\ \eta_i \left(\frac{n_i (h_i + \omega_i)}{(h_i + \lambda v_i)(\alpha_i + n_i)} \right)^{1/\alpha_i}, & \frac{n_i}{\alpha_i + n_i} > \frac{h_i + \lambda v_i}{h_i + \omega_i} \end{cases} \tag{42}$$

Following Algorithm 1, in Step 4, we successively calculate the values $j = j + 1$, $\lambda_j = \min_{i \in I_j}\{\frac{\omega_i}{v_i}\}$, $I_{j+1} = I_j - \{i \in I_j / \lambda_j = \frac{\omega_i}{v_i}\}$, and, from (42), the levels $S_i(\lambda_j)$ for every $i \in I_j$ up to $g(\lambda_j) \le 0$, with $g(\lambda)$ given in (30). Next, for these Pareto distributions, we determine the value λ_P^* such that $\lambda_P^* = \arg_{\lambda_{j-1} < \lambda \le \lambda_j}\{g(\lambda) = 0\}$. The optimal inventory levels are determined by

$$S_i^* = S_i(\lambda_P^*), \text{ if } i \in I_j \quad \text{and} \quad S_i^* = 0, \text{ if } i \notin I_j.$$

Thus, from (42), we have

$$S_i^* = S_i(\lambda_P^*) = \begin{cases} \eta_i \left(\frac{(\alpha_i + n_i)\left(\omega_i - \lambda_P^* v_i\right)}{\alpha_i (h_i + \omega_i)} \right)^{1/n_i}, & \frac{n_i}{\alpha_i + n_i} \le \frac{h_i + \lambda_P^* v_i}{h_i + \omega_i} \\ \eta_i \left(\frac{n_i (h_i + \omega_i)}{\left(h_i + \lambda_P^* v_i\right)(\alpha_i + n_i)} \right)^{1/\alpha_i}, & \frac{n_i}{\alpha_i + n_i} > \frac{h_i + \lambda_P^* v_i}{h_i + \omega_i} \end{cases} \tag{43}$$

From (33), the minimum expected inventory cost is given by

$$C^* = C(S_1^*, \dots, S_N^*) = \sum_{i=1}^{N} \frac{(h_i + \omega_i) n_i}{n_i + 1} \int_0^{S_i^*} \left(S_i^* - x_i\right) f_i(x_i) dx_i + \sum_{i=1}^{N} \frac{\omega_i n_i}{n_i + 1} \left(\mu_i - S_i^*\right) - \sum_{i=1}^{N} \frac{\lambda_P^* v_i}{n_i + 1} S_i^* + \theta \frac{A}{T_0} \tag{44}$$

Similarly to (39), we have

$$\int_0^{S_i^*} \left(S_i^* - x_i\right) f_i(x_i) dx_i = \begin{cases} 0, & S_i^* \le \eta_i \\ \frac{\eta_i^{\alpha_i}}{\alpha_i - 1} \left(S_i^*\right)^{-\alpha_i + 1} + S_i^* - \frac{\alpha_i \eta_i}{\alpha_i - 1}, & S_i^* > \eta_i \end{cases} \tag{45}$$

Table 1
Input parameters for an inventory system with $N = 6$ items whose demands follow different power pattern indices.

	Holding cost h_i (\$/unit and time)	Shortage cost ω_i (\$/unit and time)	Demand pattern index n_i	Purchasing cost c_i (\$/unit)	Selling price p_i (\$/unit)	Volume or unit storage space v_i (m^3)
Item 1	\$2.8	\$6.2	1.6	\$4	\$7	0.5
Item 2	\$1.5	\$4.2	0.4	\$7	\$11	0.7
Item 3	\$3.0	\$8.0	2.0	\$2	\$5	0.6
Item 4	\$2.4	\$3.5	1.0	\$8	\$12	0.8
Item 5	\$1.2	\$4.0	0.5	\$5	\$8	0.4
Item 6	\$3.6	\$5.4	0.8	\$3	\$6	0.6
The ordering cost is $A = \$120$ per order						

Now, to calculate the minimum expected cost, we need to partition the set of item indices $I = \{1, 2, \dots, N\}$ into two sets, which are denoted by $K_1 = \{i \in I / S_i^* \le \eta_i\}$ and $K_2 = \{i \in I / S_i^* > \eta_i\}$. Also, the inequality $S_i^* \le \eta_i$ is equivalent to the inequality $n_i/(\alpha_i + n_i) \le \left(h_i + \lambda_P^* v_i\right)/(h_i + \omega_i)$.

Thus, the minimum expected cost C^*, given in (44), is

$$C^* = \sum_{i\in K_2} (h_i + \omega_i)\frac{n_i}{n_i+1}\left(\frac{\eta_i^{\alpha_i}}{\alpha_i - 1}\left(S_i^*\right)^{-\alpha_i+1} + S_i^* - \frac{\alpha_i\eta_i}{\alpha_i - 1}\right) + \sum_{i=1}^{N}\omega_i\frac{n_i\mu_i}{n_i+1}$$
$$-\sum_{i\in K_1}\omega_i\frac{n_iS_i^*}{n_i+1} - \sum_{i\in K_2}\omega_i\frac{n_iS_i^*}{n_i+1} - \sum_{i=1}^{N}\frac{\lambda_P^* v_i}{n_i+1}S_i^* + \theta\frac{A}{T_0} \tag{46}$$

Considering (43), the minimum expected cost is reduced to

$$C^* = \sum_{i\in K_2} h_i\frac{\alpha_i S_i^*}{\alpha_i - 1} - \sum_{i\in K_2}(h_i + \omega_i)\frac{n_i\alpha_i\eta_i}{(n_i+1)(\alpha_i - 1)} - \sum_{i\in K_1}\omega_i\frac{n_iS_i^*}{n_i+1}$$
$$+\sum_{i=1}^{N}\omega_i\frac{n_i\mu_i}{n_i+1} - \sum_{i=1}^{N}\frac{\lambda_P^* v_i}{n_i+1}S_i^* + \theta\frac{A}{T_0} \tag{47}$$

From (15) and (47), the maximum expected profit EP^* per unit time is given by

$$EP^* = EP(S_1^*, \dots, S_N^*) = \sum_{i=1}^{N}\frac{(p_i - c_i)}{T_0}\mu_i - C^*$$

In the next section, we present some numerical examples to illustrate the applicability of the theoretical results obtained for the inventory model proposed in this article.

6. Numerical results

Let us suppose a company dedicated to the sell and distribution of building materials. It sells, among others, six different types of gravel. The inventory manager wants to maximize the expected profit from the storage and sale of these items. It can be assumed that the inventory system satisfies the assumptions made in this article. Thus, demands are random variables and shortages are allowed and backlogged. We study the inventory control of these items. For reasons of strategic planning, the company jointly replenishes the products every month. Thus, the scheduling period or inventory cycle is fixed and is equal to $T_0 = 1/12 = 0.0833333$ years. The replenishing cost is $A = \$120$ per order. For each of the six items, the carrying cost h_i per unit time, the shortage cost ω_i per unit time, the demand pattern index n_i, the purchasing cost c_i, the selling price p_i, and the volume or unit space v_i, with $i = 1, 2, \dots, 6$, are shown in Table 1.

Let us assume that the demand of the ith item follows a Pareto distribution with mean μ_i, shape parameter α_i and scale parameter η_i, for $i = 1, 2, \dots, 6$. Table 2 displays the values of these parameters for the six items. As all the random variables X_i are continuous, we have $\Pr(X_i = 0) = 0$, for all $i = 1, 2, \dots, N$. Hence, the expected replenishing cost is $ERC = A/T_0$.

In the following paragraphs, we calculate the optimal inventory policy for the multi-item inventory system with Pareto stochastic demands, power demand patterns and backlogged shortages.

We first need to calculate, for each item $i = 1, 2, \dots, 6$, the values of the ratios $h_i/(h_i + \omega_i)$ and $n_i/(n_i + \alpha_i)$. These values are shown in Table 3.

Next, we present below the optimal inventory policy for the multi-item inventory system with fixed inventory cycle and fully backlogged shortages, considering different storage capacities.

Table 2
Parameters of the Pareto distributions for the inventory system with $N = 6$ items.

	Scale parameter η_i	Shape parameter α_i	Average demand μ_i
Item 1	20 units	5	25 units
Item 2	8 units	5	10 units
Item 3	45 units	10	50 units
Item 4	6 units	4	8 units
Item 5	35 units	8	40 units
Item 6	75 units	4	100 units

Table 3
Values of the ratios $h_i/(h_i + \omega_i)$and $n_i/(n_i + \alpha_i)$ for the $N = 6$ items.

	Ratio $h_i/(h_i + \omega_i)$	Ratio $n_i/(n_i + \alpha_i)$
Item 1	0.311111	0.242424
Item 2	0.263158	0.0740741
Item 3	0.272727	0.166667
Item 4	0.406780	0.2
Item 5	0.230769	0.0588235
Item 6	0.4	0.166667

Table 4
Optimal inventory levels at the beginning of the inventory cycle for the system with $W = 100$ m^3.

Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
$S_i^0 = 18.8466$	$S_2^0 = 4.51945$	$S_3^0 = 42.0389$	$S_4^0 = 4.44915$	$S_5^0 = 23.3797$	$S_6^0 = 49.7424$

6.1. *Multiple-item inventory system with a storage capacity of* $W = 100$ *cubic meters*

We now suppose that the items can be stored in a warehouse with a capacity of $W = 100$ cubic meters. To obtain the optimal inventory policy, we first calculate the values S_i^0 given by (38), that is, the solution for the inventory problem when there is no limitation on the storage capacity.

From Table 3, the ratios satisfy $h_i/(h_i + \omega_i) \geq n_i/(n_i + \alpha_i)$ for any item $i = 1, 2, \ldots, 6$. Hence, from (38), the inventory levels S_i^0 are obtained by

$$S_i^0 = \eta_i \left(\frac{(\alpha_i + n_i)\omega_i}{\alpha_i (h_i + \omega_i)} \right)^{1/n_i}, \quad i = 1, 2, \ldots, 6$$

These inventory levels are shown in Table 4.

Now, we have $\sum_{i=1}^{N} v_i S_i^0 = 80.5669$. As this value is less than the capacity $W = 100$, we conclude that this is the optimal inventory policy.

Next, we calculate the minimum expected inventory cost. For that, we first determine the sets $J_1 = \{i \in I / S_i^0 \leq \eta_i\} = \{1, 2, \ldots, 6\}$ and $J_2 = \{i \in I / S_i^0 > \eta_i\} = \emptyset$. Thus, from (41), the expected total inventory cost is $C_0 = \$1661.51$ per year.

In addition, from (20) and (38), the expected holding cost is \$125.369 per year; and from (21) and (38), the expected shortage cost is \$96.1367 per year. The expected ordering cost is $A/T_0 = \$1440$ per year.

Now, taking into account the input parameters shown in Tables 1 and 2, the expected benefit per unit time obtained from the sale of items in each inventory cycle is given by

$$E\left[\sum_{i=1}^{N} \frac{(p_i - c_i)Q_i}{T_0} \right] = \sum_{i=1}^{N} \frac{(p_i - c_i)}{T_0} \mu_i = \$8604 \text{ per year}$$

Thus, from (15), the maximum expected profit per unit time is \$6942.49 per year.

6.2. *Multi-item system with storage capacity limited to* $W = 60$ *cubic meters*

Now, we consider that the items are stored in a warehouse with a capacity of $W = 60$ cubic meters. In this situation, the inventory policy given in Table 4 does not satisfy the constraint of limited storage capacity proposed in (23).

Thus, we must follow Algorithm 1 proposed in Section 4.2 to find the optimal policy. For that, let ω_i/v_i be the unit backlogging cost per storage space of each item i, with $i = 1, 2, \ldots, 6$. These ratios are shown in Table 5.

Table 5
Unit backlogging cost per storage space of the items considered in the inventory system.

Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
$\omega_1/v_1 = 12.4$	$\omega_2/v_2 = 6$	$\omega_3/v_3 = 13.333$	$\omega_4/v_4 = 4.375$	$\omega_5/v_5 = 10$	$\omega_6/v_6 = 9$

Table 6
Inventory levels $S_i(\lambda_1)$ associated with the multiplier λ_1 in the multi-item inventory system.

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
$S_i(\lambda_1) =$	14.3589	0.1725	34.4585	0	7.3975	21.6428

Table 7
Optimal inventory levels at the beginning of the inventory cycle for the system when $W = 60$ m^3.

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
$S_i^* = S_i(\lambda^*) =$	16.5723	1.34415	38.2312	2.10406	13.8402	34.3582

Table 8
Optimal inventory levels at the beginning of the inventory cycle for the system when $W = 30$ m^3.

	Item 1	Item 2	Item 3	Item 4	Item 5	Item 6
$S_i^* = S_i(\lambda^*) =$	11.5880	0	29.6396	0	2.53777	9.01186

As, in this case, the condition (23) is not hold, from step 3 we have $j = 0$, $\lambda_0 = 0$, $I_1 = \{1, \ldots, 6\}$. Next, from step 4 we calculate $j = 1$, $\lambda_1 = \min_{1\le i\le 6}\{\frac{\omega_i}{v_i}\} = \frac{\omega_4}{v_4} = 4.375$, $I_2 = \{1, 2, 3, 5, 6\}$. From (42), we must calculate the values of $S_i(\lambda)$, for $i = 1, 2, \ldots, 6$. As $\lambda > 0$, from Table 3, the ratios satisfy $(h_i + \lambda v_i)/(h_i + \omega_i) \ge n_i/(n_i + \alpha_i)$ for any item $i = 1, 2, \ldots, 6$.

Hence, from (42), the inventory levels $S_i(\lambda_1)$ are obtained by

$$S_i(\lambda_1) = \eta_i\left(\frac{(\alpha_i + n_i)(\omega_i - \lambda_1 v_i)}{\alpha_i(h_i + \omega_i)}\right)^{1/n_i}, \qquad i = 1, 2, \ldots, 6$$

These inventory levels for the multiplier λ_1 are shown in Table 6.

As $\sum_{i\in I_1} v_i S_i(\lambda_1) = 43.9200 < W = 60$, we have to go to step 6. Then, we calculate the multiplier solving the equation

$$\lambda^* = \arg_{\lambda_0<\lambda\le\lambda_1}\left\{\sum_{i\in I_1} v_i S_i(\lambda) = W\right\} = \arg_{0<\lambda\le 4.375}\left\{\sum_{i\in I_1} v_i\eta_i\left(\frac{(\alpha_i + n_i)(\omega_i - \lambda v_i)}{\alpha_i(h_i + \omega_i)}\right)^{1/n_i} = 60\right\}$$

Thus, we obtain $\lambda^* = 2.30601$. Next, from (43), we calculate the values $S_i^* = S_i(\lambda^*)$, for $i = 1, 2, \ldots, 6$. These optimal inventory levels are displayed in Table 7. From (47), we can calculate the minimum expected inventory cost C^*. For that, we first determine the sets $K_1 = \{i \in I / S_i^* \le \eta_i\} = \{1, 2, \ldots, 6\}$ and $K_2 = \{i \in I / S_i^* > \eta_i\} = \emptyset$. Thus, the expected total inventory cost is $C^* = \$1684.65$ per year.

Also, from (31) and (43), we have that the expected holding cost is \$71.5844 per year; and, from (32) and (43), the expected shortage cost is \$173.070 per year. The expected ordering cost is the same, $A/T_0 = \$1440$ per year. Hence, from (15), the maximum expected profit per unit time is \$6919.35 per year.

6.3. Multiple-item system with storage capacity limited to $W = 30$ cubic meters

Finally, suppose that the items are stored in a warehouse with a capacity of $W = 30$ cubic meters. Obviously, the values S_i^0, with $i = 1, 2, \ldots, N$, given in (38) do not satisfy the constraint (23). So, we have to apply Algorithm 1 to obtain the optimal inventory policy. Following the procedure, steps $1-3$ are similar to the above system with storage capacity $W = 60$ cubic meters. From step 4, we have $j = 1$, $\lambda_1 = \min_{1\le i\le 6}\{\frac{\omega_i}{v_i}\} = \frac{\omega_4}{v_4} = 4.375$, $I_2 = \{1, 2, 3, 5, 6\}$ and $\sum_{i\in I_1} v_i S_i(\lambda_1) = 43.9200 > W = 30$. Then, we calculate $j = 2$, $\lambda_2 = \min_{i\in I_2}\{\frac{\omega_i}{v_i}\} = \frac{\omega_2}{v_2} = 6$, $I_3 = \{1, 3, 5, 6\}$ and $\sum_{i\in I_2} v_i S_i(\lambda_2) = 33.9944 > W = 30$. Next, we continue with $j = 3$, $\lambda_3 = \min_{i\in I_3}\{\frac{\omega_i}{v_i}\} = \frac{\omega_6}{v_6} = 9$, $I_4 = \{1, 3, 5\}$ and $\sum_{i\in I_3} v_i S_i(\lambda_3) = 18.6705 < W = 30$. Then, we go to step 5 and calculate $\lambda^* = \arg_{\lambda_2<\lambda\le\lambda_3}\left\{\sum_{i\in I_3} v_i S_i(\lambda) = W\right\} = 6.70537$. Next, in step 6, we determine the optimal inventory levels by $S_i^* = S_i(\lambda^*)$ for $i \in I_3$, and $S_2^* = S_4^* = 0$. These optimal inventory levels are displayed in Table 8.

Next, we determine the sets $K_1 = \{i \in I / S_i^* \le \eta_i\} = \{1, 2, \ldots, 6\}$ and $K_2 = \{i \in I / S_i^* > \eta_i\} = \emptyset$. Thus, the expected total inventory cost is $C^* = \$1814.71$ per year. In addition, from (31) and (43), we have that the expected holding cost is \$17.9353 per

Table 9
Effects of the parameters h_i, ω_i, ν_i and η_i on the optimal multi-item inventory policy.

	Δ	$\Delta S_1^*(\%)$	$\Delta S_2^*(\%)$	$\Delta S_3^*(\%)$	$\Delta S_4^*(\%)$	$\Delta S_5^*(\%)$	$\Delta S_6^*(\%)$	$\Delta EHC^*(\%)$	$\Delta EBC^*(\%)$	$\Delta C^*(\%)$	$\Delta EP^*(\%)$
h_i	+40%	−1.93865	34.7471	−1.22104	23.6727	4.71667	−2.64758	36.8864	1.09395	1.67976	−0.408972
	+20%	−1.01506	17.5670	−0.634173	12.7894	2.48401	−1.39949	18.5578	0.545407	0.844592	−0.205633
	+10%	−0.519822	8.82159	−0.323349	6.66199	1.27575	−0.720462	9.31130	0.271724	0.423571	−0.103127
	−10%	0.546356	−8.87074	0.336649	−7.26709	−1.34841	0.766156	−9.38514	−0.268233	−0.426350	0.103804
	−20%	1.12144	−17.7572	0.687448	−15.2225	−2.77507	1.58296	−18.8548	−0.531075	−0.855739	0.208347
	−40%	8.05869	−38.4764	1.16624	−36.7043	−7.37007	2.19545	−37.6423	−1.10062	−1.71257	0.416959
ω_i	+40%	1.63908	−25.3559	1.00010	−22.6635	−4.06479	2.32774	2.13199	38.9496	4.09200	−0.996281
	+20%	0.926406	−14.7957	0.568888	−12.4885	−2.29045	1.30473	1.17424	19.4670	2.04980	−0.499064
	+10%	0.495535	−8.06311	0.305473	−6.57936	−1.22268	0.69449	0.618506	9.73156	1.02603	−0.249809
	−10%	−0.576029	9.79751	−0.358490	7.36803	1.41325	−0.797883	−0.692431	−9.72815	−1.02882	0.250488
	−20%	−1.25402	21.9042	−0.785113	15.6718	3.06433	−1.72464	−1.47258	−19.4539	−2.06113	0.501825
	−40%	−3.04842	56.8450	−1.93858	35.8735	7.35980	−4.11769	−3.36879	−38.9092	−4.14041	1.00807
ν_i	+40%	−14.3499	−89.8701	−10.6096	−107.095	−49.3039	−39.4973	−47.0226	53.0080	3.44758	−0.839383
	+20%	−7.95959	−64.5578	−5.85970	−60.6320	−29.9508	−22.8485	−29.3390	29.0354	1.73622	−0.422719
	+10%	−4.21365	−39.4603	−3.09426	−32.4733	−16.7022	−12.3846	−16.6375	15.2005	0.854633	−0.208078
	−10%	4.78795	61.8242	3.49513	37.9078	21.4267	14.8543	22.2882	−16.5785	−0.756093	0.184086
	−20%	10.2965	159.281	7.48936	82.8231	49.5044	32.9638	52.9808	−34.3417	−1.27677	0.310855
	−40%	13.7232	236.231	9.95988	111.456	68.9260	44.7760	75.1349	−44.4520	−1.37406	0.334541
η_i	+40%	19.9102	−85.8182	25.1466	−100.000	−29.0255	−15.2963	−25.8386	113.784	10.5915	47.1601
	+20%	10.4485	−57.4693	12.9684	−52.7584	−15.9409	−7.41824	−15.2068	54.8425	4.98797	23.6550
	+10%	5.36498	−33.4063	6.59632	−25.7206	−8.37247	−3.62305	−8.30126	26.7206	2.39235	11.8522
	−10%	−5.69084	45.6418	−6.85438	24.1170	9.28406	3.36887	10.0593	−24.9207	−2.13273	−11.9154
	−20%	−11.7628	107.425	−14.0085	46.2585	19.6035	6.37102	22.3846	−47.4734	−3.92592	−23.9136
	−40%	−31.7661	101.739	−34.0241	26.8733	1.35559	−13.1344	5.08094	−66.6712	−6.63344	−48.1238

year; and, from (32) and (43), the expected shortage cost is \$356.775 per year. The expected ordering cost is $A/T_0 = \$1440$ per year. In this case, from (15), the expected profit per unit time is \$6789.29 per year.

It is possible to draw the following useful findings from the numerical results. Obviously, if the storage capacity W increases, then the optimal stock levels also increase. When the storage capacity W decreases, the optimal policy suggests that, for some items, it is better not to held stock at the beginning of the inventory cycle.

It can be also seen that if the storage capacity is $W = 100$ m^3 then the maximum expected profit is \$6942.49 per year, while when the capacity is $W = 60$ m^3 the maximum expected profit is \$6919.35 per year. That is, the increasing of the storage capacity from 60 m^3 to 100 m^3 provides an expected growth of the revenue of only \$23.14 per year. In the same way, the gain of the storage capacity from 30 m^3 to 100 m^3 provides an increase of the expected profit from \$6789.29 to \$6942.49 per year, that is, only \$153.20 per year. This kind of reasoning could be useful for the decision-makers to afford or not investments for increasing their storage capacity.

These illustrative examples suppose that the demands follow Pareto probability distributions. Then the formulas deduced in Section 5 are applied. It is important to remark that similar reasoning could be performed if the demands follow other probability distributions, because the methodology proposed in Sections 3 and 4 is suitable and can be applied to any probability distribution.

7. Sensitivity analysis

In this section, we analyse the fluctuations of the optimal inventory policies when changes in some parameters of the multi-item inventory system are allowed. Initially, we consider the same parameters as in Section 6.2. Next, we calculate the variations, in percentage terms, of the optimal inventory policies, assuming that the value of each considered input parameter has changed by ±40%, ±20%, and ±10%. The results obtained can be seen in Table 9. From these results, we can establish the following observations:

1. If the unit holding costs h_i increase, then the optimal inventory levels of items 1, 3 and 6 decrease, while the optimal inventory levels of items 2, 4 and 5 increase. Also, if the parameters h_i increase, then the expected holding cost EHC^*, the expected backlogging cost EBC^* and the expected total cost C^* go up, while the expected profit EP^* goes down.
2. If the unit backlogging costs ω_i increase, then the optimal inventory levels of items 1, 3 and 6 increase, while the optimal inventory levels of items 2, 4 and 5 decrease. In addition, if the parameters ω_i increase, then the expected holding cost EHC^*, the expected backlogging cost EBC^* and the expected total cost C^* go up, while the expected profit EP^* goes down.
3. If the unit volumes of items ν_i increase, then all the optimal inventory levels S_i^* decrease. Also, if the parameters ν_i increase, then the expected holding cost EHC^* and the expected profit EP^* decrease, while the expected backlogging cost EBC^* and the expected total cost C^* go up.

4. If the scale parameters η_i of the Pareto distributions increase, then the optimal inventory levels S_1^* and S_3^* increase, while the rest of optimal inventory levels decrease. In addition, if the parameters η_i increase, then the expected holding cost EHC^* go down, while the expected backlogging cost EBC^*, the expected total cost C^* and the expected profit EP^* go up.

8. Managerial insights

With the globalization world trade and the rise of the internet and telecommunications, online sales of a wide variety of products have grown notably in recent years. The common feature of all these online sales is that customers do not receive the products at the time of purchase, but sellers deliver the goods a few days later. This situation also happens in many other in-shop sales, when the customers do not receive the purchased goods immediately. Therefore, the withdrawal of items from the inventory is not instantaneous at the purchasing time, but occurs in a period after the sale of the products.

These situations contrast with traditional models, as for example in the newsboy problem, where the realization of the demand in the inventory occurs at the same moment of purchase of the product, which leads to an immediate decreasing of the inventory level of the products.

To study these inventory systems, the model analysed in this work provides an approach for managers to find the optimal inventory levels for all the products. The proposed method also allows the decision-makers to compare the performance of different configurations of their inventory systems. Thus, the managers could decide whether it is suitable or not to afford new investments to modify the inventory system configurations, as can be to rent a larger warehouse to increase the storage capacity.

From the results proposed in this paper, the following managerial implications can be commented. When the storage capacity is fairly small, it can happen that the inventory levels of some products were zero. It does not mean that such products were off commercialization. Simply, these products are kept in sale, but they are not stored in stock. To meet the demand of these products the manager asks to the provider the required quantity by the customers.

Supposing that the optimal initial inventory level of some product was zero, if the storage capacity diminishes, then such optimal initial inventory level remains zero. If the storage capacity decreases even more, it is possible that the optimal initial inventory levels of more items become zero too. An increasing of the warehouse capacity could be a way to achieve strictly positive values for the optimal initial inventory levels of these products.

From the sensitivity analysis, the results show that the largest increase in the total cost of the system occurs when the scaling parameters of the Pareto distributions are increased.

It is also observed that if unit holding costs h_i are increased by up to 40%, then the increase in the total cost C^* is very small (less than 2%) and the decrease in the expected benefit EP^* is even smaller (less than 1%).

In addition, from the results of the sensitivity analysis, it can be checked that if the unit costs of backlogging ω_i are increased by up to 40%, then the increase in the total cost is small (less than 5%) and the decrease in the expected benefit is less than 1%.

It can also be seen that a 40% increase in item volumes leads to an increase in the total cost of less than 4% and a decrease in the expected profit of less than 1%.

In summary, from the previous results it can be deduced that the variation of the parameters h_i, ω_i and ν_i does not have a great impact on the cost of the system, and its effect is very small on the expected benefit. However, if we increase the scale parameters of the demand distributions then, although the total cost of the system increases, the expected benefit of inventory management is significantly increased.

Determining optimal policies to manage multi-item inventory systems with stochastic demands, backlogged shortages, and limited storage capacity, is an important decision that affects the economic survival of many firms, their business, job offers, and services. That happens because these firms require the efficient management of inventory systems with the characteristics previously mentioned. From practical point of view, the proposed approach provides the managers a procedure to manage these inventory systems.

9. Conclusions

The main contribution of this paper consists in developing the optimal inventory policy for multi-item inventory systems with stochastic demands, fixed scheduling period, backlogged shortages, and power demand patterns, assuming that the storage space available in the warehouse is limited. From the point of view of decision-making, incorporating stochastic demands and power demand patterns may help to better fit the evolution of the inventory levels of the products. Moreover, considering multiple items and incorporating backlogging allows more realistic inventory models to be designed. All these assumptions extend and add new contributions to other papers previously analysed. The results obtained have a direct impact on the inventory policy to follow, reducing the expected costs of inventory management.

A probabilistic inventory model has been formulated to take decisions that will maximize the expected profit per unit time. In this inventory model, demands of the items are not exactly known, but it is assumed that they follow certain probability distributions. The ways by which quantities are taken out of the inventory follow a time-dependent power function. Each of these functions represents the temporal concentration of customer demand for each item throughout the scheduling period and they are known as power demand patterns.

To solve the probabilistic multi-item inventory problem, we must calculate the initial inventory levels that maximize the expected inventory management profit of all the products. First, we determine the optimal inventory policy for a probabilistic inventory system with full backlogging, without considering the capacity of the warehouse, achieving the optimal stock levels. The optimal solution depends on the demand probability distributions, the input parameters, and the power demand pattern index of each item, i.e., the optimal policy depends on the behaviour of customer demand for each product during the inventory cycle.

Next, we analyse the model for the probabilistic multi-item inventory system with power demand patterns and full backlogging, when the warehouse used to store the items has a fixed capacity. In this situation, to obtain the optimal inventory policy we have developed an efficient algorithm.

Future research would be to study the probabilistic inventory system for multiple items considering a deterioration rate for items and power demand patterns. Another research line may be to analyse the probabilistic inventory system for multiple items and power demand patterns, assuming that shortages are lost sales. Finally, it would also be interesting to develop the optimal policy for the multi-item inventory system with stochastic demands and power demand patterns, supposing that replenishments of the products are not instantaneous and there exists a production rate for each item in the inventory model.

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Appendix

In this appendix, we give the proofs of the main results proposed in the paper.

Proof of Proposition 1.

$Z_i(S_i)$ is a strictly positive function which takes values in the interval [0,1] when $S_i \geq 0$, because

$$0 \leq 1 - \left(\frac{S_i}{x_i}\right)^{n_i} \leq 1, \qquad \text{for all} \qquad x_i \geq S_i$$

This function is continuous and differentiable on the interval $(0, b_i)$. Thus, its derivative is

$$Z_i'(S_i) = -n_i S_i^{n_i - 1} \int_{S_i}^{\infty} \frac{1}{x_i^{n_i}} f_i(x_i) dx_i < 0$$

Therefore, the function $Z_i(S_i)$ is strictly decreasing on the interval $[0, b_i)$. Moreover, for each item i, if b_i is a finite value, then $Z_i(S_i) = 0$ for all $S_i \geq b_i$. Also, the function $Z_i(S_i)$ satisfies $Z_i(0) = \int_0^{\infty} f_i(x_i) dx_i = 1$ and $\lim_{S_i \to \infty} Z_i(S_i) = 0$.

Proof of Theorem 1.

(i) The cost function (14) is a continuous function and twice differentiable. Thus, the first partial derivatives of the cost function $C(S_1, \ldots, S_N)$ with respect to the variables S_i $(i = 1, 2, \ldots, N)$ are:

$$\frac{\partial C}{\partial S_i} = (h_i + \omega_i) \int_0^{S_i} f_i(x_i) dx_i + (h_i + \omega_i) \int_{S_i}^{\infty} \left(\frac{S_i}{x_i}\right)^{n_i} f_i(x_i) dx_i - \omega_i, \qquad i = 1, 2, \ldots, N$$

The second partial derivatives of the cost function $C(S_1, \ldots, S_N)$ are given by

$$\frac{\partial^2 C}{\partial S_i^2} = (h_i + \omega_i) \int_{S_i}^{\infty} \frac{n_i S_i^{n_i - 1}}{x_i^{n_i}} f_i(x_i) dx_i, \qquad i = 1, 2, \ldots, N$$

$$\frac{\partial^2 C}{\partial S_i \partial S_j} = 0, \qquad i, j = 1, 2, \ldots, N \text{ and } i \neq j$$

Thus, the Hessian $H = H(S_1, S_2, \ldots, S_N)$ is given by

$$H = \prod_{i=1}^{N} (h_i + \omega_i) \int_{S_i}^{\infty} \frac{n_i S_i^{n_i - 1}}{x_i^{n_i}} f_i(x_i) dx_i$$

Therefore, as the second partial derivatives of the cost function with respect to S_i, for $0 < S_i < b_i$ and $i = 1, 2, \ldots, N$, are always positive and the Hessian is also positive, the cost function (14) is a strictly convex function on the interior of Ω.

(ii) Equating to zero the first partial derivatives of $C(S_1, \ldots, S_N)$ with respect to the variables S_i, we can get the optimal inventory levels $(S_1^0, S_2^0, \ldots, S_N^0)$. These optimal levels are obtained by solving the integral equation given in (17) for each item $i = 1, 2, \ldots, N$. From Proposition 1, $Z_i(S_i)$ is a strictly positive and decreasing function on $[0, b_i)$, which takes values in the interval $(0, 1]$. As the ratio $h_i/(h_i + \omega_i)$ is a positive value less than one, then the Eq. (17) has a unique positive solution S_i^0 for each $i = 1, 2, \ldots, N$.

Moreover, the above levels are always strictly positive values and less than b_i (that is, $b_i > S_i^0 > 0$, for $i = 1, 2, \ldots, N$). Thus, the optimal inventory policy $(S_1^0, S_2^0, \ldots, S_N^0)$ belongs to the interior of the region Ω.

Proof of Theorem 2.

From Proposition 1 and (29), $g(\lambda)$ is a continuous function on the interval $[0, \infty)$. Next, we prove that $g(\lambda)$ is a strictly decreasing function on the interval $(0, \lambda_{max})$, with $\lambda_{max} = \max_{1\le i\le N}\{\frac{\omega_i}{\nu_i}\}$.

Let $0 < \lambda_1 < \lambda_2 \le \lambda_{\max}$, then $I(\lambda_2) \subseteq I(\lambda_1)$ and, from Proposition 1, we have $S_i(\lambda_1) > S_i(\lambda_2)$, for $i \in I(\lambda_1)$. Thus, we obtain

$$\sum_{i\in I(\lambda_1)} \nu_i S_i(\lambda_1) > \sum_{i\in I(\lambda_2)} \nu_i S_i(\lambda_2)$$

Consequently, from (30), we get $g(\lambda_1) > g(\lambda_2)$. Moreover, from (29) and (30), $g(\lambda) = -W < 0$ for $\lambda \ge \max_{1\le i\le N}\{\frac{\omega_i}{\nu_i}\}$. Therefore, if $g(0) > 0$, as $g(\lambda)$ is a strictly decreasing function, it is deduced that the function $g(\lambda)$ has a unique root λ^* in the interval $(0, \lambda_{\max})$.

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