Finite Fields and Their Applications 91 (2023) 102272

Contents lists available at ScienceDirect

Finite Fields and Their Applications

journal homepage: www.elsevier.com/locate/ffa

Doubly and triply extended MSRD codes

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ARTICLE INFO

Article history:
Received 11 December 2022
Received in revised form 24 July 2023
Accepted 24 July 2023
Available online 7 August 2023
Communicated by Gary L. Mullen

MSC:
15B33
94B05
94B65

Keywords:
Doubly extended codes
Linearized Reed–Solomon codes
Maximum sum-rank distance codes
Sum-rank metric
Triply extended

ABSTRACT

In this work, doubly extended linearized Reed–Solomon codes and triply extended Reed–Solomon codes are generalized. We obtain a general result in which we characterize when a multiply extended code for a general metric attains the Singleton bound. We then use this result to obtain several families of doubly extended and triply extended maximum sum-rank distance (MSRD) codes that include doubly extended linearized Reed–Solomon codes and triply extended Reed–Solomon codes as particular cases. To conclude, we discuss when these codes are one-weight codes.

1. Introduction

Let $\mathbb{F}_q$ denote the finite field of size q, and denote by $\mathbb{F}_q^n$ and $\mathbb{F}_q^{m\times n}$ the spaces of row vectors of length n and matrices of size $m \times n$, respectively, over $\mathbb{F}_q$, for positive integers m and n. We also denote $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$ for a positive integer n. The Hamming metric in $\mathbb{F}_q^n$ is given by $\mathrm{d}_H(\mathbf{c}, \mathbf{d}) = |\{i \in [n] \mid c_i \neq d_i\}|$, for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^n$.

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https://doi.org/10.1016/j.ffa.2023.102272
1071-5797/

Doubly extended Reed–Solomon codes [6, Sec. 5.3] [9, Ch. 11, Sec. 5] are the linear codes in $\mathbb{F}_q^{n+2}$ given by the generator matrix

$$\left(\begin{array}{cccc|cc} 1 & 1 & \ldots & 1 & 1 & 0 \\ a_1 & a_2 & \ldots & a_n & 0 & 0 \\ a_1^2 & a_2^2 & \ldots & a_n^2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_1^{k-2} & a_2^{k-2} & \ldots & a_n^{k-2} & 0 & 0 \\ a_1^{k-1} & a_2^{k-1} & \ldots & a_n^{k-1} & 0 & 1 \end{array} \right) \in \mathbb{F}_q^{k \times (n+2)},$$

for $k \in [n]$ and distinct $a_1, a_2, \ldots, a_n \in \mathbb{F}_q^*$ (hence $n \leq q-1$ and $n+2 \leq q+1$, where equalities may be attained). One may show by using conventional polynomial results that the doubly extended Reed–Solomon code above is maximum distance separable (MDS). See [6, Th. 5.3.4]. In other words, it attains the Singleton bound for the Hamming metric. Furthermore, these codes may have length $q+1$, which is conjectured to be maximum for most values of the code dimension k. This is the well-known *MDS conjecture* (see [6, Sec. 7.4]), which has been proven for q prime [2].

Recently, a generalization of this result was given in [16] for the sum-rank metric, a metric that simultaneously generalizes the Hamming metric and the rank metric [4,5,18]. The generalization of Reed–Solomon codes to the sum-rank metric is called linearized Reed–Solomon codes, introduced in [11], which are maximum sum-rank distance (MSRD) codes, i.e., they attain the Singleton bound for the sum-rank metric. More general families of linear MSRD codes exist [12,15]. The authors of [16] introduced doubly extended linearized Reed–Solomon codes and showed, using geometric tools, that they are also MSRD.

In this work, we show how one may extend codes attaining the Singleton bound for any metric given by a weight. The metric considered for the extended codes is obtained by adding Hamming-metric components, as was done for the sum-rank metric in [16] (Section 2). In Section 3, we provide necessary and sufficient conditions for multiply extended codes to attain the Singleton bound based on the original codes. In Sections 4 and 5, we apply double and triple extensions, respectively, to the general MSRD codes obtained in [12], which include linearized Reed–Solomon codes (and therefore classical Reed–Solomon codes and Gabidulin codes [5,18]). In Section 6, we study what happens when the extended portion is not considered with Hamming-metric components, but by considering the rank metric in the whole added block, and show that doubly extended codes are no longer MSRD in general. Finally, in Section 7, we investigate when the obtained doubly and triply extended MSRD codes are one-weight codes.

We conclude this introduction by remarking that the results in this manuscript have several interesting geometric counterparts. First, two-dimensional MSRD codes correspond to the order sets of disjoint scattered linear sets on the projective line [16, Sec. 6]. When they reach the classical Singleton bound, MSRD codes correspond to order sets of scattered linear sets with respect to hyperplanes [16, Cor. 3.10]. Finally, MSRD codes correspond to order sets of disjoint maximum scattered linear sets when considering MSRD codes with certain parameters [19, Th. 7.4].

2. The Singleton bound for sums of metrics

In this manuscript, we consider metrics given by weights. Here, a weight function is a function $\mathrm{wt} : \mathbb{F}_q^n \longrightarrow \mathbb{N}$ satisfying the following properties:

1. $\mathrm{wt}(\mathbf{c}) \geq 0$, for all $\mathbf{c} \in \mathbb{F}_q^n$, and it equals 0 if, and only if, $\mathbf{c} = \mathbf{0}$.
2. $\mathrm{wt}(\lambda \mathbf{c}) = \mathrm{wt}(\mathbf{c})$, for all $\mathbf{c} \in \mathbb{F}_q^n$ and all $\lambda \in \mathbb{F}_q^*$.
3. $\mathrm{wt}(\mathbf{c} + \mathbf{d}) \leq \mathrm{wt}(\mathbf{c}) + \mathrm{wt}(\mathbf{d})$, for all $\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^n$.

Its associated metric is the function $\mathrm{d} : (\mathbb{F}_q^n)^2 \longrightarrow \mathbb{N}$ given by $\mathrm{d}(\mathbf{c}, \mathbf{d}) = \mathrm{wt}(\mathbf{c} - \mathbf{d})$, for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_q^n$. It is straightforward to prove that a metric given by a weight as above is indeed a metric (see [6, Th. 1.4.1]).

As usual, we define the minimum distance of a code $\mathcal{C} \subseteq \mathbb{F}_q^n$ (a code is just a set) with respect to d as

$$\mathrm{d}(\mathcal{C}) = \min\{\mathrm{d}(\mathbf{c}, \mathbf{d}) \mid \mathbf{c}, \mathbf{d} \in \mathcal{C}, \mathbf{c} \neq \mathbf{d}\}.$$

It is well-known that, if $\mathcal{C}$ is linear (i.e., an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^n$), then $\mathrm{d}(\mathcal{C}) = \min\{\mathrm{wt}(\mathbf{c}) \mid \mathbf{c} \in \mathcal{C} \setminus \{\mathbf{0}\}\}$, where wt is the weight giving the metric d.

We will say that a metric d satisfies the Singleton bound if

$$\mathrm{d}(\mathcal{C}) \leq n - k + 1, \tag{1}$$

where $k = \log_q |\mathcal{C}|$, for any code $\mathcal{C} \subseteq \mathbb{F}_q^n$. Any metric given by a weight that is upper bounded by the Hamming weight satisfies the Singleton bound. Many examples exist, including the Hamming metric itself, the rank metric [4,5], the sum-rank metric [11], the cover metric [18] and the multi-cover metric [13], among others.

Some of these metrics, e.g., the sum-rank metric, the multi-cover metric or the Hamming metric itself, are given by sums of other metrics. In general, given weights wt_i in $\mathbb{F}_q^{n_i}$, for $i \in [\ell]$, we may define their sum as

$$\mathrm{wt}_{\mathrm{sum}}(\mathbf{c}) = \mathrm{wt}_1(\mathbf{c}_1) + \mathrm{wt}_2(\mathbf{c}_2) + \cdots + \mathrm{wt}_\ell(\mathbf{c}_\ell),$$

for $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_\ell) \in \mathbb{F}_q^n$, where $n = n_1 + n_2 + \cdots + n_\ell$ and $\mathbf{c}_i \in \mathbb{F}_q^{n_i}$, for $i \in [\ell]$. Clearly, $\mathrm{wt}_{\mathrm{sum}}$ is a weight. We denote similarly the corresponding associated metric. It is easy to see that $\mathrm{d}_{\mathrm{sum}}$ satisfies the Singleton bound if so do the metrics d_i, for $i \in [\ell]$.

In the remainder of the manuscript, we will only consider metrics $\mathrm{d} : (\mathbb{F}_q^n)^2 \longrightarrow \mathbb{N}$ given by weights and satisfying the Singleton bound (1).

3. Multiply extended codes

In this section, we give a definition of multiply extended codes for general metrics and show that they attain the Singleton bound if so do certain codes related to the original code and the metric is extended by adding a Hamming-metric component. In Sections 4 and 5, we will particularize these results to construct doubly and triply extended MSRD codes. In the following, $\langle \cdot \rangle_{\mathbb{F}_q}$ denotes linear span over $\mathbb{F}_q$.

Theorem 1. *Let $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k \in \mathbb{F}_q^n$ be linearly independent, and let $t \in [k]$. Consider the k-dimensional linear code $\mathcal{C}_e \subseteq \mathbb{F}_q^{n+t}$ with generator matrix*

$$G_e = \left(\begin{array}{c|cccc} \mathbf{g}_1 & 1 & 0 & \ldots & 0 \\ \mathbf{g}_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_t & 0 & 0 & \ldots & 1 \\ \hline \mathbf{g}_{t+1} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{g}_k & 0 & 0 & \ldots & 0 \end{array} \right) \in \mathbb{F}_q^{k \times (n+t)}.$$

Define also the linear codes $\mathcal{C}_I = \langle \{ \mathbf{g}_i \mid i \in I \} \rangle_{\mathbb{F}_q} + \langle \mathbf{g}_{t+1}, \ldots, \mathbf{g}_k \rangle_{\mathbb{F}_q}$, and set $d_I = \mathrm{d}(\mathcal{C}_I)$, for $I \subseteq [t]$. Then it holds that $\mathrm{d}_e(\mathcal{C}_e) = \min\{ d_I + |I| \mid I \subseteq [t] \}$, where the metric $\mathrm{d}_e : (\mathbb{F}_q^{n+t})^2 \longrightarrow \mathbb{N}$ is given by

$$\mathrm{d}_e((\mathbf{c}_1, \mathbf{c}_2), (\mathbf{d}_1, \mathbf{d}_2)) = \mathrm{d}(\mathbf{c}_1, \mathbf{d}_1) + \mathrm{d}_H(\mathbf{c}_2, \mathbf{d}_2),$$

for $\mathbf{c}_1, \mathbf{d}_1 \in \mathbb{F}_q^n$ and $\mathbf{c}_2, \mathbf{d}_2 \in \mathbb{F}_q^t$.

Proof. Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_t \in \mathbb{F}_q^t$ denote the canonical basis. A codeword in $\mathcal{C}_e$ is of the form

$$\mathbf{c} = \left(\sum_{i \in I} \lambda_i \mathbf{g}_i + \sum_{j=t+1}^{k} \lambda_j \mathbf{g}_j, \sum_{i \in I} \lambda_i \mathbf{e}_i \right),$$

where $I \subseteq [t]$, $\lambda_i \in \mathbb{F}_q^*$, for $i \in I$, and $\lambda_j \in \mathbb{F}_q$, for $j = t+1, \ldots, k$. Note that possibly $I = \varnothing$. Since $\lambda_i \neq 0$ for $i \in I$, we deduce that

$$\begin{aligned} \mathrm{wt}_e(\mathbf{c}) &= \mathrm{wt}\left(\sum_{i \in I} \lambda_i \mathbf{g}_i + \sum_{j=t+1}^{k} \lambda_j \mathbf{g}_j \right) + \mathrm{wt}_H\left(\sum_{i \in I} \lambda_i \mathbf{e}_i \right) \\ &= \mathrm{wt}\left(\sum_{i \in I} \lambda_i \mathbf{g}_i + \sum_{j=t+1}^{k} \lambda_j \mathbf{g}_j \right) + |I| \\ &\geq \mathrm{d}_I + |I|. \end{aligned}$$

Therefore, we have that $\mathrm{d}_e(\mathcal{C}_e) \geq \min\{d_I + |I| \mid I \subseteq [t]\}$.

We now prove the reversed inequality. Consider a subset $J \subseteq [t]$ such that

$$\min\{d_I + |I| \mid I \subseteq [t]\} = d_J + |J|,$$

and take $\mathbf{d} = \sum_{i \in J} \lambda_i \mathbf{g}_i + \sum_{j=t+1}^{k} \lambda_j \mathbf{g}_j \in \mathcal{C}_J$ such that $\mathrm{wt}(\mathbf{d}) = d_J$, where $\lambda_i \in \mathbb{F}_q$ for $i \in J \cup \{t+1, \ldots, k\}$. Setting

$$\mathbf{c} = \left(\mathbf{d}, \sum_{i \in J} \lambda_i \mathbf{e}_i \right) \in \mathcal{C}_e,$$

we conclude that

$$\mathrm{d}_e(\mathcal{C}_e) \leq \mathrm{wt}_e(\mathbf{c}) \leq d_J + |J| = \min\{d_I + |I| \mid I \subseteq [t]\},$$

therefore $\mathrm{d}_e(\mathcal{C}_e) \leq \min\{d_I + |I| \mid I \subseteq [t]\}$ and we are done. □

We now deduce the following result on multiply extended codes that attain the Singleton bound.

Corollary 1. *With notation as in Theorem 1, the code $\mathcal{C}_e$ attains the Singleton bound for d_e if, and only if, so do the codes $\mathcal{C}_I$ for d, for all $I \subseteq [t]$.*

Proof. Note that $\dim(\mathcal{C}_e) = k$ and $\dim(\mathcal{C}_I) = k + |I| - t$, for $I \subseteq [t]$. Hence $\mathcal{C}_I$ attains the Singleton bound for d if, and only if,

$$d_I = n - (k + |I| - t) + 1 = (n + t) - k - |I| + 1.$$

We also have that $\mathcal{C}_e$ attains the Singleton bound if, and only if,

$$\begin{aligned} \mathrm{d}_e(\mathcal{C}_e) &= \min\{d_I + |I| \mid I \subseteq [t]\} \\ &= (n+t) - k + 1 \\ &= \min\{(n+t) - k - |I| + 1 + |I| \mid I \subseteq [t]\}, \end{aligned}$$

and the result follows. □

Remark 2. Setting $t = k$ and $\mathrm{d} = \mathrm{d}_H$ (i.e., $\mathrm{d}_e = \mathrm{d}_H$), then Corollary 1 recovers the well-known characterization of systematic generator matrices of MDS codes from [9, Ch. 11, Th. 8]. In other words, when $t = k$ and $\mathrm{d} = \mathrm{d}_H$, Corollary 1 states that $\mathcal{C}_e$ is MDS if, and only if, every square submatrix of G is invertible, where G is the matrix whose rows are $\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k \in \mathbb{F}_q^n$. Corollary 1 extends this result to any $t \in [k]$ and any metric d given by a weight satisfying the Singleton bound.

Finally, we note that we have a lattice of linear codes $\mathcal{C}_I \subseteq \mathbb{F}_q^n$, for $I \subseteq [t]$, with respect to inclusions or, equivalently, unions and intersections, i.e., we have the following inclusion graph:

$$\begin{array}{ccccc} & & \mathcal{C}_{I\cup J} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{C}_I & & & & \mathcal{C}_J \\ & \nwarrow & & \nearrow & \\ & & \mathcal{C}_{I\cap J}. & & \end{array}$$

By taking systematic generator matrices, we deduce that the existence of a linear code in $\mathbb{F}_q^{n+t}$ attaining the Singleton bound for d_e is equivalent to the existence of a lattice of linear codes $\mathcal{C}_I \subseteq \mathbb{F}_q^n$, for $I \subseteq [t]$, as above, attaining the Singleton bound for d. This property also holds for the dual codes, as stated in the following proposition. Here, we define the dual of a linear code $\mathcal{C} \subseteq \mathbb{F}_q^n$ as usual: $\mathcal{C}^\perp = \{\mathbf{d} \in \mathbb{F}_q^n \mid \mathbf{c} \cdot \mathbf{d}^\intercal = 0, \forall \mathbf{c} \in \mathcal{C}\}$.

Proposition 3. *Let $\mathcal{C}_I \subseteq \mathbb{F}_q^n$, for $I \subseteq [t]$, be a family of linear codes such that the map $I \mapsto \mathcal{C}_I$ is a lattice isomorphism. Define now the linear codes $\mathcal{D}_I = (\mathcal{C}_{I^c})^\perp \subseteq \mathbb{F}_q^n$, for $I \subseteq [t]$, where $I^c = [t] \setminus I$ denotes the complement of I in $[t]$. Then the map $I \mapsto \mathcal{D}_I$ is also a lattice isomorphism.*

Proof. Simply notice that, for $I, J \subseteq [t]$, we have

$$\mathcal{D}_I + \mathcal{D}_J = (\mathcal{C}_{I^c})^\perp + (\mathcal{C}_{J^c})^\perp = (\mathcal{C}_{I^c} \cap \mathcal{C}_{J^c})^\perp = (\mathcal{C}_{I^c \cap J^c})^\perp = (\mathcal{C}_{(I\cup J)^c})^\perp = \mathcal{D}_{I\cup J},$$
$$\mathcal{D}_I \cap \mathcal{D}_J = (\mathcal{C}_{I^c})^\perp \cap (\mathcal{C}_{J^c})^\perp = (\mathcal{C}_{I^c} + \mathcal{C}_{J^c})^\perp = (\mathcal{C}_{I^c \cup J^c})^\perp = (\mathcal{C}_{(I\cap J)^c})^\perp = \mathcal{D}_{I\cap J}.$$

In both lines, we use in the third equality that $I \mapsto \mathcal{C}_I$ is a lattice isomorphism. □

Assume that d is a metric such that a linear code attains the Singleton bound if, and only if, so does its dual code. In such a case, Proposition 3 states that we do not need to check the conditions in Corollary 1 for both the primary and dual codes, but only for one of them. This is the case of the sum-rank metric [10, Th. 5], and thus of the Hamming and rank metrics in particular.

4. Doubly extended MSRD codes

In this section, we generalize the construction of doubly extended linearized Reed–Solomon codes from [16] to the general family of MSRD codes from [12]. Using Corollary 1, we will show that such doubly extended MSRD codes are again MSRD.

Recall that the sum-rank metric [17] in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition (g, r) is defined as a sum of rank metrics, i.e., sum-rank weights are given by

$$\mathrm{wt}_{SR}(\mathbf{c}) = \sum_{i=1}^{g} \mathrm{wt}_R\left(\mathbf{c}^{(i)}\right),$$

for $\mathbf{c} = \left(\mathbf{c}^{(1)}, \mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(g)}\right) \in \mathbb{F}_{q^m}^n$, where $\mathbf{c}^{(i)} \in \mathbb{F}_{q^m}^r$, for $i \in [g]$, and $n = gr$. Recall that rank weights in $\mathbb{F}_{q^m}^r$ are given by $\mathrm{wt}_R(\mathbf{d}) = \dim_{\mathbb{F}_q}(\langle d_1, d_2, \ldots, d_r\rangle_{\mathbb{F}_q})$, for $\mathbf{d} = (d_1, d_2, \ldots, d_r) \in \mathbb{F}_{q^m}^r$.

We now give the definition of extended Moore matrices from [12, Def. 3.4].

Definition 4 *(Extended Moore matrices [12]).* Fix positive integers ℓ and η. Let $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ be such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$, where $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a) = a \cdot a^q \cdots a^{q^{m-1}}$, for $a \in \mathbb{F}_{q^m}$. For any $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_\eta) \in \mathbb{F}_{q^m}^\eta$ and $k \in [\ell\eta]$, we define the *extended Moore matrix* $M_k(\mathbf{a}, \boldsymbol{\beta}) \in \mathbb{F}_{q^m}^{k \times (\ell\eta)}$ by $M_k(\mathbf{a}, \boldsymbol{\beta}) =$

$$\left(\begin{array}{lll|c|lll}
\beta_1 & \ldots & \beta_\eta & \ldots & \beta_1 & \ldots & \beta_\eta \\
\beta_1^q a_1 & \ldots & \beta_\eta^q a_1 & \ldots & \beta_1^q a_\ell & \ldots & \beta_\eta^q a_\ell \\
\beta_1^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_1^{q^2} a_\ell^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_\ell^{\frac{q^2-1}{q-1}} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\beta_1^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_\eta^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_1^{q^{k-1}} a_\ell^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_\eta^{q^{k-1}} a_\ell^{\frac{q^{k-1}-1}{q-1}}
\end{array}\right),$$

and we denote by $\mathcal{C}_k(\mathbf{a}, \boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^{\ell\eta}$ the k-dimensional linear code generated by $M_k(\mathbf{a}, \boldsymbol{\beta})$ (i.e., the rows of $M_k(\mathbf{a}, \boldsymbol{\beta})$ generate the vector space $\mathcal{C}_k(\mathbf{a}, \boldsymbol{\beta})$).

The following result [12, Th. 3.12] characterizes when a code with an extended Moore matrix as generator or parity-check matrix is MSRD.

Theorem 2 *([12]). Let $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ be as in Definition 4. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$, for positive integers μ and r, and set $g = \ell\mu$. Define the $\mathbb{F}_q$-linear subspace*

$$\mathcal{H}_i = \left\langle \beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir} \right\rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}, \tag{2}$$

for $i \in [\mu]$. Given $k \in [gr]$, the code $\mathcal{C}_k(\mathbf{a}, \boldsymbol{\beta})$ from Definition 4 is MSRD over $\mathbb{F}_q$ for the length partition (g, r) if, and only if, the following two conditions hold for all $i \in [\mu]$:

1. *$\dim_{\mathbb{F}_q}(\mathcal{H}_i) = r$, and*
2. *$\mathcal{H}_i \cap \left(\sum_{j \in \Gamma} \mathcal{H}_j\right) = \{0\}$, for any set $\Gamma \subseteq [\mu]$, such that $i \notin \Gamma$ and $|\Gamma| \leq \min\{k, \mu\} - 1$.*

Several constructions of MSRD codes based on Theorem 2 were obtained in [12]. These include linearized Reed–Solomon codes [11] by taking $\mu = 1$ (in that case, Condition 2 is empty and Condition 1 means that $\beta_1, \beta_2, \ldots, \beta_r$ are $\mathbb{F}_q$-linearly independent).

For our purposes, we also need to consider the k-dimensional linear codes $\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^{\ell\eta}$ with generator matrices $M'_k(\mathbf{a},\boldsymbol{\beta}) =$

$$\left(\begin{array}{ccc|c|ccc} \beta_1^q a_1 & \ldots & \beta_\eta^q a_1 & \ldots & \beta_1^q a_\ell & \ldots & \beta_\eta^q a_\ell \\ \beta_1^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_1^{q^2} a_\ell^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_\ell^{\frac{q^2-1}{q-1}} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \beta_1^{q^k} a_1^{\frac{q^k-1}{q-1}} & \ldots & \beta_\eta^{q^k} a_1^{\frac{q^k-1}{q-1}} & \ldots & \beta_1^{q^k} a_\ell^{\frac{q^k-1}{q-1}} & \ldots & \beta_\eta^{q^k} a_\ell^{\frac{q^k-1}{q-1}} \end{array}\right),$$

for $k \in [\ell\eta]$. Observe that we have the following inclusion graph:

$$\begin{array}{ccccc} & & \mathcal{C}_k(\mathbf{a},\boldsymbol{\beta}) & & \\ & \nearrow & & \nwarrow & \\ \mathcal{C}_{k-1}(\mathbf{a},\boldsymbol{\beta}) & & & & \mathcal{D}_{k-1}(\mathbf{a},\boldsymbol{\beta}) \\ & \nwarrow & & \nearrow & \\ & & \mathcal{D}_{k-2}(\mathbf{a},\boldsymbol{\beta}). & & \end{array}$$

The codes $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})$ are MSRD given Conditions 1 and 2 in Theorem 2. We now show that the same conditions turn the codes $\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta})$ into MSRD codes.

Lemma 5. *Let ℓ, μ and r be positive integers, let $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ as in Theorem 2, and set $g = \ell\mu$. For $k \in [gr]$, $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})$ is MSRD if, and only if, so is $\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta})$, in both cases over $\mathbb{F}_q$ for the length partition (g, r).*

Proof. For $a, \beta \in \mathbb{F}_{q^m}$ and a positive integer i, we have that

$$\beta^{q^i} a^{\frac{q^i-1}{q-1}} = \beta^{q^i} a^{q^{i-1}} \cdots a^q \cdot a = \left(\beta^{q^{i-1}} a^{q^{i-2}} \cdots a^q \cdot a\right)^q a = \left(\beta^{q^{i-1}} a^{\frac{q^{i-2}-1}{q-1}}\right)^q a.$$

Hence it holds that

$$M'_k(\mathbf{a},\boldsymbol{\beta}) = M_k(\mathbf{a},\boldsymbol{\beta})^q \mathrm{diag}(a_1, \ldots, a_1 | \ldots | a_\ell, \ldots, a_\ell),$$

where $M_k(\mathbf{a},\boldsymbol{\beta})^q$ means that we raise every entry of $M_k(\mathbf{a},\boldsymbol{\beta})$ to the qth power, and $\mathrm{diag}(\cdot)$ denotes diagonal matrix. In particular, the same holds for the corresponding codes, i.e.,

$$\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta}) = \mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})^q \mathrm{diag}(a_1, \ldots, a_1 | \ldots | a_\ell, \ldots, a_\ell),$$

where $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})^q$ means that we raise every component of every codeword of $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})$ to the qth power. Now, observe that the map $\phi : \mathbb{F}_{q^m}^{gr} \longrightarrow \mathbb{F}_{q^m}^{gr}$ given by

$$\begin{aligned} &\phi\left(c_1, \ldots, c_{\mu r} | \ldots | c_{(\ell-1)(\mu r)+1}, \ldots, c_{\ell(\mu r)}\right) \\ &\quad = \left(c_1^q a_1, \ldots, c_{\mu r}^q a_1 | \ldots | c_{(\ell-1)(\mu r)+1}^q a_\ell, \ldots, c_{\ell(\mu r)}^q a_\ell\right) \end{aligned}$$

is a semilinear isometry for the sum-rank metric over $\mathbb{F}_q$ for the length partition (g,r), since $a_i \neq 0$, for $i \in [\ell]$ (see [1, Cor. 3.8]). Hence the result follows. $\square$

Remark 6. Using the skew polynomial description of the previous codes (see, e.g., [11, 14]), $\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta})$ is exactly as $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})$, except that skew polynomial powers go from 0 to $k-1$ in $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta})$, and they go from 1 to k in $\mathcal{D}_k(\mathbf{a},\boldsymbol{\beta})$. The proof of the previous lemma implicitly relies on this fact.

Therefore, we are in the situation of Corollary 1 for the sum-rank metric. For this reason, we define the following codes.

Definition 7. Let $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ be as in Definition 4. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_\eta) \in \mathbb{F}_{q^m}^\eta$ be arbitrary, for a positive integer η. For $k = 2, 3, \ldots, \ell\eta$, we define the *doubly extended Moore matrix* $M_k^e(\mathbf{a},\boldsymbol{\beta}) \in \mathbb{F}_{q^m}^{k\times(\ell\eta+2)}$ by $M_k^e(\mathbf{a},\boldsymbol{\beta}) =$

$$\left(\begin{array}{ccc|c|ccc|cc}
\beta_1 & \ldots & \beta_\eta & \ldots & \beta_1 & \ldots & \beta_\eta & 1 & 0 \\
\beta_1^q a_1 & \ldots & \beta_\eta^q a_1 & \ldots & \beta_1^q a_\ell & \ldots & \beta_\eta^q a_\ell & 0 & 0 \\
\beta_1^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_1^{\frac{q^2-1}{q-1}} & \ldots & \beta_1^{q^2} a_\ell^{\frac{q^2-1}{q-1}} & \ldots & \beta_\eta^{q^2} a_\ell^{\frac{q^2-1}{q-1}} & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\beta_1^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_\eta^{q^{k-1}} a_1^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_1^{q^{k-1}} a_\ell^{\frac{q^{k-1}-1}{q-1}} & \ldots & \beta_\eta^{q^{k-1}} a_\ell^{\frac{q^{k-1}-1}{q-1}} & 0 & 1
\end{array}\right),$$

and we denote by $\mathcal{C}_k^e(\mathbf{a},\boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^{\ell\eta+2}$ the k-dimensional linear code generated by $M_k^e(\mathbf{a},\boldsymbol{\beta})$.

Thus, by Corollary 1 and Lemma 5, we deduce the following.

Corollary 8. *Let ℓ, μ and r be positive integers, define $g = \ell\mu$ and $n = gr$, and let $\mathbf{a} = (a_1, a_2, \ldots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ as in Theorem 2. For $k = 2, 3, \ldots, n$, $\mathcal{C}_k(\mathbf{a},\boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^n$ is MSRD (i.e., Conditions 1 and 2 in Theorem 2 hold) if, and only if, $\mathcal{C}_k^e(\mathbf{a},\boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^{n+2}$ is MSRD for the extended sum-rank metric*

$$\mathrm{d}_e((\mathbf{c}, c_{n+1}, c_{n+2}), (\mathbf{d}, d_{n+1}, d_{n+2})) = \mathrm{d}_{SR}(\mathbf{c},\mathbf{d}) + \mathrm{d}_H((c_{n+1}, c_{n+2}), (d_{n+1}, d_{n+2})),$$

for $\mathbf{c},\mathbf{d} \in \mathbb{F}_{q^m}^n$ and $c_{n+1}, c_{n+2}, d_{n+1}, d_{n+2} \in \mathbb{F}_{q^m}$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition (g,r).

In particular, if $\ell = q-1$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ satisfies Conditions 1 and 2 in Theorem 2, then the doubly extended code $\mathcal{C}_k^e(\mathbf{a},\boldsymbol{\beta}) \subseteq \mathbb{F}_{q^m}^{n+2}$ is MSRD as in the corollary above, where $n = (q-1)\mu r$ and where we consider in $\mathbb{F}_{q^m}^n$ the sum-rank metric over $\mathbb{F}_q$ for the length partition (g,r), $g = (q-1)\mu$. See [12] for seven concrete explicit families of MSRD codes constructed in this way. All of them can be doubly extended as mentioned in this paragraph while preserving their MSRD property.

In particular, choosing $\mu = 1$, Corollary 8 recovers [16, Th. 4.6] as a particular case for linearized Reed–Solomon codes, which in turn recovers the classical result [6, Th. 5.3.4] for classical Reed–Solomon codes.

5. Triply extended MSRD codes

In contrast to the case of doubly extended MSRD codes (Section 4), triply extended MSRD codes are not always MSRD, as we show in this section. We will only consider 3-dimensional codes.

We start with cases where triple extension preserves the MSRD property. Notice that the case of (3-dimensional) classical Reed–Solomon codes and the Hamming metric in characteristic 2 [9, p. 326, Ch. 11, Th. 10] is recovered from the following theorem by taking $m = \mu = r = 1$ and $\beta_1 = 1$.

Theorem 3. *Let m be odd, let q be even, and set $n = (q-1)\mu r$ for positive integers μ and r. Let $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ satisfy Conditions 1 and 2 in Theorem 2. Let $\mathbf{a} = (a_1, a_2, \ldots, a_{q-1}) \in (\mathbb{F}_{q^m}^*)^{q-1}$ be such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$. The triply extended code $\mathcal{C}_e \subseteq \mathbb{F}_{q^m}^{n+3}$ with generator matrix*

$$G_e = \left(\begin{array}{ccc|c|ccc|ccc} \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} & 1 & 0 & 0 \\ a_1 \beta_1^q & \ldots & a_1 \beta_{\mu r}^q & \ldots & a_{q-1} \beta_1^q & \ldots & a_{q-1} \beta_{\mu r}^q & 0 & 1 & 0 \\ a_1^{q+1} \beta_1^{q^2} & \ldots & a_1^{q+1} \beta_{\mu r}^{q^2} & \ldots & a_{q-1}^{q+1} \beta_1^{q^2} & \ldots & a_{q-1}^{q+1} \beta_{\mu r}^{q^2} & 0 & 0 & 1 \end{array} \right) \in \mathbb{F}_{q^m}^{3 \times (n+3)}$$

is MSRD for the extended sum-rank metric

$$\mathrm{d}_e((\mathbf{c}, \mathbf{c}'), (\mathbf{d}, \mathbf{d}')) = \mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) + \mathrm{d}_H(\mathbf{c}', \mathbf{d}'),$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^n$ and $\mathbf{c}', \mathbf{d}' \in \mathbb{F}_{q^m}^3$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition (g, r), where $g = (q-1)\mu$.

Proof. By Corollary 1 and Lemma 5, we only need to show that the code with generator matrix

$$G = \left(\begin{array}{ccc|c|ccc} \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} \\ a_1^{q+1} \beta_1^{q^2} & \ldots & a_1^{q+1} \beta_{\mu r}^{q^2} & \ldots & a_{q-1}^{q+1} \beta_1^{q^2} & \ldots & a_{q-1}^{q+1} \beta_{\mu r}^{q^2} \end{array} \right) \in \mathbb{F}_{q^m}^{2 \times n}$$

is MSRD over $\mathbb{F}_q$ for the length partition (g, r).

First, since q is even, then if $a, b \in \mathbb{F}_q$ are such that $a \neq b$, then $a^2 - b^2 = (a-b)^2 \neq 0$, hence $a^2 \neq b^2$. Therefore if $i \neq j$, since $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$, we deduce that

$$N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i^{q+1}) = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i)^2 \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i)^2 = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j^{q+1}).$$

Second, since m is odd, then $\tau : \mathbb{F}_{q^m} \longrightarrow \mathbb{F}_{q^m}$ given by $\tau(a) = a^{q^2}$, for $a \in \mathbb{F}_{q^m}$, is a field automorphism such that $\{a \in \mathbb{F}_{q^m} \mid a^{q^2} = a\} = \mathbb{F}_q$. In particular, $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a) =$

$a\tau(a)\cdots\tau^{m-1}(a)$, for $a \in \mathbb{F}_{q^m}$. Hence the generator matrix G is an extended Moore matrix (Definition 4) satisfying the conditions in Theorem 2, and therefore the code it generates is MSRD and we are done. $\square$

On the other hand, when m is even or q is odd, a triply extended (full-length) linearized Reed–Solomon code is never MSRD.

Proposition 9. *Let $\beta_1, \beta_2, \ldots, \beta_m \in \mathbb{F}_{q^m}$ be $\mathbb{F}_q$-linearly independent and let $\mathbf{a} = (a_1, a_2, \ldots, a_{q-1}) \in (\mathbb{F}_{q^m}^*)^{q-1}$ be such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$. Set $n = (q-1)m$. If m is even or q is odd, then the triply extended code $\mathcal{C}_e \subseteq \mathbb{F}_{q^m}^{n+3}$ with generator matrix*

$$G_e = \left(\begin{array}{ccc|c|ccc|ccc} \beta_1 & \ldots & \beta_m & \ldots & \beta_1 & \ldots & \beta_m & 1 & 0 & 0 \\ a_1\beta_1^q & \ldots & a_1\beta_m^q & \ldots & a_{q-1}\beta_1^q & \ldots & a_{q-1}\beta_m^q & 0 & 1 & 0 \\ a_1^{q+1}\beta_1^{q^2} & \ldots & a_1^{q+1}\beta_m^{q^2} & \ldots & a_{q-1}^{q+1}\beta_1^{q^2} & \ldots & a_{q-1}^{q+1}\beta_m^{q^2} & 0 & 0 & 1 \end{array} \right) \in \mathbb{F}_{q^m}^{3\times(n+3)}$$

is not MSRD for the extended sum-rank metric

$$\mathrm{d}_e((\mathbf{c}, \mathbf{c}'), (\mathbf{d}, \mathbf{d}')) = \mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) + \mathrm{d}_H(\mathbf{c}', \mathbf{d}'),$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^n$ and $\mathbf{c}', \mathbf{d}' \in \mathbb{F}_{q^m}^3$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition $(q-1, m)$.

Proof. We first consider the case where m is even. Since $\mathbb{F}_{q^2} \subsetneq \mathbb{F}_{q^m}$ in this case, there exists an invertible matrix $A \in \mathbb{F}_q^{m\times m}$ such that the first two components of $(\beta_1, \beta_2, \ldots, \beta_m)A \in \mathbb{F}_{q^m}^m$ lie in $\mathbb{F}_{q^2}$. Since such a multiplication constitutes a linear sum-rank isometry, we may assume that $\beta_1, \beta_2 \in \mathbb{F}_{q^2}$ without loss of generality. Let

$$G = \left(\begin{array}{ccc|c|ccc} \beta_1 & \ldots & \beta_m & \ldots & \beta_1 & \ldots & \beta_m \\ a_1^{q+1}\beta_1^{q^2} & \ldots & a_1^{q+1}\beta_m^{q^2} & \ldots & a_{q-1}^{q+1}\beta_1^{q^2} & \ldots & a_{q-1}^{q+1}\beta_m^{q^2} \end{array} \right) \in \mathbb{F}_{q^m}^{2\times n}. \tag{3}$$

Since $\beta_i - \beta_i^{q^2} = 0$ ($\beta_i \in \mathbb{F}_{q^2}$), for $i = 1, 2$, we conclude that the codeword $(a_1^{q+1}, -1)G$ has sum-rank weight at most $n - 2$, hence the code generated by G is not MSRD over $\mathbb{F}_q$ for the length partition $(q-1, m)$. Thus the code generated by G_e is not MSRD with respect to d_e by Corollary 1.

We now consider the case where both q and m are odd. By assumption, we have that $\{N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \mid i \in [q-1]\} = \mathbb{F}_q^*$. Since q is odd, there exist $1 \leq i < j \leq q-1$ such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) = -N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$. In particular,

$$N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i^{q+1}) = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i)^2 = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i)^2 = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j^{q+1}).$$

Consider the matrix G as in (3). Since $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i^{q+1}) = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j^{q+1})$ and 2 and m are coprime, there exists $\beta \in \mathbb{F}_{q^m}^*$ such that $a_i^{q+1}\beta = a_j^{q+1}\beta^{q^2}$ by Hilbert's Theorem 90

[7, p. 288, Th. 6]. Now, there exist invertible matrices $A_i, A_j \in \mathbb{F}_q^{m\times m}$ such that 1 is the first component of $(\beta_1, \beta_2, \dots, \beta_m)A_i$ and β is the first component of $(\beta_1, \beta_2, \dots, \beta_m)A_j$. Let $A_l = I_m$ for $l \in [q-1] \setminus \{i, j\}$. Denoting

$$\mathrm{diag}\,(A_1, A_2, \dots, A_{q-1}) = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{q-1} \end{pmatrix} \in \mathbb{F}_q^{n\times n},$$

we deduce that $G \cdot \mathrm{diag}\,(A_1, A_2, \dots, A_{q-1})$ contains the submatrix

$$\begin{pmatrix} 1 & \beta \\ a_i^{q+1} & a_j^{q+1}\beta^{q^2} \end{pmatrix},$$

which is not invertible since $a_i^{q+1}\beta = a_j^{q+1}\beta^{q^2}$. Since multiplying by the invertible block diagonal matrix $\mathrm{diag}\,(A_1, A_2, \dots, A_{q-1}) \in \mathbb{F}_q^{n\times n}$ constitutes a linear sum-rank isometry, we deduce that the code generated by G is not MSRD over $\mathbb{F}_q$ for the length partition $(q-1, m)$. Thus the code generated by G_e is not MSRD with respect to d_e by Corollary 1. □

Remark 10. Notice that Proposition 9 works with the same proof in more general cases. Consider the sum-rank metric in $\mathbb{F}_{q^m}^n$, $n = gr$, for the length partition (g, r), $g = \ell\mu$, ℓ arbitrary with $1 \leq \ell \leq q-1$, a vector $\mathbf{a} = (a_1, a_2, \dots, a_\ell) \in (\mathbb{F}_{q^m}^*)^\ell$ such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ satisfying Conditions 1 and 2 in Theorem 2. Proposition 9 works under the following assumptions: 1) m is even and $\mathbb{F}_{q^2} \subseteq \mathcal{H}_i$ for some $i \in [\mu]$; or 2) q and m are odd, $\bigcup_{i=1}^{\mu} \mathcal{H}_i = \mathbb{F}_{q^m}$ and there exist $i \neq j$ such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq -N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$, which necessarily holds if $\ell \geq (q-1)/2$. Here, we define $\mathcal{H}_i$, for $i \in [\mu]$, as in (2). Since the linearized Reed–Solomon code case is $\mu = 1$, both conditions on the (single) subspace $\mathcal{H}_1$ hold when m is even or q and m are odd.

6. A negative result in the sum-rank metric

Up to this point, we have studied extensions of a metric d by adding a Hamming-metric component d_H. The reader may wonder if the results in Section 3 also hold if we extend d by adding another metric, for instance, the rank metric. In this section, we give a negative answer to this question by trying to doubly extend MSRD codes as in Theorem 2 (for the largest value of ℓ, i.e., $\ell = q-1$) by adding a non-trivial rank-metric block and showing that the resulting code is not MSRD even if the conditions in Corollary 1 hold.

Proposition 11. *Let $a_1, a_2, \ldots, a_{q-1} \in \mathbb{F}_{q^m}^*$ be such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ and $\mathcal{H}_i = \langle \beta_{(i-1)r+1}, \ldots, \beta_{ir} \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}$, for $i \in [\mu]$, satisfy Conditions 1 and 2 in Theorem 2. Consider the extended sum-rank metric*

$$\mathrm{d}_e((\mathbf{c}, c_{n+1}, c_{n+2}), (\mathbf{d}, d_{n+1}, d_{n+2})) = \mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) + \mathrm{d}_R((c_{n+1}, c_{n+2}), (d_{n+1}, d_{n+2})),$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^n$ and $c_{n+1}, c_{n+2}, d_{n+1}, d_{n+2} \in \mathbb{F}_{q^m}$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition (g, r), where $g = (q-1)\mu$ and $n = gr$. Let $a, b, c, d \in \mathbb{F}_{q^m}$ with $(0,0) \notin \{(a,c), (b,d), (a,b), (c,d)\}$. Then the extended 2-dimensional code $\mathcal{C}_e$ with generator matrix

$$G_e = \left(\begin{array}{ccc|ccc|c|ccc|cc} \beta_1 & \ldots & \beta_{\mu r} & \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} & a & c \\ a_1\beta_1^q & \ldots & a_1\beta_{\mu r}^q & a_2\beta_1^q & \ldots & a_2\beta_{\mu r}^q & \ldots & a_{q-1}\beta_1^q & \ldots & a_{q-1}\beta_{\mu r}^q & b & d \end{array} \right)$$

is MSRD for d_e if, and only if,

$$-\tau^{-1} \notin \bigcup_{i=1}^{q-1} \left\{ a_i \beta^{q-1} \,\middle|\, \beta \in \bigcup_{j=1}^{\mu} \mathcal{H}_j \setminus \{0\} \right\},$$

for every $\tau \in \mathbb{F}_{q^m}^*$ such that $a + \tau b$ and $c + \tau d$ are $\mathbb{F}_q$-linearly dependent. In particular, if $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$, then $\mathcal{C}_e$ is not MSRD for all $a, b, c, d \in \mathbb{F}_{q^m}$.

Proof. First of all, the reader may verify that there exists $\tau \in \mathbb{F}_{q^m}^*$ such that $a + \tau b$ and $c + \tau d$ are $\mathbb{F}_q$-linearly dependent, since $(0,0) \notin \{(a,c), (b,d), (a,b), (c,d)\}$.

Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{F}_{q^m}^n$ be the first and second rows of G_e, respectively, projected on the first n coordinates. If $\tau \in \mathbb{F}_{q^m}^*$ is such that $a + \tau b$ and $c + \tau d$ are $\mathbb{F}_q$-linearly independent, then we have that

$$\mathrm{wt}_e(\mathbf{g}_1 + \tau \mathbf{g}_2, a + \tau b, c + \tau d) \geq n + 1.$$

Therefore $\mathcal{C}_e$ is not MSRD if, and only if, $\mathrm{wt}_{SR}(\mathbf{g}_1 + \tau \mathbf{g}_2) = n - 1$, for some $\tau \in \mathbb{F}_{q^m}^*$ such that $a + \tau b$ and $c + \tau d$ are $\mathbb{F}_q$-linearly dependent. Fix one such τ. We have $\mathrm{wt}_{SR}(\mathbf{g}_1 + \tau \mathbf{g}_2) = n - 1$ if, and only if, there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{F}_q$, not all zero, such that

$$\sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k} + \tau a_i \sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k}^q = 0,$$

for some $j \in [\mu]$ and some $i \in [q-1]$. Let $\beta = \sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k} \in \mathcal{H}_j \setminus \{0\}$. Then the equation above is simply $-\tau^{-1} = a_i \beta^{q-1}$. This is possible for some $i \in [q-1]$ and some $\beta \in \mathcal{H}_j \setminus \{0\}$ if, and only if,

$$-\tau^{-1} \in \bigcup_{i=1}^{q-1} \left\{ a_i \beta^{q-1} \,\middle|\, \beta \in \bigcup_{j=1}^{\mu} \mathcal{H}_j \setminus \{0\} \right\},$$

and we are done.

Finally, assume that $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$. For $\tau \in \mathbb{F}_{q^m}^*$, there exists $i \in [q-1]$ such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(-\tau^{-1}) = N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i)$. By Hilbert's Theorem 90, there exists $\beta \in \mathbb{F}_{q^m}^* = \bigcup_{j=1}^{\mu} \mathcal{H}_j \setminus \{0\}$ such that $-\tau^{-1} = a_i \beta^{q-1}$ and we conclude that $\mathcal{C}_e$ is not MSRD when $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$. □

In the case where $(\beta_1, \beta_2, \ldots, \beta_{\mu r})$ is constructed using field reduction (as in the following lemma, see also [12, Sec. 4.1]), we have the following easy criterion to determine when $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$. Note that this lemma basically states that $\mathbb{F}_{q^m}$ can be a union of μ $\mathbb{F}_q$-linear subspaces of dimension r with pair-wise zero intersection if, and only if, $\mu = (q^m-1)/(q^r-1)$; or equivalently, $\mathbb{F}_{q^m}^*$ is as disjoint union of μ cosets of $\mathbb{F}_{q^r}^*$ if, and only if, $\mu = (q^m-1)/(q^r-1)$.

Lemma 12. *Let $m = r\rho$, for positive integers r and ρ, and let $(\beta_{(j-1)r+1}, \ldots, \beta_{jr}) = \gamma_j(\alpha_1, \ldots, \alpha_r)$, for $j \in [\mu]$, where $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_{q^r}$ are $\mathbb{F}_q$-linearly independent, and $\gamma_1, \ldots, \gamma_\mu \in \mathbb{F}_{q^m}^*$ are such that γ_i and γ_j are $\mathbb{F}_{q^r}$-linearly independent if $i \neq j$. Define $\mathcal{H}_j = \langle \beta_{(j-1)r+1}, \ldots, \beta_{jr} \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}$, for $j \in [\mu]$. In this setting, we have $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$ if, and only if, $\mu = (q^m-1)/(q^r-1)$.*

Proof. In this case, the condition $\bigcup_{j=1}^{\mu} \mathcal{H}_j = \mathbb{F}_{q^m}$ holds if, and only if, $\{[\gamma_1], \ldots, [\gamma_\mu]\} = \mathbb{P}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^m})$, where $[\gamma] = \{\lambda\gamma \mid \lambda \in \mathbb{F}_{q^r}^*\}$ is the projective point associated to $\gamma \in \mathbb{F}_{q^m}^*$ over $\mathbb{F}_{q^r}$. Now since γ_i and γ_j are $\mathbb{F}_{q^r}$-linearly independent if $i \neq j$, then $[\gamma_1], \ldots, [\gamma_\mu]$ are distinct projective points. Therefore they form the whole projective space if, and only if, there are $(q^m-1)/(q^r-1)$ of them. □

This implies that Proposition 11 holds for 2-dimensional (full-length) linearized Reed–Solomon codes (the case $r = m$ and $\mu = \rho = 1$, see [12, Sec. 4.2]) and the more general family of MSRD codes obtained from Hamming codes given in [12, Sec. 4.4], which are the longest known 2-dimensional linear MSRD codes. In other words, those two families of 2-dimensional MSRD codes may not be doubly extended as in Proposition 11. In the case of linearized Reed–Solomon codes, this could also be deduced from the results in [16]. In the case $r = 2$, it was known that the family of MSRD codes obtained from Hamming codes could not be doubly extended as in Proposition 11. This is because their number of blocks (the parameter $g = (q-1)\mu$) attains the upper bound from [3, Th. 6.12] since $g = (q-1)(q^m-1)/(q^r-1) - 1$ in this case. The fact that it may not be doubly extended for $r \geq 3$ is new.

We recall that this latter family of MSRD codes can be explicitly constructed as in Definition 4, where $\beta_1, \beta_2, \ldots, \beta_{\mu r} \in \mathbb{F}_{q^m}$ are explicitly constructed as follows (see also [12, Sec. 4.4]). Choose positive integers $m = \rho r$, $\mu = (q^m-1)/(q^r-1)$ and let $\gamma \in \mathbb{F}_{q^m}^*$ be a primitive element. Set $\gamma_i = \gamma^{(i-1)(q^r-1)}$, for $i \in [\mu]$. Then $[\gamma_1], [\gamma_2], \ldots, [\gamma_\mu]$ form the whole projective space $\mathbb{P}_{\mathbb{F}_{q^r}}(\mathbb{F}_{q^m})$ (for instance, by combining [11, Prop. 43] and [14, Th. 2.12]). Finally, set $(\beta_{(j-1)r+1}, \ldots, \beta_{jr}) = \gamma_j(\alpha_1, \ldots, \alpha_r)$, for $j \in [\mu]$, where $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_{q^r}$ form a basis of $\mathbb{F}_{q^r}$ over $\mathbb{F}_q$.

7. One-weight codes

In this section, we give necessary and sufficient conditions for the doubly extended MSRD codes from Corollary 8 to be one-weight codes (or constant-weight codes), that is, such that all of their codewords have the same weight (thus equal to the minimum distance of the code). The next proposition recovers [16, Th. 4.9] for linearized Reed–Solomon codes by taking $\mu = 1$.

Proposition 13. *Let $a_1, a_2, \ldots, a_{q-1} \in \mathbb{F}_{q^m}^*$ be such that $N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_i) \neq N_{\mathbb{F}_{q^m}/\mathbb{F}_q}(a_j)$ if $i \neq j$. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{q^m}^{\mu r}$ and $\mathcal{H}_i = \langle \beta_{(i-1)r+1}, \ldots, \beta_{ir} \rangle_{\mathbb{F}_q} \subseteq \mathbb{F}_{q^m}$, for $i \in [\mu]$, satisfy Conditions 1 and 2 in Theorem 2. Consider the extended sum-rank metric*

$$\mathrm{d}_e((\mathbf{c}, c_{n+1}, c_{n+2}), (\mathbf{d}, d_{n+1}, d_{n+2})) = \mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) + \mathrm{d}_H((c_{n+1}, c_{n+2}), (d_{n+1}, d_{n+2})),$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{q^m}^n$ and $c_{n+1}, c_{n+2}, d_{n+1}, d_{n+2} \in \mathbb{F}_{q^m}$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{q^m}^n$ over $\mathbb{F}_q$ for the length partition (g, r), where $g = (q-1)\mu$ and $n = gr$. Then the extended 2-dimensional MSRD code $\mathcal{C}_e$ with generator matrix

$$G_e = \left(\begin{array}{ccc|ccc|c|ccc|cc} \beta_1 & \ldots & \beta_{\mu r} & \beta_1 & \ldots & \beta_{\mu r} & \ldots & \beta_1 & \ldots & \beta_{\mu r} & 1 & 0 \\ a_1\beta_1^q & \ldots & a_1\beta_{\mu r}^q & a_2\beta_1^q & \ldots & a_2\beta_{\mu r}^q & \ldots & a_{q-1}\beta_1^q & \ldots & a_{q-1}\beta_{\mu r}^q & 0 & 1 \end{array} \right)$$

is a one-weight code for d_e if, and only if, $\bigcup_{i=1}^{\mu} \mathcal{H}_i = \mathbb{F}_{q^m}$.

Proof. Let $\mathbf{g}_1, \mathbf{g}_2 \in \mathbb{F}_{q^m}^{n+2}$ be the first and second rows of G_e, respectively. Since $\mathrm{d}_e(\mathcal{C}_e) = n+1$, we need to show that $\mathrm{wt}_e(\mathbf{g}_1 + \lambda \mathbf{g}_2) = n+1$, for all $\lambda \in \mathbb{F}_{q^m}^*$. Fix $\lambda \in \mathbb{F}_{q^m}^*$. We need to show that there exist $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{F}_q$, not all zero, such that

$$\sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k} + \lambda a_i \sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k}^q = 0,$$

for some $j \in [\mu]$ and some $i \in [q-1]$. Let $\beta = \sum_{k=1}^{r} \lambda_k \beta_{(j-1)r+k} \in \mathcal{H}_j \setminus \{0\}$. Then the equation above is simply $-\lambda^{-1} = a_i \beta^{q-1}$. This is possible for all $\lambda \in \mathbb{F}_{q^m}^*$ if, and only if,

$$\mathbb{F}_{q^m}^* = \bigcup_{i=1}^{q-1} \left\{ a_i \beta^{q-1} \,\middle|\, \beta \in \bigcup_{j=1}^{\mu} \mathcal{H}_j \setminus \{0\} \right\}. \tag{4}$$

Since $\beta^{q-1} = \gamma^{q-1}$ holds for $\beta, \gamma \in \mathbb{F}_{q^m}^*$ if, and only if, $\beta/\gamma \in \mathbb{F}_q^*$, it is easy to see that (4) holds if, and only if, $\bigcup_{i=1}^{\mu} \mathcal{H}_i = \mathbb{F}_{q^m}$, and we are done. □

In the case where $\boldsymbol{\beta}$ is constructed using field reduction as in Lemma 12, we see that the extended 2-dimensional MSRD code $\mathcal{C}_e$ is a one-weight code for d_e if, and only if, $\mu = (q^m - 1)/(q^r - 1)$.

In other words, 2-dimensional doubly extended linearized Reed–Solomon codes and the doubly extended MSRD codes based on Hamming codes as in [12, Sec. 4.4] are all one-weight codes for the extended metric d_e.

Finally, we show that triply extended MSRD codes are never one-weight codes for $q = 2$. Due to the results from Section 5, we only consider the case where m is odd. Notice that in this case the vector $\mathbf{a}$ is of length one and we may simply consider it as $\mathbf{a} = (1)$.

Proposition 14. *Let $q = 2$, let $m \geq 3$ be odd and set $n = \mu r$ for positive integers μ and r. Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{\mu r}) \in \mathbb{F}_{2^m}^{\mu r}$ satisfy Conditions 1 and 2 in Theorem 2. The triply extended code $\mathcal{C}_e \subseteq \mathbb{F}_{2^m}^{n+3}$ with generator matrix*

$$G_e = \left(\begin{array}{cccc|ccc} \beta_1 & \beta_2 & \ldots & \beta_{\mu r} & 1 & 0 & 0 \\ \beta_1^2 & \beta_2^2 & \ldots & \beta_{\mu r}^2 & 0 & 1 & 0 \\ \beta_1^4 & \beta_2^4 & \ldots & \beta_{\mu r}^4 & 0 & 0 & 1 \end{array} \right) \in \mathbb{F}_{2^m}^{3 \times (n+3)}$$

is MSRD but not a one-weight code for the extended sum-rank metric

$$\mathrm{d}_e((\mathbf{c}, \mathbf{c}'), (\mathbf{d}, \mathbf{d}')) = \mathrm{d}_{SR}(\mathbf{c}, \mathbf{d}) + \mathrm{d}_H(\mathbf{c}', \mathbf{d}'),$$

for $\mathbf{c}, \mathbf{d} \in \mathbb{F}_{2^m}^n$ and $\mathbf{c}', \mathbf{d}' \in \mathbb{F}_{2^m}^3$, where d_{SR} denotes the sum-rank metric in $\mathbb{F}_{2^m}^n$ over $\mathbb{F}_2$ for the length partition (μ, r).

Proof. The fact that $\mathcal{C}_e$ is MSRD for d_e is Theorem 3. Now, since $\mathrm{d}_e(\mathcal{C}_e) = n - 2$, it is enough to show that there exists a codeword $\mathbf{c} \in \mathcal{C}_e$ with $\mathrm{wt}_e(\mathbf{c}) = n$. For $\lambda, \nu \in \mathbb{F}_{2^m}^*$, let

$$\mathbf{c}_{\lambda,\nu} = (\lambda\beta_1 + \nu\beta_1^2 + \beta_1^4, \ldots, \lambda\beta_{\mu r} + \nu\beta_{\mu r}^2 + \beta_{\mu r}^4, \lambda, \nu, 1) \in \mathcal{C}_e.$$

Since $\lambda \neq 0 \neq \nu$, it holds that $\mathrm{wt}_e(\mathbf{c}_{\lambda,\nu}) < n$ if, and only if, there exists an index $i \in [\mu]$ and scalars $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{F}_2$, not all zero, such that

$$\sum_{j=1}^{r} \lambda_j \left(\lambda\beta_{(i-1)r+j} + \nu\beta_{(i-1)r+j}^2 + \beta_{(i-1)r+j}^4 \right) = 0.$$

By considering $\beta = \sum_{j=1}^{r} \lambda_j \beta_{(i-1)r+j} \in \mathbb{F}_{q^m}^*$, we have that $\mathrm{wt}_e(\mathbf{c}_{\lambda,\nu}) < n$ if, and only if, there exists an index $i \in [\mu]$ and $\beta \in \mathcal{H}_i = \langle \beta_{(i-1)r+1}, \beta_{(i-1)r+2}, \ldots, \beta_{ir} \rangle_{\mathbb{F}_2} \setminus \{0\}$ such that $\lambda\beta + \nu\beta^2 + \beta^4 = 0$, that is, $\beta^3 + \nu\beta + \lambda = 0$.

Now, since by [8, Th. 3.25] there are $(2^{3m} - 2^m)/3 > 2 \cdot 2^{2m}$ monic irreducible polynomials in $\mathbb{F}_{2^m}[x]$ of degree exactly 3, then there is at least one irreducible polynomial $f = x^3 + ax^2 + bx + c \in \mathbb{F}_{2^m}[x]$ such that $b \neq a^2$ and $b \neq 1$. Furthermore, $c \neq 0$ since f is irreducible. Define $g = f(x + a) = x^3 + (a^2 + b)x + c(b + 1)$, which is irreducible since so is f. Let $\nu = a^2 + b$ and $\lambda = c(b + 1)$, which satisfy $\lambda \neq 0 \neq \nu$. Since g is irreducible of degree 3, there is no $\beta \in \mathbb{F}_{2^m}$ such that $g(\beta) = \beta^3 + \nu\beta + \lambda = 0$. In other words, the codeword $\mathbf{c}_{\lambda,\nu} \in \mathcal{C}_e$ as above satisfies $\mathrm{wt}_e(\mathbf{c}_{\lambda,\nu}) = n$, and we are done. □

Data availability

No data was used for the research described in the article.

Acknowledgment

The author gratefully acknowledges the support from a María Zambrano contract by the University of Valladolid, Spain (Contract no. E-47-2022-0001486). The author also wishes to thank the anonymous reviewers, who helped improve the content and presentation of this manuscript.

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