Hölder regularity for abstract semi-linear fractional differential equations in Banach spaces

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Abstract

In the present work the optimal regularity, in the sense of Hölder continuity, of linear and semi–linear abstract fractional differential equations is investigated in the framework of complex Banach spaces. This framework has been considered by the authors as the most convenient to provide a posteriori error estimates for the time discretizations of such a kind of abstract differential equations. In the spirit of the classical a posteriori error estimates, under certain assumptions, the error is bounded in terms of computable quantities, in our case measured in the norm of Hölder continuous and weighted Hölder continuous functions.

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1. Introduction

A posteriori error estimates for space, time, and fully (space–time) discretizations of nonlinear partial differential equations have been widely investigated in the past and until now. However, if we restrict our attention to the framework of complex Banach spaces, and abstract formulations of nonlinear initial value problems

$$u'(t) = \mathcal{F}(u(t)), \quad 0 < t \leq T, \quad \text{with} \quad u(0) = u_0, \tag{1}$$

where $\mathcal{F} : \mathcal{B} \subset Y \to X$ is a nonlinear function, X, Y stand for two complex Banach spaces with $Y \subset X$ densely embedded, $\mathcal{B}$ is an open set, and u_0 belongs to $\mathcal{B}$, then the number of works one can find in the literature is noticeably reduced. This kind of error estimates in the framework of Banach spaces for the discretization of nonlinear problems (1) have been investigated e.g. in [15, 47, 48, 55]. In particular in [47] accretive operators in X are considered, and the notion of relaxed solutions is the key point to obtain their results; in [48] the author provides error estimates in the L^1–norm via discrete energy dissipation; in [55] the author provides error estimates for the space–time discretization of parabolic problems making use of the L^p–regularity; and in [15] the error estimates for the time discretization are based on a semi–linearization of (1)

$$u'(t) = Au(t) + F(u(t)), \quad 0 < t \leq T, \quad \text{with} \quad u(0) = u_0, \tag{2}$$

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where A is a convenient linear operator, and on the optimal regularity of the solutions in the sense of the Hölder continuity.

Notice that the idea of linearization is commonly used in the framework of Banach spaces when studying the problem (1), in particular if one studies the well–posedness [41]. Notice also that the main reason for the choice of a Banach space as the functional setting to obtain error estimates for the discretization of (2) is that this framework allows us to consider operators A within a wide set of elliptic operators beyond the classical Laplacian, and moreover the error estimates can be measured in any L^r–norm, i.e. for $1 \leq r \leq +\infty$. That is why we will opt for the functional setting of complex Banach spaces for our study.

Now, focusing on our contributions, let us replace the (integer) time derivative in (2) with a time derivative of non–integer order or in other words, let us consider the following abstract semi–linear differential equation of fractional order in time,

$$\partial_t^\beta u(t) = Au(t) + F(u(t)), \quad 0 \leq t \leq T, \quad \text{with} \quad u(0) = u_0, \tag{3}$$

where ∂_t^β stands for a time fractional derivative.

This work is motivated by the widespread use of nonlinear fractional equations of type (3) in the context of anomalous diffusion phenomenon, in fact if $1 < \beta < 2$, then it is applied as a super–diffusive model of anomalous type e.g. in heterogeneous media diffusion or in wave propagation in viscoelastic materials [4, 5, 17, 13, 21, 22, 24, 26, 27, 32, 36, 34, 31, 33, 43, 50, 51, 53, 56]. Notice that the nonlinear term $F(u)$ in (3) reflects the reaction effects in super–diffusive phenomenon. Let us highlight a particular case, which can be considered as a prototype model, that is the fractional Burgers equation [18, 37, 52] which is more closely raised in Section 2. In view of the above, in the framework of the numerical solutions it looks like clear that accurate error estimations for time discretizations of (3) draws the interest of researchers in numerical analysis. To this end, the well–posedness of the problem, and particularly the regularity of the solution is one of the key points, see e.g. the recent works [45, 44].

We must notice that the maximal regularity of linear and non–linear (semi-linear) to time fractional differential equations of type (3) has been already studied on continuous interpolation and L_p spaces (see e.g. [12, 35, 36, 49, 50]). Our work differs from those mentioned, and this is our first contribution, in that our proofs require technics allowing to state precisely all constants involved, that is they are useless results or proofs were abstract constants are provided. Our second contribution is that the maximal regularity stated for the linear problem, then extended to the non–linear one, allows us to obtain error estimates for time discretizations of the non–linear problem in the framework of the a posteriori error estimation.

As one of the main issues of this paper as we mentioned above, we will focus on the error estimates derived in the framework of the a posteriori estimation for which, to best of our knowledge, there are not so many works related to the numerical solutions of (3). Let us mention here some recent works. In [10] the authors give, by using some ideas of [30], a posteriori error estimates in the maximum norm for the equation (3) where the Banach space X stands there for the real line $X = \mathbb{R}$, and $0 < \beta < 1$. It is a well known fact that fractional differential equations in the form of (3) can be considered as Volterra equations with (possibly) a singular kernel, and having in mind this fact, in [54] the authors give a posteriori error estimates for nonlinear Volterra equations with singular kernels, but again in a finite dimensional context. More recently, in [25] the authors provide error estimates for several time discretizations in Hilbert spaces and L^r–norms, $1 < r < +\infty$, whose proof is based on the l^r–regularity of the numerical solutions. On the other hand, in [7, 11] the fractional diffusion is understood in the spatial domain (fractional Laplacian), and the authors derive a priori and a posteriori error estimates in L^2–norms, for FEMs based discretizations, and for several definitions of the non–local term. Let us mention also [3] where the fractional diffusion is once again understood in the sense of the fractional Laplacian, and where the a posteriori error estimates apply for an anisotropic FEM based discretization.

In the present work we provide a posteriori error estimates for the time discretization of an abstract semi–linear fractional equations of type (3) on the wide context of complex Banach spaces. Such estimates are based on the optimal regularity, in the sense of the Hölder continuity, of a residual function arising from a convenient continuous reconstruction of the discrete solution rather than from the discrete solution itself. These estimates are obtained via classical fixed–point theorems applied to a convenient functional, and thanks to some optimal regularity properties of the solutions of the linear equation (i.e. the equation (3) with $F(u(t)) = F(t)$) which are also proved here.

In the spirit of the a posteriori error estimates, in the present work we get fully realistic error estimates which means that all bounds and constants shown in the following sections are explicitly computed, or at least they could be explicitly computed in practical instances. This fact leads us to a presentation with a more complex notation which is the opposite what happens in classical a priori error estimates where generic constants are allowed when obtaining the error bounds.

The results shown in this work extend in some manner the ones derived in [15] for classical nonlinear parabolic problem (1), and stand for a theoretical approach to the a posteriori error estimation for the time discretization of fractional differential equations (3) in the hope that these results will be further applied in practical instances in forthcoming works. One of the relevant contributions of the present work, if compared to the related work [15], is that we take here into account the initial error of the numerical scheme, in other words the final estimates depend also on the initial error. As it can be observed in Section 4, this fact forced us to assume additional regularity assumptions on the initial data.

This paper is organized as follows. In Section 2 we describe precisely the framework where we are working on along the paper, the fractional initial value problem for which we obtain our estimates, and the hypotheses required to that end. In Section 3 we provide some optimal regularity results for the linear fractional problem, all of them oriented to the proof of the main result in Section 4 where our estimates are provided.

2. Analytic framework and notation

In this section, we give the preliminaries, the notation, and the description of the functional setting used throughout the present paper. Let $(X, \|\cdot\|_X)$ be a complex Banach space. The norm $\|\cdot\|_X$ in the Banach space X will be denoted simply by $\|\cdot\|$, if not confusing. Moreover given two Banach spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, $\mathcal{L}(X, Y)$ denotes the Banach space of all linear and bounded operators from X into Y. If $X = Y$, then we simply write $\mathcal{L}(X, X) = \mathcal{L}(X)$.

Definition 1. *A closed linear operator $A : D(A) \subset X \to X$ is called* sectorial *or θ–*sectorial *if there exist $a \in \mathbb{R}$, $M \geq 0$, and $0 < \theta < \pi/2$ such that his resolvent is analytic outside the sector*

$$a + S_\theta := \{a + z \in \mathbb{C} : |\arg(-z)| < \theta\},$$

and is bounded by

$$\|(z - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|z - a|}, \quad z \notin a + S_\theta.$$

In order to simplify the presentation of results, and without lost of generality, in the present paper we assume that $a = 0$, if not so we can take the operator $A - aI$, also sectorial, where I denotes the identity operator in X.

For a Banach space $(Y, \|\cdot\|_Y)$ and $0 < \alpha < 1$, we will denote by $C^\alpha([0, T]; Y)$ the space of all bounded α-Hölder continuous functions $g : [0, T] \to Y$, endowed with the norm

$$\|g\|_{C^\alpha([0,T];Y)} := \sup_{0 \leq t \leq T} \|g(t)\|_Y + [[g]]_{C^\alpha([0,T];Y)},$$

where $[[g]]_{C^\alpha([0,T];Y)}$ denotes the semi–norm

$$[[g]]_{C^\alpha([0,T];Y)} := \sup_{0 \leq s < t \leq T} \frac{\|g(t) - g(s)\|_Y}{(t - s)^\alpha}.$$

Moreover, if $0 < \alpha \le \gamma < 1$, then we define the space $C^\alpha_\gamma((0,T];Y)$ as the set of all bounded functions $g:(0,T]\to Y$ such that $t\mapsto t^{\gamma-\alpha}g(t)$ is α-Hölder continuous in $(0,T]$ endowed with the norm

$$\|g\|_{C^\alpha_\gamma((0,T];Y)} := \sup_{0<t\le T}\|g(t)\|_Y + [[g]]_{C^\alpha_\gamma((0,T];Y)},$$

where $[[g]]_{C^\alpha_\gamma((0,T];Y)}$ denotes the semi–norm

$$[[g]]_{C^\alpha_\gamma((0,T];Y)} := \sup_{0\le s<t\le T}\frac{s^\gamma\|g(t)-g(s)\|_Y}{(t-s)^\alpha}.$$

Let A be a linear and closed operator whose resolvent set contains the real axis $(-\infty,0]$, e.g. any sectorial operator with $a\ge 0$. For $0\le\vartheta\le 1$, we denote by X^ϑ the domain of the fractional power $\vartheta>0$ of A, that is $X^\vartheta := D(A^\vartheta)$ endowed with the graph norm $\|x\|_\vartheta = \|x\| + \|A^\vartheta x\|$ [29, 41]. In particular X^1 corresponds to the domain of A, and X^0 to the space X. Related to these spaces let us recall a classical inequality which will be useful for us in the following sections: If $0<\varepsilon<1$, and $x\in D(A)$, then there exists a constant $\kappa_\varepsilon>0$ such that (see [29, 41])

$$\|A^\varepsilon x\| \le \kappa_\varepsilon\|Ax\|^\varepsilon\|x\|^{1-\varepsilon}. \tag{4}$$

For the sake of the simplicity of the notation we will simply denote κ instead of κ_ε, for any $0<\varepsilon<1$.

Consider the nonlinear initial value problem

$$\begin{cases} u'(t) = \mathcal{F}(u(t)), & 0\le t\le T,\\ u(0) = u_0\in\mathcal{B}, \end{cases} \tag{5}$$

where $\mathcal{F}:\mathcal{B}\subset Y\to X$ is a nonlinear Fréchet differentiable function, $\mathcal{B}$ is an open set in Y, $u_0\in\mathcal{B}$, and $(X,\|\cdot\|_X), (Y,\|\cdot\|_Y)$ are two Banach spaces such that $Y\subset X$ is densely embedded.

The existence and uniqueness of solution of (5) is very well known [2, 41], and the proof in the framework of Banach spaces can be carried out making use of two facts: A linearization of (5) around a state $u^*\in\mathcal{B}$; and the optimal regularity properties of the linearized problem. In particular, the linearized problem reads

$$\begin{cases} u'(t) = Au(t) + F(u(t)), & 0\le t\le T,\\ u(0) = u_0\in\mathcal{B}, \end{cases} \tag{6}$$

where $A := \mathcal{F}_u(u^*)$, $\mathcal{F}_u$ stands for the Fréchet derivative of $\mathcal{F}$, and $F:\mathcal{B}\subset Y\to X$ is defined by $F(u) = \mathcal{F}(u) - Au$ which is Fréchet differentiable as well. Therefore it is assumed that $\mathcal{B}\subseteq D(A)$. The initial value problem (6) can be written equivalently in integral form

$$u(t) = u_0 + \int_0^t Au(s)\,\mathrm{d}s + F(u(t)), \quad 0\le t\le T, \tag{7}$$

where, for the simplicity of the notation, we denote again by F the integral in time of F in (6).

In the present work we consider the nonlinear fractional initial value problem that comes out when one replaces the integer integral in (7) by a fractional integral of order $1<\beta<2$. In fact, we consider the nonlinear fractional problem

$$u(t) = u_0 + \partial_t^{-\beta}Au(t) + F(u(t)), \quad 0\le t\le T, \quad \text{with} \quad 1<\beta<2, \tag{8}$$

where $\partial_t^{-\varrho}g(t)$ represents, for $g:(0,+\infty)\to X$, the fractional integral of order $\varrho>0$ in the variable t of g. Note that the initial condition $u(0)$ in (8) turns out to be $u_0+F(u_0)$, or simply u_0 if one assumes that $F(u_0)=0$. Moreover, since $1<\beta<2$ (that is β is greater that 1), a second initial condition could be expected which in this work is on $u'(0)$, and which for the sake of the simplicity is assumed to be zero. Also for the sake of the simplicity, and without danger of confusion with derivatives respect to other variables, we will denote $\partial^{-\varrho}$ instead of $\partial_t^{-\varrho}$. The fractional integration admits several definitions [28, 46] but we opted here for the fractional integral in the sense of Riemann–Liouville, i.e. for $\varrho > 0$

$$\partial^{-\varrho} g(t) := \int_0^t k_\varrho(t-s) g(s)\,\mathrm{d}s, \quad \text{where} \quad k_\varrho(t) := \frac{t^{\varrho-1}}{\Gamma(\varrho)}, \quad t > 0.$$

We observe that other definitions provide the same results without significant differences in the proofs.

The prototype equation we have in mind is the fractional Burgers equation [18, 37, 52]. In spite of such equation admits several formulations, we adopt the following one

$$u(x,t) = u_0(x) + \int_0^t k_\beta(t-s)\Delta u(x,s)\,\mathrm{d}s + \frac{\partial}{\partial x}(u^2(x,t)), \quad 0 \le t \le T, \quad x \in \Omega,$$

where $\Omega \subset \mathbb{R}$ denotes certain spatial domain, Δ represents the 1D Laplacian operator, the nonlinear term $\frac{\partial}{\partial x} u^2(x,\cdot)$ plays the role of $F(u)$ in (8), $1 < \beta < 2$, and where some boundary conditions are satisfied.

Other equations of type (8), also highly interesting in practical instances, can be found in the literature. Among the Burger's equations above, let us mention the recent work [1] where the author studies a fractional type approach to the Navier–Stokes equation which perfectly matches with (8).

For the sake of the simplicity of the presentation of our results, instead of the integral format (8) henceforth we adopt an integro–differential one that is

$$u'(t) = u_0 + \partial_t^{1-\beta} A u(t) + F(u(t)), \quad \text{with} \quad u(0) = u_0 \in D(A), \quad 0 \le t \le T. \tag{9}$$

Our approach requires some assumptions on the terms involved in (9), but in order to make lighter the notation, and without lost of generality in the results below, assume the following: The linearization we carried out is made around u_0 as a natural choice, i.e. $A := \mathcal{F}_u(u_0)$; F is defined and Fréchet differentiable, by simplicity in $D(A)$ (instead of $\mathcal{B} \subseteq D(A)$), And finally there holds that $F_u(u_0) = 0$. Now we are in a position to state the hypotheses will hold,

(H1) If u_0 is the initial data of (8), then there exist $R = R(u_0) > 0$ and $L = L(u_0) > 0$ such that

$$\|F_u(u_2) - F_u(u_1)\|_{\mathcal{L}(D(A),X)} \le L\|u_1 - u_2\|_Y,$$

for all $u_1, u_2 \in \mathcal{B}$ with $\|u_j - u_0\|_Y \le R$, $j = 1, 2$.

(H2) $A : D(A) \subset Y \to X$ is θ-sectorial, for some $0 < \theta < \pi/2$, such that $\theta < \pi(1 - \beta/2)$, according to the Definition 1.

(H3) The graph norm of A is equivalent to the norm of Y, that is, there exists $\gamma = \gamma(u_0) > 0$ such that

$$\frac{1}{\gamma}\|y\|_Y \le \|y\|_{D(A)} := \|y\|_X + \|Ay\|_X \le \gamma\|y\|_Y.$$

Let us mention that the existence and uniqueness of local solutions of the semi–linear problem (8) under hypotheses (H1)–(H3) can be straightforwardly deduced from results in [51] giving rise (probably) to some restrictions for the final time T. Anyhow in the rest of the paper we will assume that T satisfies such a restrictions (if the case), and the solution of (8) exists over the whole interval $[0, T]$.

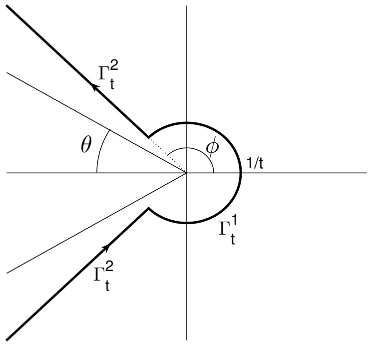

Figure 1: Complex path Γ_t.

3. The linear problem: Optimal regularity

We first consider the linear problem

$$v'(t) = \partial^{1-\beta} Av(t) + f(t), \quad \text{with} \quad v(0) = v_0, \quad 0 \le t \le T, \tag{10}$$

according to the notation of the Section 2, where $1 < \beta < 2$, and $f \in W^{1,1}([0,T], X)$ satisfying additional regularity conditions to be precisely stated below. By means of the Laplace transform it can be straightforwardly proved that there exists a family of operators $\{\mathcal{S}_\beta(t)\}_{t\ge 0} \subset \mathcal{L}(X)$, such that the solution to (10) is given by

$$v(t) = \mathcal{S}_\beta(t)v_0 + \int_0^t \mathcal{S}_\beta(t-s)f(s)\,\mathrm{d}s, \qquad 0 \le t \le T. \tag{11}$$

In fact, the inversion formula of the Laplace transform allows to write

$$\mathcal{S}_\beta(t) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{zt} z^{\beta-1}(z^\beta - A)^{-1}\,\mathrm{d}z, \qquad t \ge 0, \tag{12}$$

for a suitable complex path Γ connecting $-\mathrm{i}\infty$ and $+\mathrm{i}\infty$, positively oriented, i.e. with increasing imaginary part, and surrounding the complex sector S_θ (see [16]).

For the convenience of the proofs below, now and hereafter we set a particular choice of Γ. To be more precise, let ϕ an angle satisfying $\dfrac{\beta\pi}{2} < \phi < (\pi - \theta)$, and let Γ_t be the complex path $\Gamma_t := \Gamma_t^1 \cup \Gamma_t^2$ where (see Figure 3):

- Γ_t^1 is defined, at each time level $t > 0$, by $\gamma_t^1(\psi) = \dfrac{1}{t}\mathrm{e}^{\mathrm{i}\psi/\beta}$, for $-\phi \le \psi \le \phi$, and
- Γ_t^2 is given by $\gamma_t^2(\rho) = \rho\mathrm{e}^{\pm\mathrm{i}\phi/\beta}$, for $\dfrac{1}{t} \le \rho < +\infty$, where $\pm$ stands for the lower and upper branches of Γ_t^2 (negative and positive imaginary part) repectively.

Note that this choice of Γ_t respects hypothesis (H2) in the sense that Γ_t does not go into the sector S_θ according the choice of θ in (H2).

Next four lemmas, Lemmas 2, 4, 6, and 8, stand for technical results to be used in the proofs of the main theorems below.

Lemma 2. *Let $\mu \geq 0$. Then the following estimates hold*

$$\int_{\Gamma_t} \left| \frac{\mathrm{e}^{zt}}{z^\mu} \right| |\, dz| \leq \left(C_\beta + \frac{2\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)} \right) t^{\mu-1}, \tag{13}$$

and

$$\int_{\Gamma_t} |\mathrm{e}^{zt} z^\mu|\, |\, dz| \leq \left(C_\beta + \frac{2\Gamma(\mu+1)}{(-\cos(\phi/\beta))^{\mu+1}} \right) \frac{1}{t^{\mu+1}}, \tag{14}$$

where

$$C_\beta := \frac{1}{\beta} \int_{-\phi}^{\phi} \mathrm{e}^{\cos(\psi/\beta)}\, d\psi. \tag{15}$$

Proof of Lemma 2. In order to prove (13), we first notice that on Γ_t^1 we have

$$\int_{\Gamma_t^1} \left| \frac{\mathrm{e}^{zt}}{z^\mu} \right| |\, \mathrm{d}z| = \int_{-\phi}^{\phi} \frac{\exp(t\frac{\cos(\psi/\beta)}{t})}{\left| \frac{\exp(\mathrm{i}\mu\psi/\beta)}{t^\mu} \right|} \frac{1}{\beta t}\, \mathrm{d}\psi \leq \frac{t^{\mu-1}}{\beta} \int_{-\phi}^{\phi} \mathrm{e}^{\cos(\psi/\beta)}\, \mathrm{d}\psi = C_\beta t^{\mu-1}.$$

On the other hand, since $\cos(\phi/\beta) < 0$ we have

$$\int_{\Gamma_t^2} \left| \frac{\mathrm{e}^{zt}}{z^\mu} \right| |\, \mathrm{d}z| \;=\; 2\int_{1/t}^{\infty} \frac{\exp(t\rho\cos(\phi/\beta))}{\rho^\mu}\, \mathrm{d}\rho \leq 2t^\mu \frac{-\mathrm{e}^{\cos(\phi/\beta)}}{t\cos(\phi/\beta)} = 2t^{\mu-1} \frac{\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)},$$

which implies (13). Now, to prove (14) we observe that on Γ_t^1 we have

$$\int_{\Gamma_t^1} |\mathrm{e}^{zt}|\, |z|^\mu |\, \mathrm{d}z| = \int_{-\phi}^{\phi} \left| \exp\left(t\frac{1}{t}\mathrm{e}^{\mathrm{i}\psi/\beta} \right) \right| \frac{1}{t^\mu} \frac{1}{\beta t}\, \mathrm{d}\psi = \frac{1}{\beta t^{\mu+1}} \int_{-\phi/\beta}^{\phi/\beta} \mathrm{e}^{\cos(\psi/\beta)}\, \mathrm{d}\psi = \frac{C_\beta}{t^{\mu+1}}.$$

Finally, on Γ_t^2 we have

$$\begin{aligned}
\int_{\Gamma_t^2} |\mathrm{e}^{zt}|\, |z|^\mu |\, \mathrm{d}z| &= 2\int_{1/t}^{\infty} \left| \exp(t\rho \mathrm{e}^{\mathrm{i}\phi/\beta}) \right| |\rho^\mu \mathrm{e}^{\mathrm{i}\mu\phi/\beta}|\, \mathrm{d}\rho \\
&= 2\int_{1/t}^{\infty} \rho^\mu \mathrm{e}^{t\rho\cos(\phi/\beta)}\, \mathrm{d}\rho \\
&\leq 2\int_{0}^{\infty} \rho^\mu \mathrm{e}^{-\rho(-t\cos(\phi/\beta))}\, \mathrm{d}\rho \\
&= 2\frac{\Gamma(\mu+1)}{t^{\mu+1}(-\cos(\phi/\beta))^{\mu+1}}.
\end{aligned}$$

Remark 3. Since the cosine function is an even function and $\frac{\beta\pi}{2} < \phi < (\pi - \theta)$*, we have* $\frac{\phi}{\beta^2} < \pi$*, and then we can estimate the constant* C_β *as*

$$C_\beta = \frac{2}{\beta} \int_0^{\phi} \mathrm{e}^{\cos(\psi/\beta)}\, d\psi = 2\int_0^{\phi/\beta} \mathrm{e}^{\cos(v)}\, dv \leq 2\int_0^{\pi} \mathrm{e}^{\cos(v)}\, dv = 2\pi I_0(1) < 2\pi\cosh(1),$$

where I_0 *denotes the Bessel function of first kind (see [20, p. 336] and [40, p. 63, (6.25)]).*

Lemma 4. *Let* $0 \leq \vartheta \leq 1$*. If* $x \in X^\vartheta$*, then*

$$\|\mathcal{S}_\beta(t)x\| \leq \frac{1}{2\pi} \left(C_\beta + \frac{2\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)} \right) \left(\|x\| + \kappa\,(M+1)^{1-\vartheta} \|A^\vartheta x\| t^{\vartheta\beta} \right), \tag{16}$$

where $\mathcal{S}_\beta(t)$ *is the operator (12), for* $t > 0$*,* C_β *is the constant (15), and* κ *in given by (4).*

PROOF OF THEOREM 4. Since

$$z^\beta(z^\beta - A)^{-1} = A(z^\beta - A)^{-1} + I, \tag{17}$$

we have $z^{\beta-1}(z^\beta - A)^{-1} = \frac{1}{z}[A(z^\beta - A)^{-1} + I]$ and thus, for $x \in X$, we can write

$$\begin{aligned} \mathcal{S}_\beta(t)x &= \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_t} \frac{\mathrm{e}^{zt}}{z} x \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_t} \frac{\mathrm{e}^{zt}}{z} A(z^\beta - A)^{-1} x \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_t} \frac{\mathrm{e}^{zt}}{z} x \, \mathrm{d}z + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_t} \frac{\mathrm{e}^{zt}}{z} A^{1-\vartheta}(z^\beta - A)^{-1} A^\vartheta x \, \mathrm{d}z. \end{aligned}$$

Let $x \in X$ with $\|x\| \leq 1$. Since A is a sectorial operator, $(z^\beta - A)^{-1}x \in D(A)$, and $D(A) \subset D(A^{1-\vartheta})$, it follows from (4) that

$$\begin{aligned} \left\|A^{1-\vartheta}(z^\beta - A)^{-1}x\right\| &\leq \kappa \left\|A(z^\beta - A)^{-1}x\right\|^{1-\vartheta} \left\|(z^\beta - A)^{-1}x\right\|^{\vartheta} \\ &\leq \kappa \left(\|(z^\beta(z^\beta - A)^{-1} + I)x\|\right)^{1-\vartheta} \left(\frac{M}{|z|^\beta}\|x\|\right)^{\vartheta} \\ &\leq \kappa \left((M+1)\|x\|\right)^{1-\vartheta} \left(\frac{M}{|z|^\beta}\|x\|\right)^{\vartheta} \\ &\leq \kappa \,(M+1)^{1-\vartheta} \frac{\|x\|}{|z|^{\beta\vartheta}}. \end{aligned}$$

Therefore,

$$\left\|A^{1-\vartheta}(z^\beta - A)^{-1}\right\|_{\mathcal{L}(X)} \leq \frac{\kappa\,(M+1)^{1-\vartheta}}{|z|^{\beta\vartheta}}. \tag{18}$$

Lemma 2 allows us to obtain the following estimate for $\|\mathcal{S}_\beta(t)x\|$

$$\begin{aligned} \|\mathcal{S}_\beta(t)x\| &\leq \frac{1}{2\pi}\int_{\Gamma_t} \left|\frac{\mathrm{e}^{zt}}{z}\right| |\,\mathrm{d}z| \|x\| + \frac{1}{2\pi}\int_{\Gamma_t} \left|\frac{\mathrm{e}^{zt}}{z}\right| \left\|A^{1-\vartheta}(z^\beta - A)^{-1}\right\|_{\mathcal{L}(X)} |\,\mathrm{d}z| \|A^\vartheta x\| \\ &\leq \frac{1}{2\pi}\left(C_\beta + \frac{2\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)}\right)\|x\| + \frac{\kappa\,(M+1)^{1-\vartheta}\|A^\vartheta x\|}{2\pi}\int_{\Gamma_t}\left|\frac{\mathrm{e}^{zt}}{z^{\beta\vartheta+1}}\right| |\,\mathrm{d}z| \\ &\leq \frac{1}{2\pi}\left(C_\beta + \frac{2\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)}\right)\left(\|x\| + \kappa\,(M+1)^{1-\vartheta}\|A^\vartheta x\| t^{\vartheta\beta}\right), \end{aligned}$$

and the proof concludes.

Remark 5. If $C_0 := \frac{1}{2\pi}\left(C_\beta + \frac{2\mathrm{e}^{\cos(\phi/\beta)}}{-\cos(\phi/\beta)}\right)$, *then Lemma 4 implies for* $0 \leq t \leq T$ *that*

$$\begin{aligned} \|\mathcal{S}_\beta(t)\|_{\mathcal{L}(X^\vartheta, X)} &= \sup\{\|\mathcal{S}_\beta(t)x\| : x \in X^\vartheta, \|x\|_\vartheta \leq 1\} \\ &\leq C_0 \sup\{\|x\| + \kappa\,(M+1)^{1-\vartheta}\|A^\vartheta x\| t^{\beta\vartheta} : x \in X^\vartheta, \|x\|_\vartheta \leq 1\} \\ &\leq C_0 \max\{1, \kappa\,(M+1)^{1-\vartheta}\}(1 + t^{\beta\vartheta}). \end{aligned}$$

Lemma 6. *Let* $0 \leq \vartheta \leq 1$. *If* $x \in X^\vartheta$, *then*

$$\|A\mathcal{S}_\beta(t)x\| \leq \frac{\kappa(M+1)^{1-\vartheta}}{2\pi}\|A^\vartheta x\| \left(C_\beta + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right) t^{\beta(\vartheta-1)}, \quad 0 \leq t \leq T, \tag{19}$$

where $\mathcal{S}_\beta(t)$ *is the operator* (12).

Proof of Theorem 6. We first notice that

$$A\mathcal{S}_\beta(t)x = \frac{1}{2\pi \mathrm{i}}\int_{\Gamma_t} \mathrm{e}^{zt} z^{\beta-1} A^{1-\vartheta}(z^\beta - A)^{-1}A^{\vartheta}x\,\mathrm{d}z,$$

and by (18) and Lemma 2 we obtain

$$\begin{aligned}\|A\mathcal{S}_\beta(t)x\| &\le \frac{1}{2\pi}\int_{\Gamma_t} |\mathrm{e}^{zt}||z|^{\beta-1}\left\|A^{1-\vartheta}(z^\beta - A)^{-1}\right\|\|A^{\vartheta}x\||\,\mathrm{d}z| \\ &\le \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\|A^{\vartheta}x\|\int_{\Gamma_t}|\mathrm{e}^{zt}||z|^{\beta(1-\vartheta)-1}|\,\mathrm{d}z| \\ &\le \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\|A^{\vartheta}x\|\left(C_\beta + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)t^{\beta(\vartheta-1)},\end{aligned}$$

for all $0\le t\le T$.

Remark 7. If $C_1 := \dfrac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\left(C_\beta + \dfrac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)$, *then from Lemma 6 we obtain*

$$\|A\mathcal{S}_\beta(t)\|_{\mathcal{L}(X^\vartheta,X)} = \sup\{\|A\mathcal{S}_\beta(t)x\| : x\in X^\vartheta, \|x\|_\vartheta \le 1\} \le C_1 t^{\beta(\vartheta-1)}, \quad 0\le t\le T.$$

Lemma 8. *If $x\in X^\vartheta$ and $\vartheta > \dfrac{\beta-1}{\beta}$, then*

$$\left\|\int_0^t A\mathcal{S}_\beta(s)x\,ds\right\| \le \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\frac{\|A^\vartheta x\|}{\beta(\vartheta-1)+1}\left(C_\beta + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)t^{\beta(\vartheta-1)+1},\ 0\le t\le T, \tag{20}$$

where $\mathcal{S}_\beta(t)$ is the operator (12).

Proof of Theorem 8. Since $\beta(\vartheta-1)+1>0$, if follows from Lemma 6 that

$$\left\|\int_0^t A\mathcal{S}_\beta(s)x\,\mathrm{d}s\right\| \le \int_0^t \|A\mathcal{S}_\beta(s)x\|\,\mathrm{d}s \le C_1\|A^\vartheta x\|\int_0^t s^{\beta(\vartheta-1)}\,\mathrm{d}s = C_1\frac{\|A^\vartheta x\|}{\beta(\vartheta-1)+1}t^{\beta(\vartheta-1)+1},$$

where C_1 is the constant defined in Remark 7.

The following theorem shows the main result of this section, which provides the optimal regularity in the sense of the Hölder continuity achieved for the solution v of the linear problem (10). The assumptions of this theorem might be do not look like natural in the framework of linear problems, however these are the ones will serve as the key point in the proof of the a posteriori error estimation for the time discretization of the semi–linear problem provided in Section 4.

Theorem 9. *Let $1<\beta<2$, v_0 and f in (10) such that $v_0\in X^{1+\varepsilon}$, and $f\in C^\alpha_\gamma((0,T];X^\vartheta)$, with*

(a) $\dfrac{1-\gamma}{\beta}\le\varepsilon$.

(b) $\dfrac{\beta-1}{\beta}\le\vartheta<1$.

(c) $\alpha\le\gamma<\alpha+\beta(\vartheta-1)+1$.

Therefore, there exists a (computable) constant $K>0$ such that

$$\|v\|_{C^\alpha_\gamma((0,T];D(A))} \le K\left(\|v_0\|_{1+\varepsilon} + \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\right).$$

Proof of Theorem 9. We need to estimate $\|v\|_{C^\alpha_\gamma((0,T];D(A))}$, that is

$$\|v\|_{C^\alpha_\gamma((0,T];D(A))} = \sup_{0<t\le T}\|v(t)\|_{D(A)} + [[v]]_{C^\alpha_\gamma((0,T];D(A))}$$

First of all note that the solution v of (10) can be written as

$$\begin{aligned} v(t) &= \mathcal{S}_\beta(t)v_0 + \int_0^t \mathcal{S}_\beta(t-s)[f(s)-f(t)]\,\mathrm{d}s + \int_0^t \mathcal{S}_\beta(t-s)f(t)\,\mathrm{d}s \\ &= \mathcal{S}_\beta(t)v_0 + \int_0^t \mathcal{S}_\beta(t-s)[f(s)-f(t)]\,\mathrm{d}s + \int_0^t \mathcal{S}_\beta(s)f(t)\,\mathrm{d}s. \end{aligned} \tag{21}$$

In order to find the constant K, we will divide the proof in two parts.

Part I: We first estimate

$$\sup_{0<t\le T}\|v(t)\|_{D(A)} = \sup_{0<t\le T}\|v(t)\| + \sup_{0<t\le T}\|Av(t)\|.$$

Step 1: Estimation of $\sup_{0<t\le T}\|v(t)\|$.

Since $v_0 \in D(A^{1+\varepsilon}) \subset D(A)$, by Lemma 4 with $\vartheta = 1$ we have

$$\|\mathcal{S}_\beta(t)v_0\| \le C_0(\|v_0\| + \kappa\|Av_0\|t^\beta) \le C_0\max\{1,\kappa T^\beta\}\|v_0\|_{D(A)},$$

where C_0 is the constant defined in Remark 5. On the other hand, by Remark 5,

$$\begin{aligned} \left\|\int_0^t \mathcal{S}_\beta(t-s)[f(s)-f(t)]\,\mathrm{d}s\right\| &\le \int_0^t \|\mathcal{S}_\beta(t-s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(t)-f(s)\|_\vartheta\,\mathrm{d}s \\ &\le C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}(1+T^{\beta\vartheta})\int_0^t \|f(t)-f(s)\|_\vartheta\,\mathrm{d}s \\ &\le 2C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}(1+T^{\beta\vartheta})\int_0^t \sup_{0\le t\le T}\|f(t)\|_\vartheta\,\mathrm{d}s \\ &\le 2C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}(1+T^{\beta\vartheta})T\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}. \end{aligned}$$

Finally, similar computations now for the third term in (21) show that

$$\left\|\int_0^t \mathcal{S}_\beta(s)f(t)\,\mathrm{d}s\right\| \le C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}T(1+T^{\beta\vartheta})\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}, \quad 0\le t\le T.$$

We conclude that,

$$\sup_{0<t\le T}\|v(t)\| \le C_0\max\{1,\kappa T^\beta\}\|v_0\|_{D(A)} + 3C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}T(1+T^{\beta\vartheta})\|g\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

Step 2: Estimation of $\sup_{0<t\le T}\|Av(t)\|$.

Since $v_0 \in X^{1+\varepsilon} \subset D(A)$, from Lemma 6 with $\vartheta = 1$ we obtain that

$$\|A\mathcal{S}_\beta(t)v_0\| \le C_2\|Av_0\| \le C_2\|v_0\|_{D(A)},$$

where $C_2 := \dfrac{\kappa}{2\pi}(C_\beta + 2)$. On the other hand, Remark 7 implies

$$\begin{aligned}
\left\|\int_0^t A\mathcal{S}_\beta(t-s)[f(s)-f(t)]\,\mathrm{d}s\right\| &\leq \int_0^t \|A\mathcal{S}_\beta(t-s)\|_{\mathcal{L}(X^\vartheta,X)}\|f(t)-f(s)\|_\vartheta\,\mathrm{d}s\\
&\leq C_1\int_0^t (t-s)^{\beta(\vartheta-1)}\|f(t)-f(s)\|_\vartheta\,\mathrm{d}s\\
&= C_1\int_0^t (t-s)^{\beta(\vartheta-1)+\alpha}s^{-\gamma}\frac{s^\gamma\|f(t)-f(s)\|_\vartheta}{(t-s)^\alpha}\,\mathrm{d}s\\
&\leq C_1\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)}\int_0^t (t-s)^{\beta(\vartheta-1)+\alpha}s^{-\gamma}\,\mathrm{d}s\\
&= C_1\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)}t^{\alpha+\beta(\vartheta-1)+1-\gamma}B(\alpha+\beta(\vartheta-1)+1,1-\gamma),
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes the Beta function, and C_1 stands for the constant defined in Remark 7. The condition (c) of the present theorem on the parameters α,β,ϑ and γ implies that $\alpha+\beta(\theta-1)+1-\gamma\geq 0$, therefore

$$\left\|\int_0^t A\mathcal{S}_\beta(t-s)[f(s)-f(t)]\,\mathrm{d}s\right\| \leq C_1\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)}T^{\alpha+\beta(\vartheta-1)+1-\gamma}B(\alpha+\beta(\vartheta-1)+1,1-\gamma).$$

Finally since $f(t)\in X^\vartheta$, for $0\leq t\leq T$, we have by Lemma 6

$$\begin{aligned}
\left\|\int_0^t A\mathcal{S}_\beta(s)f(t)\,\mathrm{d}s\right\| &\leq C_1\frac{\|A^\vartheta f(t)\|}{\beta(\vartheta-1)+1}t^{\beta(\vartheta-1)+1}\\
&\leq \frac{C_1}{\beta(\vartheta-1)+1}t^{\beta(\vartheta-1)+1}\|f(t)\|_\vartheta\\
&\leq \frac{C_1}{\beta(\vartheta-1)+1}T^{\beta(\vartheta-1)+1}\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)}.
\end{aligned}$$

Therefore, if $C_3 := C_1T^{\alpha+\beta(\vartheta-1)+1-\gamma}B(\alpha+\beta(\vartheta-1)+1,1-\gamma)+\dfrac{C_1}{\beta(\vartheta-1)+1}T^{\beta(\vartheta-1)+1}$, then

$$\sup_{0<t\leq T}\|Av(t)\|\leq C_2\|v_0\|_{D(A)}+C_3\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)}. \tag{22}$$

Moreover since $\|v_0\|_{D(A)}\leq\|v_0\|_{D(A^{1+\varepsilon})}$, from Steps 1 and 2 we conclude that

$$\sup_{0<t\leq T}\|v(t)\|_{D(A)}\leq D_1\|v_0\|_{D(A^{1+\varepsilon})}+D_2\|f\|_{C_\gamma^\alpha((0,T];X^\vartheta)},$$

where $D_1 := C_0\max\{1,\kappa T^\beta\}+C_2$, and $D_2 := 3C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}T(1+T^{\beta\vartheta})+C_3$. This finishes the proof or Part I.

Part II: Here we estimate

$$[[v]]_{C_\gamma^\alpha((0,T];D(A))} = \sup_{0\leq s<t\leq T}\frac{s^\gamma\|v(t)-v(s)\|_{D(A)}}{(t-s)^\alpha},$$

by considering separately $\displaystyle\sup_{0\leq s<t\leq T}\frac{s^\gamma\|v(t)-v(s)\|}{(t-s)^\alpha}$, and $\displaystyle\sup_{0\leq s<t\leq T}\frac{s^\gamma\|Av(t)-Av(s)\|}{(t-s)^\alpha}$.

Step 1: Estimation of $\sup\limits_{0\le s<t\le T} \dfrac{s^\gamma \|v(t)-v(s)\|}{(t-s)^\alpha}$.

First, we notice that, for $0<s<t$, $v_0\in X^{1+\varepsilon}\subset D(A)$, and $f\in C^\alpha_\gamma((0,T];X^\vartheta)$, we have from (11)

$$v(t)-v(s)=(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))v_0+\int_0^s[\mathcal{S}_\beta(t-r)-\mathcal{S}_\beta(s-r)]f(r)\,\mathrm{d}r+\int_s^t\mathcal{S}_\beta(t-r)f(r)\,\mathrm{d}r. \tag{23}$$

Let us consider the first term in (23). By applying (17) once again, and making the change of variable $w=sz/t$ (but preserving by simplicity the notation with z), we have that

$$\begin{aligned}
&(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))v_0\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta(z^\beta-A)^{-1}v_0\,\mathrm{d}z-\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_s}\frac{\mathrm{e}^{zs}}{z}z^\beta(z^\beta-A)^{-1}v_0\,\mathrm{d}z\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta(z^\beta-A)^{-1}v_0\,\mathrm{d}z-\frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta(z^\beta-(s/t)^\beta A)^{-1}v_0\,\mathrm{d}z\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta\left\{(z^\beta-A)^{-1}-(z^\beta-(s/t)^\beta A)^{-1}\right\}v_0\,\mathrm{d}z\\
&= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}\frac{t^\beta-s^\beta}{t^\beta}z^\beta A(z^\beta-A)^{-1}(z^\beta-(s/t)^\beta A)^{-1}v_0\,\mathrm{d}z.
\end{aligned}$$

Observe that,

$$z^\beta A(z^\beta-A)(z^\beta-(s/t)^\beta A)^{-1}=\Big(I+(s/t)^\beta(z^\beta-(s/t)^\beta A)^{-1}A\Big)\Big(z^\beta(z^\beta-A)^{-1}-I\Big),$$

therefore, by Lemma 2 and the sectoriality bounds,

$$\begin{aligned}
&\|(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))v_0\|\\
&= \frac{1}{2\pi}\frac{t^\beta-s^\beta}{t^\beta}\Big\|\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}\Big((z^\beta(z^\beta-A)^{-1}-I)+\frac{s^\beta}{t^\beta}(z^\beta(z^\beta-A)-I)(z^\beta-(s/t)^\beta A)^{-1}A\Big)v_0\,\mathrm{d}z\Big\|\\
&\le \frac{1}{2\pi}\frac{t^\beta-s^\beta}{t^\beta}(M+1)\left(\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|}\|v_0\|\,|\mathrm{d}z|+\frac{s^\beta}{t^\beta}\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|}\|(z^\beta-(s/t)^\beta A)^{-1}\|\,\|Av_0\|\,|\mathrm{d}z|\right)\\
&\le \frac{1}{2\pi}\frac{t^\beta-s^\beta}{t^\beta}(M+1)\left(\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|}\|v_0\|\,|\mathrm{d}z|+\frac{Ms^\beta}{t^\beta}\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|^{\beta+1}}\|Av_0\|\,|\mathrm{d}z|\right)\\
&\le C_0(M+1)\frac{t^\beta-s^\beta}{t^\beta}(1+Ms^\beta)\|v_0\|_{1+\varepsilon}.
\end{aligned}$$

Hence,

$$\frac{\|(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))v_0\|s^\gamma}{(t-s)^\alpha}\le C_0(M+1)\frac{s^\gamma(t^\beta-s^\beta)}{t^\beta(t-s)^\alpha}(1+Ms^\beta)\|v_0\|_{1+\varepsilon}.$$

Here note that, by the boundness of the function $(1-x^\beta)/(1-x)^\alpha$, for $0\le x<1$ (x here plays the role of s/t), under the hypothesis (c) of the present Theorem, we have

$$\frac{s^\gamma(t^\beta-s^\beta)}{t^\beta(t-s)^\alpha}\le\frac{1-(s/t)^\beta}{(1-s/t)^\alpha}t^{\gamma-\alpha}\le T^{\gamma-\beta}.$$

Therefore,

$$\frac{\|(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))v_0\|s^\gamma}{(t-s)^\alpha}\le C_0(M+1)\Big(T^{\gamma-\alpha}+MT^{\gamma+\beta-\alpha}\Big)\|v_0\|_{1+\varepsilon},$$

where have to note that, according the hypothesis (c), $\gamma-\alpha\geq 0$, and of course $\gamma+\beta-\alpha\geq 0$.

Now, we estimate the norm of the second term in (23), that is

$$\left\|\int_0^s (\mathcal{S}_\beta(t-r)-\mathcal{S}_\beta(s-r))f(r)\,\mathrm{d}r\right\|,\qquad 0\leq s<t.$$

Observe that for $x\in X^\vartheta$, and $t>s$, the same ideas as in the previous bounds lead to the following expression

$$\begin{aligned}(\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))x &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta\Big((z^\beta-A)^{-1}-(z^\beta-(s/t)A)^{-1}\Big)x\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\frac{t^\beta-s^\beta}{t^\beta}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}z^\beta(z^\beta-(s/t)^\beta A)^{-1}A^{1-\vartheta}(z^\beta-A)^{-1}A^\vartheta x\,\mathrm{d}z.\end{aligned}$$

Then the next bounds follows from Lemma 2, and (18)

$$\begin{aligned}\|\mathcal{S}_\beta(t)-\mathcal{S}_\beta(s))x\| &\leq \frac{1}{2\pi}(1-(s/t)^\beta)M\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|}\|A^{1-\vartheta}(z^\beta-A)^{-1}\|\,\|A^\vartheta x\|\,\mathrm{d}z\\ &\leq \frac{M(1+M)^{1-\vartheta}\kappa}{2\pi}\left(C_\beta+\frac{2\mathrm{e}^{\cos(\varphi/\beta)}}{-\cos(\varphi/\beta)}\right)\|A^\vartheta x\|(1-(s/t)^\beta)t^{\beta\vartheta}.\end{aligned}\tag{24}$$

Straightforwardly we have from (24)

$$\begin{aligned}&\left\|\frac{s^\gamma}{(t-s)^\alpha}\int_0^s(\mathcal{S}_\beta(t-r)-\mathcal{S}_\beta(s-r))x\,\mathrm{d}r\right\|\\ &\leq \frac{M(1+M)^{1-\vartheta}\kappa}{2\pi}\left(C_\beta+\frac{2\mathrm{e}^{\cos(\varphi/\beta)}}{-\cos(\varphi/\beta)}\right)\|A^\vartheta x\|\int_0^s\frac{s^\gamma\left(1-\frac{(s-r)^\beta}{(t-r)^\beta}\right)(t-r)^{\beta\vartheta}}{(t-s)^\alpha}\,\mathrm{d}r\\ &\leq C_0M(1+M)^{1-\vartheta}\kappa\|A^\vartheta x\|t^{\gamma-\alpha+\beta\vartheta+1},\end{aligned}$$

where we applied again the straightforward bound

$$\int_0^s\frac{s^\gamma\left(1-\frac{(s-r)^\beta}{(t-r)^\beta}\right)(t-r)^{\beta\vartheta}}{(t-s)^\alpha}\,\mathrm{d}r\leq t^{\gamma-\alpha+\beta\vartheta+1}$$

Therefore, since $\gamma-\alpha+\vartheta\beta+1\geq 0$ by the condition (c) of the present Theorem, and since $f(r)\in X^\vartheta$, for $0\leq r\leq s$, and $\|A^\vartheta f(r)\|\leq\sup_{0\leq r\leq T}\|f(r)\|_\vartheta\leq\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}$,

$$\left\|\int_0^s\frac{s^\gamma(\mathcal{S}_\beta(t-r)-\mathcal{S}_\beta(s-r))f(r)}{(t-s)^\alpha}\,\mathrm{d}r\right\|\quad\leq\quad C_0M(1+M)^{1-\vartheta}\kappa T^{\gamma-\alpha+\beta\vartheta+1}\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

Finally, we estimate the norm of the third term in (23). Again, since $f(r)\in X^\vartheta$, and $\|A^\vartheta f(r)\|\leq\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}$, for $r>0$, the Lemma 4 implies that

$$\begin{aligned}\|\mathcal{S}_\beta(t-r)f(r)\| &\leq C_0\left(\|f(r)\|+\kappa\,(M+1)^{1-\vartheta}\|A^\vartheta f(r)\|(t-r)^{\vartheta\beta}\right)\\ &\leq C_0\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\max\{1,\kappa\,(M+1)^{1-\vartheta}\}\left(1+(t-r)^{\vartheta\beta}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\int_s^t \mathcal{S}_\beta(t-r)f(r)\,\mathrm{d}r\right\| &\le \int_s^t \|\mathcal{S}_\beta(t-r)f(r)\|\,\mathrm{d}r\\ &\le C_0\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\max\{1,\kappa(M+1)^{1-\vartheta}\}\int_s^t 1+(t-r)^{\vartheta\beta}\,\mathrm{d}r\\ &\le C_0\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\max\{1,\kappa(M+1)^{1-\vartheta}\}\left((t-s)+\frac{(t-s)^{\vartheta\beta+1}}{\beta\vartheta+1}\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\left\|\int_s^t \frac{s^\gamma\mathcal{S}_\beta(t-r)f(r)}{(t-s)^\alpha}\,\mathrm{d}r\right\| &\le C_0\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\max\{1,\kappa\,(M+1)^{1-\vartheta}\}\left((t-s)^{1-\alpha}s^\gamma+\frac{(t-s)^{\vartheta\beta+1-\alpha}s^\gamma}{\beta\vartheta+1}\right)\\ &\le C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}\left(T^{1-\alpha+\gamma}+\frac{T^{\vartheta\beta+1-\alpha+\gamma}}{\beta\vartheta+1}\right)\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.\end{aligned}$$

We conclude that

$$\sup_{0\le s<t\le T}\frac{s^\gamma\|v(t)-v(s)\|}{(t-s)^\alpha}\le C_4\|v_0\|_{1+\varepsilon}+C_5\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

where $C_4 := C_0(M+1)\Big(T^{\gamma-\alpha}+MT^{\gamma+\beta-\alpha}\Big)$ and

$$C_5 := C_0M(1+M)^{1-\vartheta\kappa T^{\gamma-\alpha+\beta\vartheta+1}}+C_0\max\{1,\kappa\,(M+1)^{1-\vartheta}\}\left(T^{1-\alpha+\gamma}+\frac{T^{\vartheta\beta+1-\alpha+\gamma}}{\beta\vartheta+1}\right).$$

Step 2: Estimation of $\sup_{0\le s<t\le T}\dfrac{s^\gamma\|Av(t)-Av(s)\|}{(t-s)^\alpha}$.

In order to obtain this estimation, we first notice that, for $0<s<t$, and $v_0\in X^{1+\varepsilon}$ we can write

$$Av(t)-Av(s)=(A\mathcal{S}_\beta(t)-A\mathcal{S}_\beta(s))v_0+\int_0^s A\mathcal{S}_\beta(r)(f(t-r)-f(s-r))\,\mathrm{d}r+\int_s^t A\mathcal{S}_\beta(r)f(s-r)\,\mathrm{d}r. \quad (25)$$

Repeating previous ideas, the following equality is straightforward,

$$(A\mathcal{S}_\beta(t)-A\mathcal{S}_\beta(s))v_0 = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_t}\frac{\mathrm{e}^{zt}}{z}\frac{t^\beta-s^\beta}{t^\beta}z^\beta A(z^\beta-A)^{-1}(z^\beta-(s/t)^\beta A)^{-1}Av_0\,\mathrm{d}z.$$

Now observe that $A(z^\beta-A)^{-1}Av_0=A^{1-\varepsilon}(z^\beta-A)^{-1}A^{1+\varepsilon}v_0$, and that $D(A)\subset X^{1-\varepsilon}$. Therefore, in similar fashion as in (18)

$$\left\|A(z^\beta-A)^{-1}Av_0\right\|\le\frac{\kappa(M+1)^{1-\varepsilon}\|A^{1+\varepsilon}v_0\|}{|z|^{\beta\varepsilon}}. \quad (26)$$

By Lemma 2, and (26), we have

$$\begin{aligned}\|(A\mathcal{S}_\beta(t)-A\mathcal{S}_\beta(s))v_0\| &\le \frac{M\kappa(1+M)^{1+\varepsilon}}{2\pi}\|A^{1+\varepsilon}v_0\|\frac{t^\beta-s^\beta}{t^\beta}\int_{\Gamma_t}\frac{|\mathrm{e}^{zt}|}{|z|^{1+\beta\varepsilon}}\,\mathrm{d}z\\ &\le \frac{M\kappa(1+M)^{1+\varepsilon}}{2\pi}\|A^{1+\varepsilon}v_0\|\frac{t^\beta-s^\beta}{t^{\beta-\beta\varepsilon}}.\end{aligned}$$

Having in mind that by assumption (c) of this theorem there holds $\gamma+\beta\varepsilon-1\geq 0$, we obtain

$$\begin{aligned}\frac{s^{\gamma}\|(A\mathcal{S}_\beta(t)-A\mathcal{S}_\beta(s))v_0\|}{(t-s)^{-\alpha}} &\leq \frac{M\kappa(1+M)^{1+\varepsilon}}{2\pi}\|A^{1+\varepsilon}v_0\|\frac{(t^\beta-s^\beta)s^\gamma}{t^{\beta-\beta\varepsilon}(t-s)^\alpha}\\ &\leq \frac{M\kappa(1+M)^{1+\varepsilon}}{2\pi}\|A^{1+\varepsilon}v_0\|T^{\gamma-\alpha+\beta\varepsilon}\\ &= C_6\|A^{1+\varepsilon}v_0\|,\end{aligned}$$

where

$$C_6:=\frac{M\kappa(1+M)^{1+\varepsilon}}{2\pi}T^{\gamma-\alpha+\beta\varepsilon}$$

To estimate the second term in (25) we notice that

$$\begin{aligned}A\mathcal{S}_\beta(r)(f(t-r)-f(s-r)) &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_r}\mathrm{e}^{zr}z^{\beta-1}(z^\beta-A)^{-1}(f(t-r)-f(s-r))\,\mathrm{d}z\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_r}\mathrm{e}^{zr}z^{\beta-1}A^{1-\vartheta}(z^\beta-A)^{-1}A^{\vartheta}(f(t-r)-f(s-r))\,\mathrm{d}z,\end{aligned}$$

which implies by (18) and Lemma 2 that

$$\begin{aligned}&\|A\mathcal{S}_\beta(r)(f(t-r)-f(s-r))\|\\ &\leq \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\|A^{\vartheta}(f(t-r)-f(s-r))\|\int_{\Gamma_r}|\mathrm{e}^{zr}||z|^{\beta(1-\vartheta)-1}|\,\mathrm{d}z|\\ &\leq \left(C_\beta+\frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)r^{\beta(\vartheta-1)}\frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\|A^{\vartheta}(f(t-r)-f(s-r))\|.\end{aligned}$$

On the other hand, we notice that

$$\|A^{\vartheta}(f(t-r)-f(s-r))\| = \frac{\|A^{\vartheta}(f(t-r)-f(s-r))\|}{(t-s)^\alpha}(s-r)^\gamma\,\frac{(t-s)^\alpha}{(s-r)^\gamma}\leq\frac{(t-s)^\alpha}{(s-r)^\gamma}\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

Hence,

$$\begin{aligned}&\|A\mathcal{S}_\beta(r)(f(t-r)-f(s-r))\|\\ &\leq \left(C_\beta+\frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)\frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}\,\frac{(t-s)^\alpha}{(s-r)^\gamma}r^{\beta(\vartheta-1)}\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.\end{aligned}$$

Since $\beta(\vartheta-1)+1\geq 0$ we have

$$\int_0^s(s-r)^{-\gamma}r^{\beta(\vartheta-1)}\,\mathrm{d}r=s^{\beta(\vartheta-1)+1-\gamma}B(1-\gamma,\beta(\vartheta-1)+1),$$

and we obtain that

$$\int_0^s\|A\mathcal{S}_\beta(r)(f(t-r)-f(s-r))\|\,\mathrm{d}r\leq C_7(t-s)^\alpha s^{\beta(\vartheta-1)+1-\gamma}\|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)},$$

where

$$C_7:=\left(C_\beta+\frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}}\right)\frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi}B(1-\gamma,\beta(\vartheta-1)+1).$$

Thus,

$$\int_0^s \left\| A\mathcal{S}_\beta(r) \frac{s^\gamma f(t-r) - f(s-r)}{(t-s)^\alpha} \right\| \mathrm{d}r \leq C_6 T^{\beta(\vartheta-1)+1} \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

Finally, to estimate the third integral in (25) we write

$$A\mathcal{S}_\beta(r) f(t-r) \quad = \quad \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r} \mathrm{e}^{zr} z^{\beta-1} A^{1-\vartheta}(z^\beta - A)^{-1} A^\vartheta f(t-r)\,\mathrm{d}z,$$

and since, $\|A^\vartheta f(t-r)\| \leq \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}$, we obtain by (18) and Lemma 2 that

$$\begin{aligned} \|A\mathcal{S}_\beta(r) f(t-r)\| &\leq \frac{1}{2\pi} \int_{\Gamma_r} |\mathrm{e}^{zr}||z|^{\beta-1} \|A^{1-\vartheta}(z^\beta - A)^{-1}\| \|A^\vartheta f(t-r)\| |\,\mathrm{d}z| \\ &\leq \left(C_\beta + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}} \right) \frac{\kappa(M+1)^{1-\vartheta}}{2\pi} r^{\beta(\vartheta-1)} \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}, \end{aligned} \tag{27}$$

which implies, for $0 \leq s < t \leq T$, that

$$\int_s^t \|A\mathcal{S}_\beta(r) f(t-r)\|\,\mathrm{d}r \quad \leq \quad C_8 (t^{\beta(\vartheta-1)+1} - s^{\beta(\vartheta-1)+1}) \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)},$$

where,

$$C_8 := \left(C_\beta + \frac{2\Gamma(\beta(1-\vartheta))}{(-\cos(\phi/\beta))^{\beta(1-\vartheta)}} \right) \frac{\kappa\,(M+1)^{1-\vartheta}}{2\pi[\beta(\vartheta-1)+1]}.$$

One more time, according the hypothesis (c) of the present Theorem, the function $\dfrac{1-x^{\beta(\vartheta-1)+1}}{(1-x)^\alpha}$ is bounded by 1, for $0 \leq x < 1$, where once again x plays the role here of s/t. This allows us to conclude from (27) that,

$$\frac{s^\gamma}{(t-s)^\alpha} \int_s^t \|A\mathcal{S}_\beta(r) f(t-r)\|\,\mathrm{d}r \quad \leq \quad C_8 s^\gamma t^{\beta(\vartheta-1)+1-\alpha} \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)} \leq C_8 T^{\beta(\vartheta-1)+1+\gamma-\alpha} \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

Now, we notice that $\|A^{1+\varepsilon} v_0\| \leq \|v_0\|_{1+\varepsilon}$ and hence

$$\sup_{0 \leq s < t \leq T} \frac{s^\gamma \|Av(t) - Av(s)\|}{(t-s)^\alpha} \leq C_6 \|v_0\|_{1+\varepsilon} + C_{10} \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)},$$

where $C_{10} := C_7 T^{\beta(\vartheta-1)+1} + C_8 T^{\beta(\vartheta-1)+1+\gamma-\alpha}$.

From Steps 1 and 2 of the Part II of this proof we have

$$\sup_{0 \leq s < t \leq T} \frac{s^\gamma \|v(t) - v(s)\|_{D(A)}}{(t-s)^\alpha} \leq D_3 \|v_0\|_{1+\varepsilon} + D_4 \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)}.$$

where $D_3 = C_4 + C_6$ and $D_4 := C_5 + C_{10}$. This concludes the proof or Part II.

From Part I and II, we obtain that

$$\|v\|_{C^\alpha_\gamma((0,T];D(A))} \leq (D_1 + D_3) \sup_{0 < t \leq T} \|v(t)\|_{D(A)} + (D_2 + D_4) \sup_{0 \leq s < t \leq T} \frac{s^\gamma \|v(t) - v(s)\|_{D(A)}}{(t-s)^\alpha},$$

and therefore there exists a constant $K := \max\{D_1 + D_3, D_2 + D_4\}$ such that

$$\|v\|_{C^\alpha_\gamma((0,T];D(A))} \leq K \left(\|v_0\|_{1+\varepsilon} + \|f\|_{C^\alpha_\gamma((0,T];X^\vartheta)} \right),$$

and the proof concludes.

Remark 10. Observe that all constants shown in the proof of Theorem 9 are in fact computable, which is an essential requirement in the main result of next section.

4. A posteriori error estimates for the time discretization

Let $\{U_n\}_{n=1}^N$ be a time discretization of (9) at time levels $0 = t_0 < t_1 < t_2 < \ldots < t_N = T$, where U_n stands for the approximation to the continuous solution $u(t)$ in t_n, i.e. $U_n \approx u(t_n)$, $1 \le n \le N$. Moreover denote $I_n = [t_{n-1}, t_n]$, and $\tau_n := t_n - t_{n-1}$, for $1 \le n \le N$. A lot of time discretizations of (9) (but also for (8)) have been studied in the literature, e.g. convolution quadrature based methods [6, 14, 42], numerical inversion of the Laplace transform [9, 38], collocation methods [8], Adomian decomposition methods [19, 23], and so many others. Without loss of generality we can assume that the numerical method that provides the numerical solution above admits the format

$$U_n = U_0 + \sum_{j=0}^{n} q_{n-j} U_j + \tau_n F(U_n), \qquad 1 \le n \le N, \tag{28}$$

for certain weights $\{q_j\}_{n=0}^N$ where each q_n depends in some manner on τ_n. In particular, if one can combine the backward Euler method for the time derivative, and a convolution quadrature methods with constant step size, such numerical method admits the formulation (28) (see [16, 39]).

Note that the nonlinearity F of (8), and henceforth of (28), typically obliges to assume some restrictions on the largest time step $\tau_{max} := \max_{0 \le n \le N}\{\tau_n\}$, however this fact is not relevant for our purposes and therefore we will assume in the rest of the section that τ_{max} is small enough.

Anyhow, since our results make use of a convenient continuous reconstruction of the discrete solution, rather than of the discrete solution itself, and since the convergence order of the method is not the subject of this work, the numerical scheme chosen does not deserve further attention.

An important issue in our study is the regularity of the terms involved, not only in the continuous equation but also in the numerical scheme. In fact, the nonlinear character of (8) makes expected that some regularity on the discrete solution $\{U_n\}_{n=0}^N$ is required, but even more, the fractional nature of the integral term involved in (8) will make expected as well some additional regularity conditions. To be more precise, assume that

$$\{U_n\}_{n=0}^N \subset X^{1+\vartheta}, \qquad \text{with} \qquad \frac{\beta - 1}{\beta} \le \vartheta < 1, \tag{29}$$

where β is the order of integration in (8). In the integer case, i.e. if $\beta = 1$, ϑ can be 0 and therefore spatial regularity is not longer needed. This is consistent with the results achieved in [15] where it is merely required that $\{U_n\}_{n=0}^N \subset \mathcal{B}$.

Special attention must be paid to the regularity of the numerical initial data U_0, and more precisely, since the estimates we show below takes into account the contribution of the initial error $e_0 := \mathcal{U}(0) - u_0 = U_0 - u_0$, special attention must be paid to the regularity of the initial error e_0 rather than of U_0. In particular, we assume that for certain $0 < \varepsilon < 1$, to be determined below, we have

$$e_0 \in X^{1+\varepsilon}. \tag{30}$$

Our estimates are obtained from a convenient continuous reconstruction of the numerical solution. In this way, we define the continuous piecewise polynomial function

$$\mathcal{U} : [0, T] \to X^{1+\vartheta}, \qquad \mathcal{U} \in \mathcal{C}^1((0, T), X^{1+\vartheta}), \tag{31}$$

satisfying for $1 \le n \le N$,

- $\mathcal{U}|_{I_n} \in \mathbb{P}_3(I_n, X^{1+\vartheta})$.
- $\mathcal{U}(t_n) = U_n$.
- $\mathcal{U}'|_{I_n}(t_n) = \mathcal{U}'|_{I_{n+1}}(t_n)$, for $1 \le n \le N-1$, and $\mathcal{U}'(0) = \mathcal{U}'(T) = 0$,

where $\mathbb{P}_3(I_n, X^{1+\vartheta})$ is the set of all $X^{1+\vartheta}$-valued polynomials of degree less or equal to 3 defined in I_n.

Let $e : [0,T] \to \mathcal{B}$ be the error function defined as $e(t) := \mathcal{U}(t) - u(t)$. Then, there exists a computable residual function $\mathcal{R} : [0,T] \to \mathcal{B}$ such that e is the solution of

$$e'(t) = \partial^{1-\beta} Ae(t) + G(t, e(t)) + \mathcal{R}(t), \qquad e(0) = e_0, \qquad 0 \le t \le T, \tag{32}$$

where $G : [0,T] \times \mathcal{B} \to X$ is the function defined by

$$G(t, w) := F(\mathcal{U}(t)) - F(\mathcal{U}(t) - w), \quad 0 \le t \le T. \tag{33}$$

Note that $G(t, e(t)) = F(\mathcal{U}(t)) - F(\mathcal{U}(t) - e(t) = F(\mathcal{U}(t)) - F(u(t))$, for $0 \le t \le T$. Moreover, $\mathcal{R}$ is in fact computable since it can be expressed in term of computable quantities as

$$\mathcal{R}(t) = \mathcal{U}'(t) - \partial^{1-\beta} A\mathcal{U}(t) - F(\mathcal{U}(t)), \qquad \mathcal{U}(0) = U_0, \qquad 0 \le t \le T,$$

or in other words, there holds

$$\mathcal{U}'(t) = \partial^{1-\beta} A\mathcal{U}(t) + F(\mathcal{U}(t)) + \mathcal{R}(t), \qquad \mathcal{U}(0) = U_0, \qquad 0 \le t \le T.$$

The proof on the main result in this paper is based on the application of a fixed point theorem over the linear problem

$$e'(t) = +\partial^{1-\beta} Ae(t) + G(t, w(t)) + \mathcal{R}(t), \qquad e(0) = e_0, \qquad 0 \le t \le T, \tag{34}$$

for a given w belonging to a suitable functional space to be described below, in such a manner that the fixed point of (34) stands for the solution of (32). Here $G(t, w(t)) + \mathcal{R}(t)$ plays the role of $f(t)$ in (10), is for that the regularity of such a term is one of the key points for our purposes.

On the one hand, the regularity of $\mathcal{R}$ is straightforward having in mind that $\mathcal{U} \in \mathcal{C}^1((0,T), X^{1+\vartheta})$, the linear structure of the numerical scheme assumed in (28), and Hypothesis (H1) on the Lipchitz continuity of F_u. In fact we have that $\mathcal{R} \in C^\alpha_\gamma((0,T]; X^\vartheta)$.

The regularity of $G(t, w(t))$ is not so trivial and it is shown in Lemma 11 below. To this end we need to state a suitable set of functions, in fact let $0 < \rho < 1$ be a constant such that

$$\rho \le \frac{1}{2} R(u_0), \tag{35}$$

where $R(u_0)$ is the constant given in Hypothesis (H1), and define the set of functions

$$\mathcal{Y}_\rho := \Big\{ w \in C^\alpha_\gamma((0,T]; D(A)) : w(0) = e_0, \text{ and } \|w\|_{C^\alpha_\gamma((0,T];D(A))} < \rho \Big\}.$$

Moreover, in addition to (H1)–(H3) we assume that

(H4) The reconstruction $\mathcal{U}$ defined in (31) satisfies

$$\|\mathcal{U}(\cdot) - u_0\|_{C^\alpha_\gamma((0,T];D(A))} \le \rho.$$

In order to formulate all our results in terms of truly computable terms one can express Hypothesis (H4) depending on U_0 instead of u_0 . In that case small changes in the proof lead to the same result, however for the sake of the simplicity of the presentation we assume Hypothesis (H4) as stated above.

Lemma 11. *Let $\mathcal{U}$ be the continuous reconstruction (31) satisfying Hypothesis (H4), and ρ satisfying the condition (35). Assume also that α, β, γ, and ϑ satisfy the conditions (a)–(c) of Theorem 9. Then $G(\cdot, w(\cdot)) \in C^\alpha_\gamma((0,T]; X^\vartheta)$, for every $w \in \mathcal{Y}_\rho$, and there holds*

$$\|G(\cdot, w(\cdot))\|_{C^\alpha_\gamma((0,T];X^\vartheta)} \le \Lambda \|w\|_{C^\alpha_\gamma((0,T];D(A))},$$

where $\Lambda := \frac{9L\rho}{2}$, and $L = L(u_0)$ is the constant given in (H1).

Proof of Theorem 11. Let $w \in \mathcal{Y}_\rho$. Since F is Frechét differentiable and $F_u(u_0) = 0$, we can write

$$G(t, w(t)) \quad = \quad F(\mathcal{U}(t)) - F(\mathcal{U}(t) - w(t)) = \int_0^1 [F_u(\mathcal{U}(t) - (1-\tau)w(t)) - F_u(u_0)]\, \mathrm{d}\tau\, w(t).$$

By Hypothesis (H1) we have

$$\begin{aligned}
\|G(t, w(t))\|_\vartheta &\leq \int_0^1 \|F_u(\mathcal{U}(t) - (1-\tau)w(t)) - F_u(u_0)\|_{\mathcal{L}(D(A), X^\vartheta)}\, \mathrm{d}\tau \|w(t)\|_{D(A)} \\
&\leq L \int_0^1 \left[\|\mathcal{U}(t) - u_0\|_{D(A)} + (1-\tau)\|w(t)\|_{D(A)}\right] \mathrm{d}\tau \sup_{0<t\leq T} \|w(t)\|_{D(A)} \\
&\leq L \int_0^1 \left[\sup_{0<t\leq T} \|\mathcal{U}(t) - u_0\|_{D(A)} + (1-\tau) \sup_{0<t\leq T} \|w(t)\|_{D(A)}\right] \mathrm{d}\tau \|w\|_{C^\alpha_\gamma((0,T];D(A))} \\
&\leq L \int_0^1 \left[\|\mathcal{U} - u_0\|_{C^\alpha_\gamma((0,T];D(A))} + (1-\tau)\|w\|_{C^\alpha_\gamma((0,T];D(A))}\right] \mathrm{d}\tau \|w\|_{C^\alpha_\gamma((0,T];D(A))} \\
&\leq L \int_0^1 \rho + (1-\tau)\rho\, \mathrm{d}\tau \|w\|_{C^\alpha_\gamma((0,T];D(A))} \\
&= \frac{3L\rho}{2} \|w\|_{C^\alpha_\gamma((0,T];D(A))}.
\end{aligned}$$

On the other hand, if $w \in \mathcal{Y}_\rho$, and $0 \leq s < t \leq T$, then

$$G(t, w(t)) - G(s, w(s)) = [F(\mathcal{U}(t)) - F(\mathcal{U}(s))] - [F(\mathcal{U}(t) - w(t)) - F(\mathcal{U}(s) - w(s))].$$

Once again, since F is Frechét differentiable and $F_u(u_0) = 0$, we obtain

$$\begin{aligned}
G(t, w(t)) - G(s, w(s)) \quad = \quad & \int_0^1 [F_u(\mathcal{U}(t) - (1-\tau)w(t)) - F_u(\mathcal{U}(s) - (1-\tau)w(s))]\, \mathrm{d}\tau\, w(t) \\
& + \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w(s)) - F_u(u_0)]\, \mathrm{d}\tau [w(t) - w(s)],
\end{aligned}$$

which implies

$$\begin{aligned}
& \|G(t, w(t)) - G(s, w(s))\|_\vartheta \\
\leq\ & \int_0^1 \|F_u(\mathcal{U}(t) - (1-\tau)w(t)) - F_u(\mathcal{U}(s) - (1-\tau)w(s))\|_{\mathcal{L}(D(A), X^\vartheta)}\, \mathrm{d}\tau \|w(t)\|_{D(A)} \\
& + \int_0^1 \|F_u(\mathcal{U}(s) - (1-\tau)w(s)) - F_u(v_0)\|_{\mathcal{L}(D(A), X^\vartheta)}\, \mathrm{d}\tau \|w(t) - w(s)\|_{D(A)} \\
\leq\ & L \int_0^1 \|(\mathcal{U}(t) - \mathcal{U}(s)) - (1-\tau)(w(t) - w(s))\|_{D(A)}\, \mathrm{d}\tau \|w\|_{C^\alpha_\gamma((0,T];D(A))} \\
& + L \int_0^1 \|(\mathcal{U}(s) - u_0) - (1-\tau)w(s)\|_{D(A)}\, \mathrm{d}\tau \|w(t) - w(s)\|_{D(A)}.
\end{aligned}$$

Moreover, we notice that by Hypothesis (H4)

$$\frac{s^\gamma \|\mathcal{U}(t) - \mathcal{U}(s)\|_{D(A)}}{(t-s)^\alpha} = \frac{s^\gamma \|(\mathcal{U}(t) - u_0) - (\mathcal{U}(s) - u_0)\|_{D(A)}}{(t-s)^\alpha} \leq \|\mathcal{U} - u_0\|_{C^\alpha_\gamma((0,T];D(A))} < \rho. \tag{36}$$

Henceforth, since $\|w\|_{C_\gamma^\alpha((0,T];D(A))} < \rho$

$$\begin{aligned}
\frac{s^\gamma \|G(t,w(t)) - G(s,w(s))\|_\vartheta}{(t-s)^\alpha} &\le L\left[\rho + \int_0^1 (1-\tau)\frac{s^\gamma\|w(t)-w(s)\|_{D(A)}}{(t-s)^\alpha}\,\mathrm{d}\tau\right]\|w\|_{C_\gamma^\alpha((0,T];D(A))} \\
&\quad + L\left[\rho + \int_0^1 (1-\tau)\|w(s)\|_{D(A)}\,\mathrm{d}\tau\right]\frac{s^\gamma\|w(t)-w(s)\|_{D(A)}}{(t-s)^\alpha} \\
&\le 2L\left[\rho + \int_0^1 (1-\tau)\,\mathrm{d}\tau \|w\|_{C_\gamma^\alpha((0,T];D(A))}\right]\|w\|_{C_\gamma^\alpha((0,T];D(A))} \\
&= 2L\left[\rho + \frac{\rho}{2}\right]\|w\|_{C_\gamma^\alpha((0,T];D(A))} \\
&= 3L\rho\|w\|_{C_\gamma^\alpha((0,T];D(A))}.
\end{aligned}$$

Since $\|G(\cdot,w(\cdot))\|_{C_\gamma^\alpha((0,T];X^\vartheta)} = \sup_{0<t\le T}\|G(t,w(t))\|_\vartheta + \sup_{0\le s<t\le T}\frac{s^\gamma\|G(t,w(t))-G(s,w(s))\|_\vartheta}{(t-s)^\alpha}$ we obtain

$$\|G(\cdot,w(\cdot))\|_{C_\gamma^\alpha((0,T];X^\vartheta)} \le \frac{9}{2}L\rho\|w\|_{C_\gamma^\alpha((0,T];D(A))},$$

and the proof is finished.

The next theorem shows the main result of this paper.

Theorem 12. *Let $u,\mathcal{U} : [0,T] \to \mathcal{B}$ be the solution of (5), and the continuous reconstruction (31) respectively, such that u satisfies Hypotheses (H1)–(H3), and $\mathcal{U}$ Hypothesis (H4).*

Let $\alpha,\beta,\gamma,\varepsilon,$ and ϑ be positive constants satisfying (a)–(c) of Theorem 9, and $\rho > 0$ satisfying

$$\rho < \frac{1}{6KL}, \tag{37}$$

where K is the constant obtained in Theorem 9 and $L = L(u_0)$ is the Lipschitz constant of Hypothesis (H1). If there holds (30), and the residual $\mathcal{R}$ defined in (32) satisfies

$$\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C_\gamma^\alpha((0,T];X^\vartheta)} \le \frac{\rho}{K}\left(1 - \frac{9KL\rho}{2}\right), \tag{38}$$

then there exists a computable constant $C > 0$ such that

$$\|\mathcal{U} - u\|_{C_\gamma^\alpha((0,T];D(A))} \le C\left(\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C_\gamma^\alpha((0,T];X^\vartheta)}\right). \tag{39}$$

Proof of Theorem 12. First of all, recall that $e(t)$ stands for the solution of equation (32), i.e.

$$e'(t) = \partial^{1-\beta}Ae(t) + G(t,w(t)) + \mathcal{R}(t), \quad e(0) = e_0, \quad 0 \le t \le T, \tag{40}$$

for each $w \in \mathcal{Y}_\rho$. Notice that for the brevity of the notation we avoided the dependence of e on w. Let $\Psi : \mathcal{Y}_\rho \to \mathcal{Y}_\rho$ be the operator defined by $\Psi(w) = h$, for $w \in \mathcal{Y}_\rho$, where h is the solution of the linear equation

$$h'(t) = \partial^{1-\beta}Ah(t) + G(t,w(t)) + \mathcal{R}(t), \quad h(0) = e_0, \quad 0 \le t \le T.$$

Recall that the fixed point of Ψ is the solutions of the equation (32) in $\mathcal{Y}_\rho$, and therefore, in order to prove the theorem we will show in two steps that Ψ has a unique fixed point in $\mathcal{Y}_\rho$.

Step 1: Let us show that $\Psi(\mathcal{Y}_\rho) \subset \mathcal{Y}_\rho$.

If we take $w \in \mathcal{Y}_\rho$, then by Lemma 11 the function $G(\cdot, w(\cdot))$ belongs to $C^\alpha_\gamma((0,T];X^\vartheta)$. Since $\mathcal{R} \in C^\alpha_\gamma((0,T];X^\vartheta)$ we have that the function $f(t) := G(t,w(t)) + \mathcal{R}(t)$ belongs to $C^\alpha_\gamma((0,T];X^\vartheta)$ as well. Therefore, by Theorem 9, and Lemma 11, $\Psi(w) \in C^\alpha_\gamma((0,T];D(A))$, and

$$\begin{aligned} \|\Psi(w)\|_{C^\alpha_\gamma((0,T];D(A))} &= \leq K\Big(\|e_0\|_{1+\varepsilon} + \|G(\cdot,w(\cdot)) + \mathcal{R}\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\Big) \\ &\leq K\left(\|e_0\|_{1+\varepsilon} + \frac{9L\rho}{2}\|w\|_{C^\alpha_\gamma((0,T];D(A))} + \|\mathcal{R}\|_{C^\alpha_\gamma((0,T];X^\vartheta)}\right). \end{aligned} \tag{41}$$

Since $\|w\|_{C^\alpha_\gamma((0,T];D(A))} < \rho$ we obtain by Lemma 11 that

$$\|\Psi(w)\|_{C^\alpha_\gamma((0,T];D(A))} \leq K\left(\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C^\alpha_\gamma((0,T];X^\vartheta)} + \frac{9L\rho}{2}\rho\right). \tag{42}$$

The assumption (38) implies $\|\Psi(w)\|_{C^\alpha_\gamma((0,T];D(A))} < \rho$, and therefore $\Psi(\mathcal{Y}_\rho) \subset \mathcal{Y}_\rho$.

Step 2: Ψ is a contraction on $C^\alpha_\gamma((0,T];D(A))$.

We need to prove that if $w_1, w_2 \in C^\alpha_\gamma((0,T];D(A))$ with $\|w_i\|_{C^\alpha_\gamma((0,T];D(A))} < \rho$, $i = 1,2$, then

$$\|\Psi(w_2) - \Psi(w_1)\|_{C^\alpha_\gamma((0,T];D(A))} \leq c\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}$$

for certain constant $0 < c < 1$. In fact, let $w_j \in \mathcal{Y}_\rho$ and $h_j = \Psi(w_j)$, be the solutions of

$$h'_j(t) = \partial^{1-\beta} A h_j(t) + G(t, w_j(t)) + \mathcal{R}(t), \quad e(0) = e_0, \quad 0 \leq t \leq T,$$

for $j = 1,2$, respectively. Then, for $h(t) := h_2(t) - h_1(t)$, and applying (11), we have

$$\Psi(w_2(t)) - \Psi(w_1(t)) = \int_0^t \mathcal{S}_\beta(t-s)\big(G(t,w_2(s)) - G(t,w_1(s))\big)\,\mathrm{d}s, \quad 0 \leq t \leq T,$$

or in other words, $h(t)$ is the solution of a linear equation (10) with $v_0 = 0$, and $f(t) = G(t,w_2(t)) - G(t,w_1(t))$. Hence, by Theorem 9, there holds

$$\|\Psi(w_2) - \Psi(w_1)\|_{C^\alpha_\gamma((0,T];D(A))} \leq K\|G(\cdot,w_2(\cdot)) - G(\cdot,w_1(\cdot))\|_{C^\alpha_\gamma((0,T];X^\vartheta)}. \tag{43}$$

Now, we will estimate $\|G(\cdot,w_2(\cdot)) - G(\cdot,w_1(\cdot))\|_{C^\alpha_\gamma((0,T];X^\vartheta)}$. From the definition of function G we have

$$\|G(t,w_2(t)) - G(t,w_1(t))\|_\vartheta = \|F(\mathcal{U}(t) - w_2(t)) - F(\mathcal{U}(t) - w_1(t))\|_\vartheta,$$

and by Hypotheses (H1)–(H4)

$$\begin{aligned} &\|F(\mathcal{U}(t) - w_2(t)) - F(\mathcal{U}(t) - w_1(t))\|_\vartheta \\ &= \left\|\int_0^1 F_u(\mathcal{U}(t) - \tau w_2(t) - (1-\tau)w_1(t))\,\mathrm{d}\tau(w_2(t) - w_1(t))\right\|_\vartheta \\ &\leq \int_0^1 \|F_u(\mathcal{U}(t) - \tau w_2(t) - (1-\tau)w_1(t)) - F_u(u_0)\|_{\mathcal{L}(D(A),X^\vartheta)}\,\mathrm{d}\tau\|w_2(t) - w_1(t)\|_{D(A)} \\ &\leq L\int_0^1 \|(\mathcal{U}(t) - u_0) - \tau w_2(t) - (1-\tau)w_1(t)\|_{D(A)}\,\mathrm{d}\tau\|w_2(t) - w_1(t)\|_{D(A)} \\ &\leq L\left[\|\mathcal{U} - u_0\|_{C^\alpha_\gamma((0,T];D(A))} + \int_0^1 \tau\|w_2\|_{C^\alpha_\gamma((0,T];D(A))}\,\mathrm{d}\tau + \int_0^1 (1-\tau)\|w_1\|_{C^\alpha_\gamma((0,T];D(A))}\,\mathrm{d}\tau\right]\cdot \\ &\quad \|w_2(t) - w_1(t)\|_{D(A)} \\ &\leq L\left[\rho + \frac{\rho}{2} + \frac{\rho}{2}\right]\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))} \\ &\leq 2L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}. \end{aligned}$$

We conclude that

$$\sup_{0<t\leq T} \|G(t,w_2(t)) - G(t,w_1(t))\|_\vartheta \leq 2L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}. \tag{44}$$

On the other hand,

$$\begin{aligned}
&(G(t,w_2(t)) - G(t,w_1(t))) - (G(s,w_2(s)) - G(s,w_1(s))) \\
&= \quad [(F(\mathcal{U}(t)) - F(\mathcal{U}(t) - w_2(t))) - (F(\mathcal{U}(t)) - F(\mathcal{U}(t) - w_1(t)))] \\
&\quad - [(F(\mathcal{U}(s)) - F(\mathcal{U}(s) - w_2(s))) - (F(\mathcal{U}(s)) - F(\mathcal{U}(s) - w_1(s)))] \\
&= \quad \int_0^1 [F_u(\mathcal{U}(t) - (1-\tau)w_2(t)) - F_u(\mathcal{U}(t) - (1-\tau)w_1(t))]\, \mathrm{d}\tau (w_2(t) - w_1(t)) \\
&\quad - \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))]\, \mathrm{d}\tau (w_2(s) - w_1(s)) \\
&= \quad \int_0^1 [F_u(\mathcal{U}(t) - (1-\tau)w_2(t)) - F_u(\mathcal{U}(t) - (1-\tau)w_1(t))]\, \mathrm{d}\tau (w_2(t) - w_1(t)) \\
&\quad + \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))]\, \mathrm{d}\tau (w_2(t) - w_1(t)) \\
&\quad - \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))]\, \mathrm{d}\tau (w_2(t) - w_1(t)) \\
&\quad - \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))]\, \mathrm{d}\tau (w_2(s) - w_1(s)) \\
&:= \quad I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We first estimate $\|I_2 + I_4\|_\vartheta$. We notice that

$$I_2 + I_4 \quad = \quad \int_0^1 [F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))]\, \mathrm{d}\tau [(w_2 - w_1)(t) - (w_2 - w_1)(s)],$$

which implies

$$\begin{aligned}
\|I_2 + I_4\|_\vartheta \quad &\leq \quad \int_0^1 \|F_u(\mathcal{U}(s) - (1-\tau)w_2(s)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))\|_{\mathcal{L}(D(A),X^\vartheta)}\, \mathrm{d}\tau \cdot \\
&\qquad \|(w_2 - w_1)(t) - (w_2 - w_1)(s)\|_{D(A)} \\
&\leq \quad L\int_0^1 (1-\tau)(\|w_1(s)\|_{D(A)} + \|w_2(s)\|_{D(A)})\, \mathrm{d}\tau \|(w_2 - w_1)(t) - (w_2 - w_1)(s)\|_{D(A)} \\
&\leq \quad L\int_0^1 (1-\tau)(\|w_1\|_{C^\alpha_\gamma((0,T];D(A))} + \|w_2\|_{C^\alpha_\gamma((0,T];D(A))})\, \mathrm{d}\tau \|(w_2 - w_1)(t) - (w_2 - w_1)(s)\|_{D(A)} \\
&\leq \quad L\rho\|(w_2 - w_1)(t) - (w_2 - w_1)(s)\|_{D(A)}.
\end{aligned}$$

Hence,

$$\frac{s^\gamma \|I_2 + I_4\|_\vartheta}{(t-s)^\alpha} \quad \leq \quad L\rho \frac{s^\gamma \|(w_2 - w_1)(t) - (w_2 - w_1)(s)\|_{D(A)}}{(t-s)^\alpha} \leq L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}.$$

The estimate of the norm of $I_1 + I_3$ follows from Hypotheses (H1)–(H4):

$$\begin{aligned}
\|I_1 + I_3\|_\vartheta &\leq \Bigg[\int_0^1 \|F_u(\mathcal{U}(t) - (1-\tau)w_2(t)) - F_u(\mathcal{U}(s) - (1-\tau)w_2(s))\|_{\mathcal{L}(D(A),X^\vartheta)}\, \mathrm{d}\tau \\
&\quad + \int_0^1 \|F_u(\mathcal{U}(t) - (1-\tau)w_1(t)) - F_u(\mathcal{U}(s) - (1-\tau)w_1(s))\|_{\mathcal{L}(D(A),X^\vartheta)}\, \mathrm{d}\tau \Bigg] \cdot \\
&\quad \|w_2(t) - w_1(t)\|_{D(A)} \\
&\leq L\Bigg[\int_0^1 \|\mathcal{U}(t) - \mathcal{U}(s)\|_{D(A)} + (1-\tau)\|w_2(t) - w_2(s)\|_{D(A)}\, \mathrm{d}\tau \\
&\quad + \int_0^1 \|\mathcal{U}(t) - \mathcal{U}(s)\|_{D(A)} + (1-\tau)\|w_1(t) - w_1(s)\|_{D(A)}\, \mathrm{d}\tau \Bigg] \|w_2(t) - w_1(t)\|_{D(A)} \\
&\leq L\left[2\|\mathcal{U}(t) - \mathcal{U}(s)\|_{D(A)} + \frac{1}{2}\|w_2(t) - w_2(s)\|_{D(A)} + \frac{1}{2}\|w_1(t) - w_1(s)\|_{D(A)} \right] \cdot \\
&\quad \|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}.
\end{aligned}$$

From (36) we have

$$\begin{aligned}
\frac{s^\gamma \|I_1 + I_3\|_\vartheta}{(t-s)^\alpha} &\leq L\left[2\rho + \frac{1}{2}\frac{s^\gamma\|w_2(t) - w_2(s)\|_{D(A)}}{(t-s)^\alpha} + \frac{1}{2}\frac{s^\gamma\|w_1(t) - w_1(s)\|_{D(A)}}{(t-s)^\alpha} \right] \|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))} \\
&\leq L\left[2\rho + \frac{1}{2}(\|w_1\|_{C^\alpha_\gamma((0,T];D(A))} + \|w_2\|_{C^\alpha_\gamma((0,T];D(A))}) \right] \|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))} \\
&\leq 3L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}.
\end{aligned}$$

Therefore,

$$\sup_{0\leq s<t\leq T} \frac{s^\gamma \| (G(t,w_2(t)) - G(t,w_1(t))) - (G(s,w_2(s)) - G(s,w_1(s))) \|_\vartheta}{(t-s)^\alpha} \leq 4L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}. \tag{45}$$

We conclude from (44) and (45) that

$$\|G(\cdot, w_2(\cdot)) - G(\cdot, w_1(\cdot))\|_{C^\alpha_\gamma((0,T];X^\vartheta)} \leq 6L\rho\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))}. \tag{46}$$

From (43) we obtain

$$\|\Psi(w_2) - \Psi(w_1)\|_{C^\alpha_\gamma((0,T];D(A))} < 6L\rho K\|w_2 - w_1\|_{C^\alpha_\gamma((0,T];D(A))},$$

and since $6L\rho K < 1$ as assumed in (37), we have that Ψ is a contraction

Therefore, Ψ has a unique fixed point $e \in \mathcal{Y}_\rho$, that is, $e \in C^\alpha_\gamma((0,T];D(A))$ with $\|e\|_{C^\alpha_\gamma((0,T];D(A))} < \rho$ and $\Psi(e) = e$. Moreover, by (37) and (42), and since $1 - \dfrac{9KL\rho}{2} > 0$, we have that

$$\|e\|_{C^\alpha_\gamma((0,T];D(A))} \leq C\left(\|e_0\|_{1+\varepsilon} + \|\mathcal{R}\|_{C^\alpha_\gamma((0,T];X^\vartheta)} \right),$$

where $C := \dfrac{K}{1 - \frac{9KL\rho}{2}}$ stands for the computable constant predicted in the statement of the theorem, and which concludes the proof.

As a final remark in this work notice that our estimates take into account not only the computable residual $\mathcal{R}$ but also the contribution of the initial error e_0. This is meaningful in the context of partial differential equations where the exact evaluation of the initial data u_0 is often unachievable, in other words U_0 does not always coincide exactly with u_0. Unfortunately the contribution of the initial error in the final estimate forces us to demand certain regularity to e_0.

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