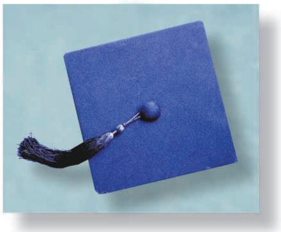

Educator's Corner

The Basics of Transmission Line Theory in Four Arrows

■ Julio Sánchez-Curto

Transmission line (TL) theory can be considered a simplified and consistent framework within general electromagnetic theory, so the governing equations of the line and their solutions can be viewed as more tractable and understandable versions of their corresponding electromagnetic counterparts [1]. This is particularly appealing for undergraduate students in electrical engineering since the vector nature of the field vanishes, and the propagation phenomenon in the line can be understood in terms of forward and backward scalar voltage waves. The presentation of the Smith chart [2], [3] within this context gives a boost to this initial attraction since the chart beautifully replaces all complex calculations involved in TL analysis with mere geometrical considerations.

Any simplification process, however, usually veils the theory that underlies it. This is precisely what, in my opinion, usually happens whenever an undergraduate student tries to match or relate the theoretical side of the problem, summarized in the extent formulae of TL theory, to its practical side, namely, the chart-based geometrical solution of electromagnetic problems [4]. The student does not perceive such relationships, leaving the theory apart, because doing so is considered useless when compared to the powerful chart. After years of lecturing on this subject, we have turned to very simple linear algebra ideas to fill this gap in such a way that the everlasting search for simplicity demanded by students and the teacher's aim to transmit the profound meaning of a subject can be simultaneously satisfied.

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An Introductory Example

The evolution of a pendulum in a lossless medium for low oscillations and the voltage in a lossless TL in the microwave regime are ruled by the same ordinary differential equation. While $\ddot{x}(t)+x(t)=0$ governs, in normalized units, the former, the latter is described by $\ddot{v}(z)+\beta^2 v(z)=0$, where β is the constant phase [5], [6], [7]. The role of time t and displacement $x(t)$ from the equilibrium position [see Figure 1(a)] is replaced by the line length z and voltage $v(z)$ so that all results and properties, such as periodicity and phase difference between the solutions, can be extrapolated from one scenario to the other [8]. This is particularly useful when it comes to teaching, where the simplicity of the pendulum example facilitates the comprehension of the propagation phenomenon in the lossless TL. This occurs, for instance, when one analyses a particular solution of the pendulum, which, in its easiest form, is $x(t)=\sin(t)$ and $s(t)=\dot{x}(t)=\cos(t)$, where $s(t)$ is the pendulum speed.

Taking into account Euler's formula, the solution can be expressed in matrix form as a linear combination of two complex exponentials:

$$\begin{bmatrix} x(t) \\ s(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{jt} \\ e^{-jt} \end{bmatrix}. \tag{1}$$

They represent two turns with the same angular velocity but in opposite directions. As Figure 1(b) and (c) show, clockwise (blue) and counterclockwise (red) turns are combined so that their sum for each t gives the displacement and speed, respectively.

Although (1) may merely seem an alternative way of representing the solution, such a decomposition, however, lies at the core of the complex analysis of TLs in the microwave regime. In fact, the complex exponentials of (1) resemble the incident and reflected voltage waves that, propagating in opposite directions, play an essential role in the complex analysis of lossless TLs. In the rigorous search for a result like (1), one must turn to linear algebra, as described in the next section.

The Linear Algebra Approach

The telegrapher equations [9], [10], in their simplest form, describe the evolution of voltages and currents in lossless TLs under sinusoidal steady state

$$\begin{bmatrix} \dot{v}(z) \\ \dot{i}(z) \end{bmatrix} = \begin{bmatrix} 0 & -j\omega L \\ -j\omega C & 0 \end{bmatrix}\begin{bmatrix} v(z) \\ i(z) \end{bmatrix} = A\begin{bmatrix} v(z) \\ i(z) \end{bmatrix} \tag{2}$$

where $\dot{v}(z)$ and $\dot{i}(z)$ denote the derivatives of the voltage and current phasors along the line z. In (2), ω is the angular frequency; L and C are the inductance and capacitance per unit length, respectively; and A is a 2×2 complex matrix that gives the evolution of $v(z)$ and $i(z)$ in the TL [11], [12]. Equation (2) is usually transformed into a Helmholtz-type wave equation [5], [6], [7] whose solutions perfectly describe the propagation phenomenon inside the TL. We instead preserve the matrix form analysis [11], [12] because it is not these well-known solutions but, rather, the decomposition of A that really matters here.

A little linear algebra shows that A decomposes as $AP = PD$ [13], where

$$D = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ -Z_0^{-1} & Z_0^{-1} \end{bmatrix} \text{ and } P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -Z_0 \\ 1 & Z_0 \end{bmatrix}. \tag{3}$$

In (3), $\gamma = j\beta = j\omega\sqrt{LC}$ and $Z_0 = \sqrt{LC^{-1}}$ are the propagation constant and line impedance, respectively. While $\pm\gamma$ represents the eigenvalues, the columns of P are the eigenvectors of the system. For this problem, however, we define $M = P^{-1}$ so that $A = M^{-1}DM$, and one can get the commutative diagram presented in Figure 2, where $\mathbb{C}$ denotes the set of complex numbers.

Figure 2 suggests that, instead of solving the TL in terms of voltage and current (through A), the process can be carried out in terms of D, where the eigenfunctions of the system are the incident $v^{+}(z)$ and reflected $v^{-}(z)$ voltages in the line. Following the commutative diagram of Figure 2, one can express not only the general solution of the lossless TL,

$$\begin{bmatrix} v(z) \\ i(z) \end{bmatrix} = M^{-1}\begin{bmatrix} v^{-}(z) \\ v^{+}(z) \end{bmatrix} = M^{-1}e^{Dz}\begin{bmatrix} v^{-}(0) \\ v^{+}(0) \end{bmatrix} = \begin{bmatrix} e^{\gamma}z & e^{-\gamma z} \\ -Z_0^{-1}e^{\gamma z} & Z_0^{-1}e^{-\gamma z} \end{bmatrix}\begin{bmatrix} v^{-}(0) \\ v^{+}(0) \end{bmatrix} \tag{4}$$

but also its ABCD parameters,

$$\begin{bmatrix} v(z) \\ i(z) \end{bmatrix} = e^{Az}\begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = M^{-1}e^{Dz}M\begin{bmatrix} v(0) \\ i(0) \end{bmatrix} = \begin{bmatrix} \cos(\beta z) & -jZ_0\sin(\beta z) \\ -jZ_0^{-1}\sin(\beta z) & \cos(\beta z) \end{bmatrix} \times \begin{bmatrix} v(0) \\ i(0) \end{bmatrix} \tag{5}$$

as a linear combination of the matrixes presented in (3).

Example: The Boundary Conditions

Figure 2 also provides any student a systematic way of calculating the particular solution of (4) when the boundary conditions at the load and generator are imposed. Since they are expressed in terms of Ohm's law relating voltages and currents, they must be transformed into the eigenfunctions domain through M, as the vertical arrows in Figure 3 demonstrate. Figure 3(a) represents a lossless TL loaded with an antenna whose impedance at a given frequency is Z_L so that the boundary condition at $z = 0$ reads $v(0) = Z_L i(0)$. Following Figure 2, one gets

$$\begin{bmatrix} v^{-}(z) \\ v^{+}(z) \end{bmatrix} = M\begin{bmatrix} v(z) \\ i(z) \end{bmatrix} \Rightarrow \begin{bmatrix} v^{-}(0) \\ v^{+}(0) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} Z_L i(0) \\ i(0) \end{bmatrix} \Rightarrow \rho_L = \frac{v^{-}(0)}{v^{+}(0)} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{6}$$

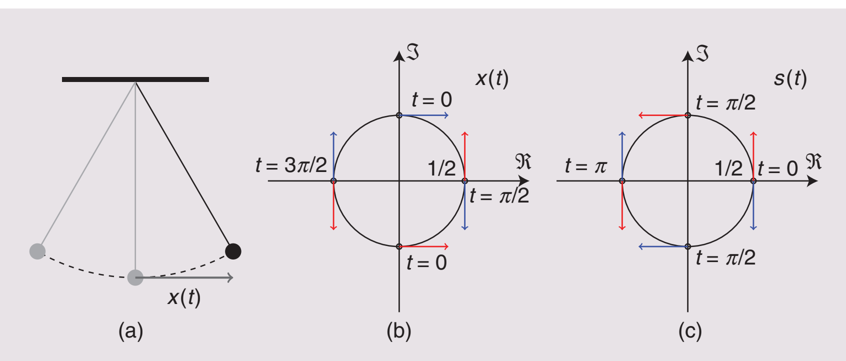

Figure 1. *(a) The pendulum for low oscillations. The time evolution of the complex exponentials to give (b) x(t) and (c) s(t).*

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "A"] \arrow[d, "M"'] & \mathbb{C}^2 \\ \mathbb{C}^2 \arrow[r, "D"'] & \mathbb{C}^2 \arrow[u, "M^{-1}"'] \end{tikzcd}$$

Figure 2. *The commutative diagram for solving (2). The colors are consistent with those of Figure 4.*

where the reflection coefficient at the load ρ_L represents the boundary condition expressed in terms of the eigenfunctions.

This procedure is also valid for the boundary condition at the generator, as Figure 3(b) illustrates. A power generator oscillating at ω with an internal impedance Z_g and providing a sinusoidal signal of amplitude V_g is connected to the line so that the boundary condition at the line input $z = -l$ reads $V_g = i(-l)Z_g + v(-l)$. The incident voltage wave $v^+(-l)$ can be calculated following the same steps:

$$\begin{bmatrix} v^-(-l) \\ v^+(-l) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -Z_0 \\ 1 & Z_0 \end{bmatrix}\begin{bmatrix} V_g - Z_g i(-l) \\ i(-l) \end{bmatrix}$$
$$\Rightarrow v^+(-l) = \frac{(1-\rho_g)V_g/2}{1-\rho_g\rho(-l)},$$
$$\rho_g = \frac{Z_g - Z_0}{Z_g + Z_0} \tag{7}$$

so (4) is fully characterized.

TL Formulae in Four Arrows

Leaving aside the mathematical side of the problem, we turn to Figure 2 because it resembles the three-step procedure of Figure 4(a) that illustrates how the calculation of the input impedance Z_{in} of a loaded TL is not carried out in terms of impedances but in terms of the reflection coefficient. The well-known geometrical solution to this problem is illustrated in Figure 4(b), where colors have been chosen to emphasize the relationship between both figures. While the vertical arrows of Figure 2 map voltages and currents into the eigenfunctions and vice versa, their corresponding counterparts in Figure 4(a) map their ratios, i.e., impedances into reflection coefficients and vice versa.

The relationship between both mappings can be formally established when one takes into account the homeomorphism of the linear group of invertible 2×2 complex matrixes into the set of Möbius transformations [14], [15]. In other terms, one can associate an invertible 2×2 complex matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{8}$$

onto the Möbius transformation

$$\frac{ax+b}{cx+d} \tag{9}$$

where x is a complex variable and $ad - bc \neq 0$. Since each matrix in (3) is nonsingular, we get, by mere inspection, the result shown in the right column of Table 1.

We have preserved the conventional notation for impedances and reflection coefficients (Z and ρ) in the right column of Table 1 so that any student can recognize, at first sight, the formulae involved in TL complex analysis. While (11) and (13) account for the generalized reflection coefficient and impedance, respectively, (10) and its explicit inverse, (12), are the transformations that originate the Smith chart. Each equation at the right column of Table 1 is fully equivalent to its corresponding matrix, so each colored arrow or point in Figures 2, 4(a), and 4(b) can now be associated to a complex function. The TL analysis formulae and the chart-based solution of Figure 4(b) are simply two sides of the

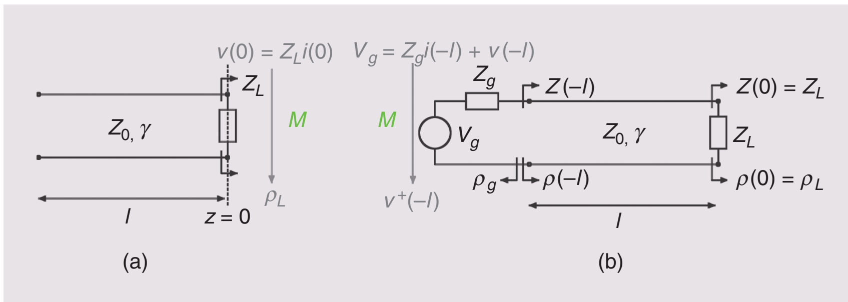

Figure 3. *The boundary conditions at the (a) load and (b) generator.*

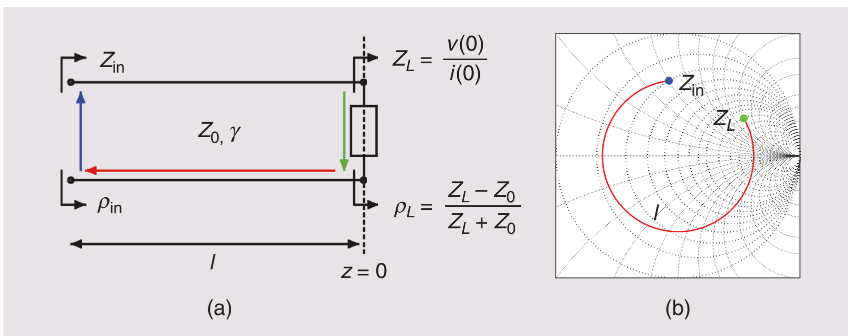

Figure 4. *The (a) calculation of Z_{in} based on the reflection coefficient and (b) its geometrical solution.*

TABLE 1. The matrixes resulting from the diagonalization of A (left column) and their corresponding functions in the complex analysis of TLs (right column).

Matrixes Involved in the Solution of (2)	Complex Functions Involved in the Calculation of Z_{in} in Figure 4	
$M = \frac{1}{2}\begin{bmatrix} 1 & -Z_0 \\ 1 & Z_0 \end{bmatrix}$	$\frac{Z - Z_0}{Z + Z_0}$	(10)
$e^{Dz} = \begin{bmatrix} e^{j\beta z} & 0 \\ 0 & e^{-j\beta z} \end{bmatrix}$	$e^{j2\beta z}\rho$	(11)
$M^{-1} = \begin{bmatrix} 1 & 1 \\ -Z_0^{-1} & Z_0^{-1} \end{bmatrix}$	$Z_0 \frac{1+\rho}{1-\rho}$	(12)
$M^{-1}e^{Dz}M = \begin{bmatrix} \cos(\beta z) & -jZ_0\sin(\beta z) \\ -jZ_0^{-1}\sin(\beta z) & \cos(\beta z) \end{bmatrix}$	$Z_0 \frac{Z\cos(\beta z) - jZ_0\sin(\beta z)}{Z_0\cos(\beta z) - jZ\sin(\beta z)}$	(13)

same coin, which constitutes the main idea of this work.

Conclusions

We have presented a complementary approach to the complex analysis of TLs, based on the diagonalization of the 2×2 complex matrix inherent to the telegrapher equations. The analysis is summarized in the four arrows of a commutative diagram that allows any undergraduate student to identify each complex function of TL analysis with its equivalent action on the Smith chart, thus bringing together the theoretical and practical sides of the problem. This work can also be applied to an analogous scenario, which is usually taught in any course in electrical engineering, i.e., the normal incidence of plane waves at planar interfaces separating two homogeneous media.

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