JORDAN ALGEBRAS AT JORDAN ELEMENTS OF SEMIPRIME RINGS WITH INVOLUTION

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Abstract. In this paper we determine the Jordan algebras associated to ad-nilpotent elements of index at most 3 in Lie algebras R^- and $\mathrm{Skew}(R,*)$ for semiprime rings R without or with involution $*$. To do so we first characterize these ad-nilpotent elements.

Key words: Lie algebra, ring with involution, ad-nilpotent element, local algebra, skew-symmetric elements.

2010 Mathematics Subject Classification: 17C50, 17B05, 17C10, 17B65.

1. Introduction

Local algebras at elements were introduced by Meyberg in a nonassociative context [29], and they have also proved to be a very useful tool in the setting of associative systems, see [14]. In the Jordan setting they were used by Zelmanov as a minor part of his brilliant classification of Jordan systems [33, 34, 35], and were revisited by D'Amour and McCrimmon in [10]. They have played a prominent role in the structure theory of Jordan systems, mainly due to the fact that nice properties flow between the system and its local algebras. Thus, D'Amour and McCrimmon extended a substantial part of Zelmanov's results to arbitrary quadratic Jordan systems by making use of local algebras [11]. On the other hand, ad-nilpotent elements of index at most 3 (here called Jordan elements) play a fundamental role in the proof of Kostrikin's conjecture that any finite-dimensional simple nondegenerate Lie algebra (over a field of characteristic greater than 5) is classical [8, 30]. Jordan elements are also of great importance in the Lie inner ideal structure of associative rings [7].

The analogue of local algebras for Lie algebras was introduced by the second two authors and A. Fernández López in [13]. They showed that it is possible to attach a Jordan algebra to any Jordan element of a Lie algebra. Since their introduction, these Jordan algebras have proven to be very useful: they inherit good properties from the Lie algebra itself, such as nondegeneracy [13], strong primeness [16] and even local finiteness [20], so the structure theory of Jordan systems can be transferred to the original Lie algebra. For example, in [15] the authors

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The first author was supported by the MEC through the FPU grant AP2009-4848, by the Universidad de Málaga through a contrato-puente, and partially supported by the Junta de Andalucía FQM264.

The second author was partially supported by the MEC and Fondos FEDER MTM2010-16153, and by the Junta de Andalucía FQM264.

The third author was partially supported by the MEC and Fondos FEDER MTM2010-19482, and by the Junta de Andalucía FQM264.

revisited the celebrated paper of Zelmanov [32] by using Jordan algebras of Lie algebras. Moreover, Jordan elements have been used as tools in the study of infinite-dimensional Lie algebras. Indeed, they are important in the recent description of simple, infinite-dimensional, locally finite and locally nondegenerate Lie algebras with Jordan elements, see [20, Theorem 1], and A. Baranov and J. Rowley gave a characterization of infinite locally finite simple diagonal Lie algebras in terms of inner ideals, which are closely related to Jordan elements, see [4].

We focus on two types of Lie algebras coming from the associative context: R^- and $\mathrm{Skew}(R,*)$ for centrally closed semiprime rings R. For these Lie algebras, we highlight the recent works [1] and [3]. Our aim is, on the one hand, to describe Jordan elements of semiprime rings, and on the other hand, to describe the Jordan algebras of Lie algebras of the form R^- and $\mathrm{Skew}(R,*)$ for a centrally closed semiprime ring R at those Jordan elements. We characterize those Jordan algebras in terms of local algebras of the original ring (in the case of R^-) and in terms of local algebras of the symmetric Martindale ring of quotients of R for the case with involution.

The first step in this project is to associate a nilpotent element to any Jordan element. Jordan elements are directly associated to a particular case of nilpotent derivations, and this has been a topic of interest since the 1960's. In 1963, I. N. Herstein showed that any ad-nilpotent element a of index n in a simple ring R of characteristic zero or greater than n gives rise to a nilpotent element $a-\lambda$ for some λ in the center of R. Moreover, he showed that the index of nilpotency of such element is less than or equal to $\lfloor\frac{n+1}{2}\rfloor$, see [21, Theorem in page 84]. This result of Herstein was generalized by W. S. Martindale and C. R. Miers in 1983 ([25, Corollary 1]) to prime rings of characteristic greater than n. This time the nilpotent element has the form $a-\lambda$ for an element λ in the extended centroid of R. Later on, in 1992, this same result was studied by P. Grzeszczuk for the case of semiprime rings. He showed that any nilpotent derivation in a semiprime ring is an inner derivation in a semiprime subring of the right Martindale ring of quotients of R and is induced by a nilpotent element in such subring, see [19, Corollary 8].

Since we are interested in Jordan elements, our nilpotent derivations have the form ad_a with $\mathrm{ad}_a^3=0$, so all the previously mentioned results apply directly to Jordan elements of simple, prime or semiprime rings respectively. Nevertheless, for the sake of completeness, we include in this paper an alternative proof of the form of Jordan elements of R^- for semiprime rings R. On the other hand, when dealing with rings with involution $*$, apart from the Lie algebra R^- it is natural to study the Lie algebra of skew-symmetric elements $\mathrm{Skew}(R,*)$. The nilpotent derivations of the skew elements of prime rings with involution were studied in the 1990's by W. S. Martindale and C. R. Miers, who showed that if R is a prime ring with involution of characteristic zero which is not an order in a 4-dimensional central simple Lie algebra and has some inner derivation ad_a with $\mathrm{ad}_a^n=0$, then there exists an element λ in the extended centroid of R such that either $(a-\lambda)^{\lfloor\frac{n+1}{2}\rfloor}=0$ or the involution is of the first kind and $a^{\lfloor\frac{n+1}{2}\rfloor+1}=0$, see [26, Main Theorem]. As far as we know this result has not been extended to semiprime rings yet. In this paper we prove an analogue of it for Jordan elements of the skew elements of a semiprime ring, showing that either they have the form $a-\lambda$ with λ in the extended centroid and $(a-\lambda)^2=0$ or a has index of nilpotency 3 and R satisfies a generalized polynomial identity.

These results on Jordan elements make possible to classify the Jordan algebras at Jordan elements of the Lie algebras of R^- and $\mathrm{Skew}(R,*)$ type. For the R^- case we show the following result:

Lemma 5.1 *Let R be a centrally closed semiprime ring free of 2 and 3-torsion and let $a \in R$ be a Jordan element. Then there exists a' in R such that $(R^-)_a \cong (R_{a'})^+$, i.e., the Jordan algebra of the Lie algebra R^- at a is isomorphic to the symmetrization of a local algebra of the ring R.*

For rings with involution we relate the Jordan algebras of the Lie algebras of their skew elements with local algebras of their symmetric Martindale ring of quotients.

Theorem 5.8 *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R,*)$ be a Jordan element. Let $Q_m^s(R)$ be the symmetric Martindale ring of quotients of R. Then there exist two idempotents $e, f \in \mathrm{Sym}(C(R),*)$ that decompose $Q_m^s(R)$ as a sum of three orthogonal ideals $Q_m^s(R) = eQ_m^s(R) \oplus fQ_m^s(R) \oplus (1-e-f)Q_m^s(R)$, and an element $\lambda \in e\,\mathrm{Skew}(C(R),*)$ such that $a = ea + fa + (1-e-f)a \in \mathcal{K} := \mathrm{Skew}(Q_m^s(R),*)$, $\mathcal{K}_a \cong \mathcal{K}_{ea} \oplus \mathcal{K}_{(1-e-f)a}$ and*

(i) *$\mathcal{K}_{ea} \cong \mathcal{K}_{ea-e\lambda} \cong \mathrm{Sym}(Q_m^s(R)_{ea-e\lambda},*)$.*
(ii) *$\mathcal{K}_{(1-e-f)a}$ is a nondegenerate Jordan algebra of quadratic form.*
(iii) *$\mathcal{K}_{fa} = 0$.*

In this case we have to resort to $Q_m^s(R)$ to assure that the Jordan algebra of type (ii) is "complete"; for R, the associated Jordan algebras are just "forms" of these type. This is due to the element $(1-e-f)a$ being von Neumann regular in $Q_m^s(R)$ but not necessarily in R, in which all that we know is that $(1-e-f)a$ is regular when multiplied by certain central idempotents. Indeed, similar results could be proved inside the orthogonal completion of R.

In the particular case of prime rings with involution, our result on the Jordan algebras of $\mathrm{Skew}(R,*)$ at Jordan elements gives rise to the next result, which appears in [9]. Note that it is not necessary to extend R to its symmetric Martindale ring of quotients.

Corollary 5.9 *Let R be a centrally closed $*$-prime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R,*)$ be a Jordan element. Then we have one of the next mutually exclusive possibilities:*

(i) *There exists $\lambda \in \mathrm{Skew}(C(R),*)$ such that $(a-\lambda)^2 = 0$ and therefore $K_a \cong K_{a-\lambda} \cong \mathrm{Sym}(R_{a-\lambda},*)$.*
(ii) *$a^3 = 0$, $a^2 \neq 0$ and K_a is a Clifford Jordan algebra.*
(iii) *$a \in Z(K)$ but $a \notin Z(R)$ and therefore $K_a = \{0\}$.*

2. Preliminaries

2.1. Throughout this paper and at unless otherwise specified, we will be dealing with Lie algebras L, rings R and Jordan algebras J free of 2 and 3-torsion. As usual, $[x,y]$ will denote the Lie bracket of two elements x, y of L, with ad_x the adjoint map determined by x; the product of two elements x, y of R will be written by juxtaposition, xy; the Jordan product of two elements x, y of J will be denoted by $x \bullet y$, with U-operator $U_x y := 2x \bullet (x \bullet y) - x^2 \bullet y$. The reader is referred to [22, 28] for basic results, notation and terminology on Lie algebras and Jordan algebras respectively. Nevertheless, we will stress some notions and basic properties of both kinds of algebras.

Any ring R, which can be seen as an associative algebra over $\mathbb{Z}$, gives rise to:

(1) A Lie algebra R^- with Lie bracket $[x,y] := xy - yx$, for all $x, y \in R$.
(2) A subalgebra $K := \mathrm{Skew}(R,*)$ of the Lie algebra R^-, when R has an involution $*$.
(3) A Jordan algebra R^+ with Jordan product $x \bullet y := \frac{1}{2}(xy + yx)$, called the symmetrization of R.
(4) A subalgebra $\mathrm{Sym}(R,*)$ of the Jordan algebra R^+, when R has an involution $*$.

Notice that for every element $x \in R$ we can express $2x$ as a sum of the skew-symmetric element $x - x^*$ and the symmetric element $x + x^*$. Moreover, since our rings are free of 2-torsion $2x = 0$ implies $x = 0$.

2.2. Let V be a module over a ring of scalars Φ with $\frac{1}{2} \in \Phi$, let $Q : V \to \Phi$ be a quadratic form on V, i.e., a form such that $Q(\alpha v) = \alpha^2 Q(v)$, and let $Q(v,w) := Q(v+w) - Q(v) - Q(w)$ be the associated bilinear form on $V \times V$. Let $c \in V$ be such that $Q(c) = 1$. Let $T : V \to \Phi$ be defined by $T(v) := Q(c,v)$, called a trace form. Then we can define the unitary Jordan algebra $\mathfrak{Jord}(Q,c) := V$ as a Φ-module with unit $1 := c$ and product

$$x \bullet y := \frac{1}{2}(T(x)y + T(y)x - Q(x,y)1).$$

This Jordan algebra is called a Jordan algebra of quadratic form or a quadratic factor, and satisfies the second-degree equation $x^2 - T(x)x + Q(x)1 = 0$, see [28, page 75]. If Φ is a field and the quadratic form is nondegenerate, then $\mathfrak{Jord}(Q,c)$ is a Jordan algebra called the Clifford Jordan algebra associated to the quadratic form Q, which is simple whenever the dimension of V is different from 1, see [28, page 97].

2.3. A ring R is semiprime if for every nonzero ideal I of R we have $I^2 := \{\sum_i x_i y_i \mid x_i, y_i \in I\} \neq 0$, and it is prime if $IJ := \{\sum_i y_i x_i \mid y_i \in I, x_i \in J\} \neq 0$ for every pair of nonzero ideals I, J of R. It is well known that a ring R is prime if and only if $xRy \neq 0$ for arbitrary nonzero elements $x, y \in R$, and semiprime if and only if it is nondegenerate, i.e., if $xRx \neq 0$ for every nonzero element $x \in R$. Moreover, if R is a semiprime ring and $x, y \in R$ satisfy $xRy = 0$ then the ideals $\mathrm{Id}_R(x)$, $\mathrm{Id}_R(y)$ are orthogonal. We will use this property without mentioning it.

2.4. Given an ideal I of R, the annihilator of I in R is the set

$$\mathrm{Ann}_R(I) := \{z \in R \mid zI = Iz = 0\}.$$

The annihilator of an ideal I of R is an ideal of R. Moreover, if R is semiprime,

- $\mathrm{Ann}_R(I) := \{z \in R \mid zIz = 0\}$.
- $I \cap \mathrm{Ann}_R(I) = 0$.
- An ideal I of R is essential (for every nonzero ideal J of R, $I \cap J \neq 0$) if and only if $\mathrm{Ann}_R(I) = 0$.

2.5. Let a be an element in a ring R. The additive group $(R,+)$ endowed with the a-homotope product $x \cdot_a y := xay$ becomes a ring R^a called the homotope of R at a. The set $\mathrm{Ker}(a) := \{x \in R \mid axa = 0\}$ is an ideal of R^a. Meyberg's local rings are the quotients $R_a := R^a / \mathrm{Ker}(a)$ for $a \in R$ with product given by $\tilde{x} \circ \tilde{y} := xay + \mathrm{Ker}(a)$ ([29]). If R has an involution $*$ and $a \in \mathrm{Skew}(R,*)$ then R_a is also a ring with involution $*$ given by $\tilde{x}^* := \widetilde{-x^*}$.

In [14] the authors proved that R_a is isomorphic to the submodule aRa, endowed with the product $axa \centerdot aya := axaya$ via the mapping φ given by $\tilde{x} = x + \operatorname{Ker}(a) \mapsto axa$. In [18] the authors generalized this notion: Let R be a subring of a ring Q. If $a \in Q$ is such that $RaR \subset R$ then aRa can be regarded as a subring of $\varphi(Q_a)$. This ring will be called the generalized local ring of R at a, and will be denoted by R_a.

2.6. Let L be a Lie algebra over a ring of scalars Φ such that $\frac{1}{2}, \frac{1}{3} \in \Phi$. We say that an element a in L is a Jordan element if a is ad-nilpotent of index no greater than 3, i.e., if $\operatorname{ad}_a^3 = 0$. Every Jordan element gives rise to a Jordan algebra, called the Jordan algebra of L at a, see [13]: Let L be a Lie algebra and let $a \in L$ be a Jordan element. Then L with the new product $x \bullet y := \frac{1}{2}[[x,a],y]$ is an algebra such that

$$\operatorname{Ker}(a) := \{z \in L \mid [a,[a,z]] = 0\}$$

is an ideal of $(L, \bullet)$. Moreover, $L_a := (L/\operatorname{Ker}(a), \bullet)$ is a Jordan algebra. In this Jordan algebra the U-operator and the triple product have these nice expressions:

$$U_{\overline{x}}\overline{y} = \frac{1}{4}\overline{\operatorname{ad}_x^2 \operatorname{ad}_a^2 y}, \quad \text{for all } x, y \in L, \quad \text{and}$$

$$\{\overline{x}, \overline{y}, \overline{z}\} = -\frac{1}{2}\overline{[x, [\operatorname{ad}_a^2 y, z]]} \quad \text{for all } x, y, z \in L.$$

A Lie algebra is nondegenerate if and only if L_a is nonzero for every Jordan element $a \in L$. In particular, L_a inherits nondegeneracy from L [13, 2.15(i)].

In the following two paragraphs we will review the concepts of right Martindale ring of quotients and symmetric Martindale ring of quotients. The theory of rings of quotients has its origins between 1930 and 1940 in the works of O. Ore and K. Osano on the construction of the total ring of fractions. Martindale rings of quotients were introduced by W.S. Martindale in 1969 for prime rings [24]. This concept was designed for applications to rings satisfying a generalized polynomial identity (GPI for short). In 1972, A. Amitsur generalized the construction of Martindale rings of quotients to the setting of semiprime rings ([2]).

2.7. Given a ring R we define a permissible map of R as a pair (I, f) where I is an essential ideal of R and f is a monomorphism of right R-modules. For permissible maps (I, f) and (J, g) of R, define a relation $\equiv$ by $(I, f) \equiv (J, g)$ if there exists an essential ideal K of R, contained in $I \cap J$, such that $f(x) = g(x)$ for all $x \in K$. It is easy to see that this is an equivalence relation. The quotient set $Q_m^r(R)$ will be called the right Martindale ring of quotients of R. If R is a semiprime ring $Q_m^r(R)$ has a ring structure coming from the addition of homomorphisms and from the composition of restrictions of homomorphisms, see [5]:

- $[I, f] + [J, g] := [I \cap J, f + g]$,
- $[I, f] \cdot [J, g] := [(I \cap J)^2, f \circ g]$.

Note that if R is a semiprime ring then the map $f : R \to Q_m^r(R)$ defined by $f(r) := [R, \lambda_r]$, where $\lambda_r : R \to R$ is defined by $\lambda_r(x) := rx$, is a monomorphism of associative rings, i.e., R can be considered as a subring of its right Martindale ring of quotients. Moreover, given any $0 \neq q := [I, f] \in Q_m^r(R)$ we have that $0 \neq qI \subset R$. Therefore every subring S of $Q_m^r(R)$ which contains R is semiprime because every nonzero ideal of S has nonzero intersection with R.

2.8. Given a ring R, the symmetric Martindale ring of quotients of R is defined as
$Q_m^s(R) := \{q \in Q_m^r(R) |$ there exists an essential ideal I of R such that $qI + Iq \subset R\}$.
If R is semiprime then $Q_m^s(R)$, which is a subring of $Q_m^r(R)$ containing R, is a semiprime ring. Moreover, if R is a semiprime ring with involution $*$ then we can extend $*$ to $Q_m^s(R)$, and this extension is unique: given $q := [I, f] \in Q_m^s(R)$, $q^* := [U, g]$ where $U := I \cap I^*$ is an essential $*$-ideal of R such that $qU + Uq \subset R$, and $g(u) := f(u^*)^*$ for every $u \in U$.

2.9. The extended centroid $C(R)$ of a semiprime ring R is defined as the center of its symmetric Martindale ring of quotients. The extended centroid of a prime ring is a field, and the extended centroid of a semiprime ring is a commutative and unital von Neumann regular ring. In particular, if R is semiprime, $C(R)$ is a semiprime ring without nilpotent elements.

The central closure of R, denoted by $\hat{R}$, is defined (see [25]) as the subring of $Q_m^s(R)$ generated by R and $C(R)$, i.e., $\hat{R} := C(R)R + C(R)$, and can be seen as a $C(R)$-algebra. Therefore we can consider R contained in $\hat{R}$. Moreover, since $\hat{R}$ contains R and is contained in $Q_m^r(R)$, if R is semiprime then $\hat{R}$ is also semiprime. We say that R is centrally closed if it coincides with its central closure $\hat{R}$. In particular, the ring $\hat{R}$ is centrally closed with center equal to its extended centroid, i.e., $Z(\hat{R}) = C(R)$. If R is a centrally closed semiprime ring with involution then R^- is a Lie algebra over $C(R)$; if in addition R has an involution, then $\mathrm{Skew}(R, *)$ is a Lie algebra over $\mathrm{Sym}(C(R), *)$.

2.10. For a semiprime ring R, given an element $\lambda \in C(R)$ there exists a unique $\lambda' \in C(R)$ such $\lambda\lambda'\lambda = \lambda$ and $\lambda' = \lambda'\lambda\lambda'$ (indeed, if $\lambda = \lambda\lambda'\lambda = \lambda\mu\lambda$, $\lambda' = \lambda'\lambda\lambda'$ and $\mu = \mu\lambda\mu$ then $\lambda' = \lambda^2(\lambda')^3 = \lambda^2\mu^3 = \mu$). Such unique element λ' will be called the partner of λ. Let us define $e_\lambda := \lambda\lambda'$. Then e_λ is an idempotent of $C(R)$ such that $e_\lambda\lambda = \lambda$.

If R has no k-torsion for some $k \in \mathbb{N}$ then k is invertible in $C(R)$: for $k = k \cdot 1 \in C(R)$ there exists $k' \in C(R)$ such that $kk'k = k$, so $k(k'k - 1) = 0$ and $kk' = 1$, thus $k' = \frac{1}{k} \in C(R)$. In particular, when R is a semiprime ring with involution $*$ and no 2-torsion, every element $x \in \hat{R}$ can be expressed as $x = x_s + x_k$ with $x_s := \frac{1}{2}(x + x^*) \in \mathrm{Sym}(\hat{R}, *)$ and $x_k := \frac{1}{2}(x - x^*) \in \mathrm{Skew}(\hat{R}, *)$. We will use this property without mentioning it.

Moreover, if R is a semiprime ring with involution $*$ and $\lambda \in \mathrm{Skew}(C(R), *)$ then $\lambda = -\lambda^* = -(\lambda\lambda'\lambda)^* = \lambda(-\lambda')^*\lambda$ and $(-\lambda')^*\lambda(-\lambda')^* = (\lambda'\lambda^*\lambda')^* = (-\lambda')^*$, which imply by uniqueness of λ' that $\lambda' = (-\lambda')^* \in \mathrm{Skew}(C(R), *)$. In this case $e_\lambda = \lambda\lambda'$ is a symmetric idempotent of $C(R)$.

Lemma 2.11. *Let $(R, *)$ be a semiprime ring with involution free of 2-torsion and let $a \in \mathrm{Skew}(R, *)$ and $\lambda \in C(R)$ be such that $a - \lambda$ is nilpotent. Then $\lambda \in \mathrm{Skew}(C(R), *)$. Moreover, if $a - \mu$ is nilpotent with $\mu \in C(R)$, then $\lambda = \mu$.*

Proof. If $a - \lambda$ and $a - \mu$ are nilpotent elements of $\hat{R}$, since they commute, $a - \lambda - (a - \mu) = \mu - \lambda$ is a nilpotent element in the semiprime commutative ring $C(R)$. Therefore $\lambda = \mu$. Now, if $a - \lambda$ is nilpotent then $(a - \lambda)^* = -(a + \lambda^*)$ is nilpotent and therefore $a + \lambda^*$ is nilpotent, which implies that $\lambda^* = -\lambda$. $\square$

The following result is an analogue of the Lie Jacobson-Morozov lemma for the setting of rings. Its proof will appear in [9, Lemma 2.2].

Lemma 2.12. *Let R be an algebra over a ring of scalars Φ and let $a, b \in R$ be such that $a^2 = 0$ and $aba = a$. Then there exists $c \in R$ such that $aca = a$, $cac = c$ and $c^2 = 0$. Moreover, if R has an involution $*$, $\frac{1}{2} \in \Phi$ and $a \in \mathrm{Skew}(R, *)$ (respectively $a \in \mathrm{Sym}(R, *)$) then c can be taken in $\mathrm{Skew}(R, *)$ (respectively $c \in \mathrm{Sym}(R, *)$).*

Under the conditions of the previous lemma, the elements a and c will be called twins (as done in [9]).

We will use the following results due to Beidar, Martindale and Mikhalev ([6, Theorem 2.3.3, Theorem 2.3.9]).

Theorem 2.13. *For any semiprime ring R with extended centroid $C(R)$ and any $a_1, \ldots, a_n \in R$, if $a_1 \notin \sum_{i=2}^n C(R)a_i$ in $\hat{R}$ then there exist $r_j, s_j \in R$ with $j = 1, 2, \ldots, m$ such that $\sum_{j=1}^m r_j a_1 s_j \neq 0$ and $\sum_{j=1}^m r_j a_k s_j = 0$ for $k = 2, \ldots, n$.*

Corollary 2.14. *Let R be a semiprime ring with extended centroid $C(R)$. Let $a_i, b_i \in R$ for $i = 1, 2, \ldots, n$ be such that $\sum_{i=1}^n a_i x b_i = 0$ for every $x \in R$. In addition, suppose that every nonzero ideal contained in $\mathrm{Id}_R(a_1)$ has nonzero intersection with $\mathrm{Id}_R(b_1)$ (this happens in particular if $\mathrm{Id}_R(a_1) \subset \mathrm{Id}_R(b_1)$). Then there exist $\lambda_i \in C(R)$ for $i = 2, \ldots, n$ such that $a_1 = \sum_{i=2}^n \lambda_i a_i$.*

Proof. By 2.13, if $a_1 \notin \sum_{i=2}^n C(R)a_i$ there exist $r_j, s_j \in R$, $j = 1, \ldots, m$, such that $\sum_{j=1}^m r_j a_1 s_j \neq 0$ and $\sum_{j=1}^m r_j a_k s_j = 0$ for $k = 2, \ldots, n$. In the identity $\sum_{i=1}^n a_i x b_i = 0$ replace x by $s_j x$ and multiply on the left by r_j. We have

$$0 = \sum_{i=1}^n \sum_{j=1}^m r_j a_i s_j x b_i = \sum_{j=1}^m r_j a_1 s_j x b_1,$$

which implies that the ideal generated by $\sum_{j=1}^m r_j a_1 s_j$, $I := \mathrm{Id}_R(\sum_{j=1}^m r_j a_1 s_j)$, is orthogonal to the ideal generated by b_1. Therefore $I \cap \mathrm{Id}_R(b_1)$ is a nonzero ideal (by hypothesis) of zero square, a contradiction because R is semiprime. $\square$

Proposition 2.15. *Let R be a centrally closed semiprime ring free of 2-torsion with extended centroid $C(R)$. Then for any subset $V \subset R$ there exists a unique idempotent $e \in C(R)$ such that:*

(a) *$ev = v$ for all $v \in V$,*
(b) *the annihilator in $C(R)$ of V is $\mathrm{Ann}_{C(R)}(V) = (1 - e)C(R)$,*
(c) *the annihilator in R of the ideal generated by V is $\mathrm{Ann}_R(RVR) = (1-e)R$, and*
(d) *the ideal generated by V is essential in eR.*

Moreover, when R has an involution $*$ and $V \subset \mathrm{Sym}(R, *) \cup \mathrm{Skew}(R, *)$, then $e \in \mathrm{Sym}(C(R), *)$.

Proof. The first part of the proof follows as in [6, Theorem 2.3.9(i)] with the obvious changes. When R has an involution we can decompose $e = e_k + e_s$ with $e_k \in \mathrm{Skew}(C(R), *)$ and $e_s \in \mathrm{Sym}(C(R), *)$. If $V \subset \mathrm{Sym}(R, *) \cup \mathrm{Skew}(R, *)$, for every $v \in V$ we have that $ev = v$ implies $e_k v = 0$ because if $v \in \mathrm{Sym}(R, *)$ then $e_k v \in \mathrm{Skew}(R, *)$, while if $v \in \mathrm{Skew}(R, *)$ then $e_k v \in \mathrm{Sym}(R, *)$. Therefore $e_k \in \mathrm{Ann}_{C(R)}(V) = (1 - e)C(R)$, i.e., $e_k e = 0$ and $e_k^2 = e_k e_s = 0$, and therefore $e = e^2 = (e_k + e_s)^2 = e_s^2 \in \mathrm{Sym}(C(R), *)$. $\square$

3. Jordan elements of R^-

The first part of the following lemma appears in [12, Proposition 3.6(1)]. We include it here for the sake of completeness.

Lemma 3.1. *Let R be a semiprime ring and let $a \in R$ be such that $\operatorname{ad}_a^3(R) \subset Z(R)$. Then $\operatorname{ad}_a^3(R) = 0$. Moreover, if R is free of 3-torsion, then $(\operatorname{ad}_a^2 x)(\operatorname{ad}_a^2 y) = 0$ for every $x, y \in R$.*

Proof. For every $x \in R$ we have by the Leibniz Rule that

$$0 = [\operatorname{ad}_a^3(xa), x] = [\operatorname{ad}_a^3(x)a, x] = \operatorname{ad}_a^3(x)[a, x],$$

Therefore $0 = \operatorname{ad}_a^2(\operatorname{ad}_a^3(x)[a, x]) = (\operatorname{ad}_a^3(x))^2$, which implies, since R is semiprime and $\operatorname{ad}_a^3(x) \in Z(R)$, that $\operatorname{ad}_a^3(x) = 0$. Now,

$$0 = \operatorname{ad}_a^3(x \operatorname{ad}_a(y)) = \sum_{i=0}^{3} \binom{3}{i} \operatorname{ad}_a^i(x) \operatorname{ad}_a^{4-i}(y) = 3 \operatorname{ad}_a^2(x) \operatorname{ad}_a^2(y),$$

so $(\operatorname{ad}_a^2 x)(\operatorname{ad}_a^2 y) = 0$ for every $y \in R$. □

Let R be a ring and let $a \in R$ be such that there exists $\lambda \in C(R)$ with $(a-\lambda)^2 = 0$. Then it is easy to see that a is a Jordan element of R: if $b := a - \lambda$, for all $x \in R$ we have

$$\operatorname{ad}_a^3(x) = \operatorname{ad}_{a-\lambda}^3(x) = \operatorname{ad}_b^3(x) = b^3x - 3b^2xb + 3bxb^2 - xb^3 = 0 \tag{1}$$

where the calculations are done in the central closure $\hat{R}$ of R.

The converse was proved in [21, Theorem in page 84] and [7, Theorem 3.2] (in a more general form) for simple rings, and extended later by Martindale and Miers [25, Corollary 1] to prime rings and by Grzeszczuk [19, Corollary 8] to semiprime rings. We include here a simpler proof for the semiprime case.

Theorem 3.2. *Let R be a semiprime ring free of 2,3-torsion, and let $a \in R$ be a Jordan element of R^-. Then there exists $\lambda \in C(R)$, the extended centroid of R, such that $(a - \lambda)^2 = 0$ in the central closure $\hat{R}$ of R.*

Proof. By replacing R by $\hat{R}$ we may assume, without loss of generality, that R is centrally closed. By 3.1 for every $x, y \in R$ we have that $(\operatorname{ad}_a^2 x)(\operatorname{ad}_a^2 y) = 0$, i.e.,

$$0 = (\operatorname{ad}_a^2 x)(\operatorname{ad}_a^2 y) = a^2x(\operatorname{ad}_a^2 y) + ax(-2a \operatorname{ad}_a^2 y) + x(a^2 \operatorname{ad}_a^2 y). \tag{2}$$

We claim that $a^2 = \mu a + \tau$ for some $\mu, \tau \in C(R)$. Otherwise, by 2.13 applied to $a^2, a, 1 \in R$ there would exist $r_j, s_j \in R$, $j = 1, \ldots, m$, such that $\sum_{j=1}^m r_j a^2 s_j \neq 0$ and $\sum_{j=1}^m r_j a^k s_j = 0$ for $k = 0, 1$. In formula (2), for each $j = 1, \ldots, m$ replace x by $s_j x$ and multiply on the left by r_j. We obtain

$$0 = \sum_{j=1}^{m} r_j \left(a^2 s_j x(\operatorname{ad}_a^2 y) + a s_j x(-2a \operatorname{ad}_a^2 y) + s_j x(a^2 \operatorname{ad}_a^2 y)\right) = \sum_{j=1}^{m} (r_j a^2 s_j) x(\operatorname{ad}_a^2 y)$$

which says that, for each $y \in R$,

$$\operatorname{ad}_a^2 y = a^2y + ya^2 - 2aya \in \operatorname{Ann}_R(I)$$

where I denotes the ideal generated by $\sum_{j=1}^m r_j a^2 s_j$. For each $j = 1, \dots, m$ replace y by s_j and multiply on the left by r_j. Then by semiprimeness of R we obtain

$$\sum_{j=1}^m r_j a^2 s_j = \sum_{j=1}^m r_j(a^2 s_j + s_j a^2 - 2as_j a) \in I \cap \operatorname{Ann}_R(I) = 0,$$

which is a contradiction. Therefore, there exist $\mu, \tau \in C(R)$ such that $a^2 = \mu a + \tau$.

Substituting $a^2 = \mu a + \tau$ in the expansion of $\operatorname{ad}_a^3(x) = 0$, since $a^3 = (\mu^2 + \tau)a + \mu\tau$, we obtain

$$\begin{aligned} 0 = \operatorname{ad}_a^3(x) &= a^3x - 3a^2xa + 3axa^2 - xa^3 = ((\mu^2+\tau)a + \mu\tau)x - 3(\mu a + \tau)xa \\ &+ 3ax(\mu a + \tau) - x((\mu^2+\tau)a + \mu\tau) = (\mu^2 + 4\tau)ax - (\mu^2+4\tau)xa \\ &= [(\mu^2 + 4\tau)a, x] \end{aligned}$$

for every $x \in R$, which proves that $(\mu^2 + 4\tau)a$ is a central element of R, i.e.,

$$(\mu^2 + 4\tau)a \in C(R).$$

Set $\alpha := \mu^2 + 4\tau$. Now apply 2.10 to get its partner α' and the central idempotent e_α. Recall that $e_\alpha \alpha = \alpha$. Define the idempotent $f_\alpha := 1 - e_\alpha$ and the element $\lambda := e_\alpha a + \frac{1}{2} f_\alpha \mu$, which is central since $e_\alpha a = \alpha'(\alpha a)$ and $\alpha a \in C(R)$. Then

$$\begin{aligned} (a-\lambda)^2 &= (a - e_\alpha a - \frac{1}{2} f_\alpha \mu)^2 = (f_\alpha a - \frac{1}{2} f_\alpha \mu)^2 = f_\alpha(a^2 - \mu a + \frac{1}{4}\mu^2) \\ &= f_\alpha(\mu a + \tau - \mu a + \frac{1}{4}\mu^2) = f_\alpha(\tau + \frac{1}{4}\mu^2) = \frac{1}{4} f_\alpha \alpha = 0. \end{aligned}$$ □

4. Jordan elements of $\operatorname{Skew}(R, *)$

Proposition 4.1. *Let R be a ring with involution $*$ free of 2-torsion and let $K := \operatorname{Skew}(R, *)$. Let $a \in K$ be a Jordan element of K. Then, for every $x, y \in K$ and $\lambda \in \operatorname{Skew}(Z(R), *)$ we have:*

(a) *a^2 is a Jordan element of R, i.e., $\operatorname{ad}_{a^2}^3(R) = 0$.*
(b) *λa is a Jordan element of R, i.e., $\operatorname{ad}_{\lambda a}^3(R) = 0$.*
(c) *If in addition R is free of 3-torsion, then $\operatorname{ad}_a^2(x)\operatorname{ad}_a^2(y)\operatorname{ad}_a^2(x) = 0$.*

Proof. (a) Let $x \in R$. Then $2x = x_k + x_s$ where $x_k := x - x^* \in K$ and $x_s := x + x^* \in \operatorname{Sym}(R, *)$. Now, since $\operatorname{ad}_a^3(x_s)a = \operatorname{ad}_a^3(x_s a)$ and $a\operatorname{ad}_a^3(x_s) = \operatorname{ad}_a^3(ax_s)$, we have

$$\begin{aligned} 2\operatorname{ad}_{a^2}^3(x) = \operatorname{ad}_{a^2}^3(2x) &= \sum_{i=0}^3 \binom{3}{i} a^i \operatorname{ad}_a^3(x_k + x_s)a^{3-i} = \sum_{i=0}^3 \binom{3}{i} a^i \operatorname{ad}_a^3(x_k)a^{3-i} + \\ &+ \sum_{i=0}^3 \binom{3}{i} a^i \operatorname{ad}_a^3(x_s)a^{3-i} = 0 + \sum_{i=0}^3 \binom{3}{i} a^i \operatorname{ad}_a^3(x_s)a^{3-i} \\ &= \operatorname{ad}_a^3(x_s)a^3 + 3a\operatorname{ad}_a^3(x_s)a^2 + 3a^2\operatorname{ad}_a^3(x_s)a + a^3\operatorname{ad}_a^3(x_s) \\ &= \operatorname{ad}_a^3(x_s a + ax_s)a^2 + 2a\operatorname{ad}_a^3(x_s a + ax_s)a + a^2\operatorname{ad}_a^3(x_s a + ax_s) = 0 \end{aligned}$$

since $x_s a + ax_s \in K$, so $\operatorname{ad}_{a^2}^3(R) = 0$.

(b) We have

$$2\operatorname{ad}_{\lambda a}^3(x) = \operatorname{ad}_{\lambda a}^3(2x) = \lambda^3 \operatorname{ad}_a^3(x_k + x_s) = \lambda^3 \operatorname{ad}_a^3(x_s) = \lambda^2 \operatorname{ad}_a^3(\lambda x_s) = 0$$

since $\lambda x_s \in K$. So $\mathrm{ad}^3_{\lambda a}(R) = 0$.

(c) Since $x, y \in K$ we have that $x \ \mathrm{ad}^2_a(y)\, x \in K$ and therefore

$$0 = \mathrm{ad}^4_a(x \ \mathrm{ad}^2_a(y)\, x) = 6\, \mathrm{ad}^2_a(x)\, \mathrm{ad}^2_a(y)\, \mathrm{ad}^2_a(x). \qquad \square$$

The next theorem produces a decomposition of a Jordan element a of $\mathrm{Skew}(R, *)$ for a semiprime ring R with involution $*$. Essentially, $\hat{R}$ can be decomposed as a direct sum of five orthogonal ideals such that a lies in the sum of four of them. This makes possible to write the Jordan element a as a sum of four elements: a central element of R (see 4.2 case (i)), a "purely" central element of $\mathrm{Skew}(R, *)$ (see 4.2 case (ii)), and an ad-nilpotent element of index 3 that again is decomposed as a sum of two different elements: a Jordan element of $\mathrm{Skew}(R, *)$ which is a Jordan element of R (see 4.2 case (iii)) and a "Clifford" element of $\mathrm{Skew}(R, *)$ (see 4.2 case (iv)).

Note that there are no ad-nilpotent elements of index 2 in $\mathrm{Skew}(R, *)$ when R is a semiprime ring with involution $*$, see [17, Corollary 4.8].

Theorem 4.2. *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $K := \mathrm{Skew}(R, *)$. Let $a \in K$ be a Jordan element of K. Then there exists a complete system of five orthogonal idempotents $e_i \in \mathrm{Sym}(C(R), *)$, $i = 1, \cdots, 5$, that decomposes the central closure $R = \bigoplus_{i=1}^5 e_i R$ as a sum of five orthogonal ideals in such a way that*

- (i) *$e_1 a$ is a central (skew) element of R,*
- (ii) *$e_2 a$ is a central (skew) element of $\mathrm{Skew}(R, *)$, $e_2 R$ is a PI-algebra that satisfies the standard identity S_4, $\mathrm{Skew}(e_2 R, *)$ is an abelian Lie algebra such that $\mathrm{Skew}(e_2 R, *)^2 \subseteq Z(R)$,*
- (iii) *there exists a unique $\lambda \in e_3 \,\mathrm{Skew}(C(R), *)$ such that $(e_3 a - e_3 \lambda)^2 = 0$,*
- (iv) *$(e_4 a)^3 = 0$, and if $e_4 \neq 0$ then $(e_4 a)^2$ generates an essential ideal in $e_4 R$ and R satisfies a nonzero GPI, and*
- (v) *$a = \sum_{i=1}^4 e_i a$, $e_5 a = 0$, and the ideal generated by a is essential in $\bigoplus_{i=1}^4 e_i R$.*

Moreover,

- (*) *if $h \in \mathrm{Sym}(C(R), *)$ is an idempotent such that $ha \in Z(\mathrm{Skew}(R, *))$ then*

$$(e_1 + e_2)h = h.$$

Proof. Note that every idempotent $e \in \mathrm{Sym}(C(R), *)$ decomposes R as a sum of two orthogonal ideals $R = eR \oplus (1-e)R$ and both eR and $(1-e)R$ are semiprime rings with involution with Jordan elements ea (of $\mathrm{Skew}(eR, *)$) and $(1-e)a$) (of $\mathrm{Skew}((1-e)R, *)$). Moreover $C(eR) \subset C(R)$ and $C((1-e)R) \subset C(R)$.

(1) At this point we are going to decompose a as the sum of a central element and some other element that we will call b. Given the set $T := [a, \mathrm{Skew}(R, *)] \cup [a, \mathrm{Sym}(R, *)]$, by 2.15 there exists an idempotent $u \in \mathrm{Sym}(C(R), *)$ such that $uy = y$ for every $y \in T$ and therefore $uy = y$ for every $y \in [a, R]$, and $\mathrm{Ann}_R(\mathrm{Id}_R([a, R])) = (1-u)R$, because $\mathrm{Ann}_R(\mathrm{Id}_R([a, R])) = \mathrm{Ann}_R(\mathrm{Id}_R(T))$. Let $e_1 := 1 - u$. We can decompose

$$R = e_1 R \oplus (1 - e_1)R$$

and then we can work with

$$R^{(1)} := (1 - e_1)R, \quad K^{(1)} := \mathrm{Skew}(R^{(1)}, *), \text{ and } b := (1 - e_1)a,$$

where $R^{(1)}$ is semiprime with involution and b is a Jordan element of $K^{(1)}$. Notice that $[e_1 a, R] = (1 - u)[a, R] = 0$, so $e_1 a$ is a central (skew) element of R.

(2) Secondly, we are going to decompose b as the sum of a "purely" central element of $\mathrm{Skew}(R,*)$ and a remainder that we call c. We will also prove $(*)$ in (2.2), and show that c satisfies a polynomial identity of degree 4 over $C(R)$. Given the set $[b,K^{(1)}]$, by 2.15 there exists an idempotent $v \in \mathrm{Sym}(C(R^{(1)}),*)$ such that $vy = y$ for every $y \in [b,K^{(1)}]$ and $\mathrm{Ann}_{R^{(1)}}(\mathrm{Id}_{R^{(1)}}([b,K^{(1)}])) = (1-v)R^{(1)}$. Therefore, if we define $e_2 := (1-e_1)(1-v)$ we have that

$$R = e_1R \oplus e_2R \oplus (1-e_1-e_2)R,$$

and analogously we can work with

$$R^{(2)} := (1-e_1-e_2)R, \quad K^{(2)} := \mathrm{Skew}(R^{(2)},*), \text{ and } c := (1-e_1-e_2)a.$$

Notice that e_2a is a central (skew) element of $\mathrm{Skew}(R,*)$ because $[e_2a,K] = (1-v)(1-e_1)[a,K] = (1-v)[(1-e_1)a,(1-e_1)K] = (1-v)[b,K^{(1)}] = 0$.

(2.1) e_2R is a PI-algebra that satisfies the standard identity S_4 and $\mathrm{Skew}(e_2R,*)$ is an abelian Lie algebra: $[e_2a,R]$ generates an essential ideal in e_2R (because its annihilator is zero) and therefore given any $y \in e_2R$ there exists $0 \neq z \in \mathrm{Id}_{e_2R}(y) \cap \mathrm{Id}_{e_2R}(e_2[a,R])$. Moreover, since e_2R is semiprime there exists a $*$-prime ideal I_α of e_2R such that $z \notin I_\alpha$, see [27]. In particular $\overline{e_2a} \in Z(\mathrm{Skew}(R/I_\alpha,*))$ since $[e_2a,K] = 0$, but $\overline{e_2a} \notin Z(R/I_\alpha)$ since $[e_2a,R] \subseteq I_\alpha$ implies $z \in I_\alpha$, a contradiction. This implies by [17, Proposition 4.6] that the involution of R/I_α is of the first kind, R/I_α is an order in a simple algebra Q with $\dim_{Z(Q)}(Q) \leq 4$, and $\mathrm{Skew}(R/I_\alpha,*)$ is an abelian Lie algebra. In particular R/I_α is a PI-algebra that satisfies the standard identity S_4. Now, since the intersection of all I_α in these conditions is zero (see [27]), we have that $[\mathrm{Skew}(e_2R,*),\mathrm{Skew}(e_2R,*)] \subset \bigcap_\alpha I_\alpha = 0$ and hence $\mathrm{Skew}(e_2R,*)$ is an abelian Lie algebra. Moreover, e_2R is a subdirect product of PI-algebras that satisfy the standard identity S_4 and hence e_2R satisfies S_4.

Now, if $k \in \mathrm{Skew}(e_2R,*)$ and $x \in e_2R$ we have that $2x = x_k + x_h$, where $x_k = x - x^* \in \mathrm{Skew}(e_2R,*)$ and $x_h = x + x^* \in \mathrm{Sym}(e_2R,*)$ and

$$\begin{aligned}2[k^2,x] &= [k^2,2x] = k[k,2x] + [k,2x]k = k[k,x_h] + [k,x_h]k = [k^2,x_h] \\ &= k^2x_h - x_hk^2 = k^2x_h + kx_hk - kx_hk - x_hk^2 = [k,kx_h + x_hk] = 0\end{aligned}$$

because $kx_h + x_hk \in \mathrm{Skew}(e_2R,*)$. Therefore $[k^2,e_2R] = 0$ and $k^2 \in Z(e_2R) \subset Z(R)$.

(2.2) If $h \in \mathrm{Sym}(C(R),*)$ is an idempotent such that $ha \in Z(\mathrm{Skew}(R,*))$ then $(1-e_1)h \in \mathrm{Ann}_{R^{(1)}}(\mathrm{Id}_{R^{(1)}}([b,K^{(1)}]) = (1-v)R$ and therefore $(1-e_1)h = (1-v)(1-e_1)h = e_2h$, which implies that $h = e_1h + (1-e_1)h = (e_1+e_2)h$, proving $(*)$.

(2.3) There exist $\mu,\gamma \in \mathrm{Skew}(C(R^{(2)}),*)$ and $\tau \in \mathrm{Sym}(C(R^{(2)}),*)$ such that $c^3 = \mu c^2 + \tau c + \gamma$: For every $x \in R^{(2)}$, $c\,\mathrm{ad}_c^3(x) + \mathrm{ad}_c^3(x)c = \mathrm{ad}_c^3(cx+xc)$ and, since $x = x_k + x_s$ for some $x_k \in \mathrm{Skew}(R^{(2)},*)$ and some $x_s \in \mathrm{Sym}(R^{(2)},*)$ (see 2.10),

$$\begin{aligned}c\,\mathrm{ad}_c^3(x) + \mathrm{ad}_c^3(x)c &= c\,\mathrm{ad}_c^3(x_k+x_s) + \mathrm{ad}_c^3(x_k+x_s)c = c\,\mathrm{ad}_c^3(x_s) + \mathrm{ad}_c^3(x_s)c \\ &= \mathrm{ad}_c^3(cx_s + x_sc) = 0,\end{aligned}$$

so we have

$$0 = c\,\mathrm{ad}_c^3(x) + \mathrm{ad}_c^3(x)c = c^4x - 2c^3xc + 2cxc^3 - xc^4. \tag{I}$$

Now, by 4.1(a) and 3.2 applied to the Jordan element c^2 of $R^{(2)}$ there exists $\alpha \in C(R^{(2)})$ such that $(c^2-\alpha)^2 = c^4 - 2\alpha c^2 + \alpha^2 = 0$. So if we substitute c^4 in formula (I),

$$0 = c^4x - 2c^3xc + 2cxc^3 - xc^4 = 2\alpha c^2x - 2\alpha xc^2 - 2c^3xc + 2cxc^3.$$

Then, by 2.14, since $\mathrm{Id}_{R^{(2)}}(c^3) \subset \mathrm{Id}_R(c)$ there exist $\mu, \tau, \gamma \in C(R^{(2)})$ such that $c^3 = \mu c^2 + \tau c + \gamma$. Finally, if we decompose $\mu = \mu_s + \mu_k$, $\tau = \tau_s + \tau_k$, $\gamma = \gamma_s + \gamma_k$ with $\mu_s, \tau_s, \gamma_s \in \mathrm{Sym}(C(R^{(2)}), *)$ and $\mu_k, \tau_k, \gamma_k \in \mathrm{Skew}(C(R^{(2)}), *)$ (see 2.10) then we can decompose $\mu c^2 + \tau c + \gamma$ in its symmetric and skew parts. Its symmetric part is $\mu_s c^2 + \tau_k c + \gamma_s$ and its skew part is $\mu_k c^2 + \tau_s c + \gamma_k$, but since c^3 is skew-symmetric, $c^3 = \mu_k c^2 + \tau_s c + \gamma_k$. For convenience let us rename μ_k as μ, τ_s as τ and γ_k as γ.

(3) The elements μ and γ make it possible to decompose c as an element of type (iii) and some other element that we will call d, see (3.1). Initially $d^3 = \tau d$ but indeed $d^3 = 0$, see (3.3).

(3.1) For every $\lambda \in \mathrm{Skew}(C(R^{(2)}), *)$ there exists $\alpha_\lambda \in e_\lambda \mathrm{Skew}(C(R^{(2)}), *)$ such that

$$e_\lambda(c - \alpha_\lambda)^2 = (e_\lambda c - e_\lambda \alpha_\lambda)^2 = 0,$$

where the idempotent e_λ is defined in 2.10: By 4.1(b), λc is a Jordan element of $R^{(2)}$ and therefore $\lambda' \lambda c = e_\lambda c$ is again a Jordan element of $R^{(2)}$. Now, by 3.2 and 2.11 there exists $\beta_\lambda \in \mathrm{Skew}(C(R^{(2)}), *)$ such that $(e_\lambda c - \beta_\lambda)^2 = 0$. So $0 = e_\lambda(e_\lambda c - \beta_\lambda)^2 = (e_\lambda c - e_\lambda \beta_\lambda)^2$, and $\alpha_\lambda := e_\lambda \beta_\lambda \in e_\lambda \mathrm{Skew}(C(R^{(2)}), *)$ is the desired element.

(3.2) There exist an idempotent $w \in \mathrm{Sym}(C(R^{(2)}), *)$ and $\alpha \in \mathrm{Skew}(C(R^{(2)}), *)$ such that $c = wc + (1-w)c$ with $(wc - w\alpha)^2 = 0$ and $(1-w)c^3 = (1-w)\tau c$: Given $\mu \in \mathrm{Skew}(C(R^{(2)}), *)$, by (3.1) there exists $\alpha_\mu \in e_\mu \mathrm{Skew}(C(R^{(2)}), *)$ such that $c = e_\mu c + (1 - e_\mu)c$ with $(e_\mu c - \alpha_\mu c)^2 = 0$. If we multiply formula $c^3 = \mu c^2 + \tau c + \gamma$ of (2.3) by $1 - e_\mu$ we get

$$((1 - e_\mu)c)^3 = (1 - e_\mu)\tau c + (1 - e_\mu)\gamma. \tag{II}$$

Let us consider $\delta := (1 - e_\mu)\gamma \in \mathrm{Skew}(C(R^{(2)}), *)$. By (3.1) there exists $\alpha_\delta \in e_\delta \mathrm{Skew}(C(R^{(2)}), *)$ such that $c = e_\delta c + (1 - e_\delta)c$ with $(e_\delta c - \alpha_\delta c)^2 = 0$, and multiplying formula (II) by $1 - e_\delta$ we get

$$((1 - e_\mu)(1 - e_\delta)c)^3 = (1 - e_\mu)(1 - e_\delta)\tau c. \tag{III}$$

Note that e_δ, e_μ are two central orthogonal idempotents such that $c = e_\mu c + e_\delta c + (1 - e_\mu - e_\delta)c$ with $(e_\mu c - \alpha_\mu)^2 = 0$, $(e_\delta c - \alpha_\delta)^2 = 0$ and $(1 - e_\mu - e_\delta)c = (1 - e_\mu - e_\delta)\tau c$

Putting $w := e_\delta + e_\mu$ and $\alpha := \alpha_\mu + \alpha_\delta$ we have that $(wc - \alpha)^2 = 0$ and $(1 - w)c^3 = (1 - w)\tau c$. Since $w \in \mathrm{Sym}(C(R^{(2)}), *)$, it is orthogonal to e_1 and e_2 and $wc = wa$.

We can decompose R as

$$R = e_1 R \oplus e_2 R \oplus wR \oplus (1 - e_1 - e_2 - w)R,$$

so we can do as before and work with

$$R^{(3)} := (1 - e_1 - e_2 - w)R, \quad K^{(3)} := \mathrm{Skew}(R^{(3)}, *), \text{ and}$$

$$d := (1 - e_1 - e_2 - w)a = (1 - w)c.$$

Notice that $d^3 = \tau d$ and $(1-w)\tau \in \mathrm{Sym}(C(R^{(3)}), *)$. For convenience let us rename $(1 - w)\tau$ as τ.

(3.3) Let us prove that $d^3 = 0$: If $\tau = 0$ there is nothing to prove; otherwise consider e_τ, so that $d = e_\tau d + (1 - e_\tau)d$ with $(1 - e_\tau)d^3 = 0$ and $e_\tau d^3 = \tau d$ with τ

an invertible element in $e_\tau R^{(3)}$. If we multiply formula (I) by $(1-e_1-e_2-w)e_\tau$ and substitute $e_\tau d^3 = \tau d$ we get

$$\begin{aligned}0 &= (1-e_1-e_2-w)e_\tau(c^4x - 2c^3xc + 2cxc^3 - xc^4)\\ &= e_\tau(d^4x - 2d^3xd + 2dxd^3 - xd^4) = \tau[d^2, x].\end{aligned}$$

So $\tau d^2 \in Z(R^{(3)})$ and since τ is invertible in $R^{(3)}$, $e_\tau d^2 \in Z(R^{(3)})$. Let us denote $\xi := e_\tau d^2 \in Z(R^{(3)}) \subset C(R^{(3)})$. Now, since $e_\tau d$ is a Jordan element of $e_\tau K^{(3)}$ we have that, for every $x \in K^{(3)}$,

$$\begin{aligned}0 &= \mathrm{ad}^3_{e_\tau d}(x) = e_\tau(d^3x - 3d^2xd + 3dxd^2 - xd^3)\\ &= \xi(dx - 3xd + 3dx - xd) = 4\xi[d, x].\end{aligned}$$

So $\xi d \in Z(R^{(3)}) \subset C(R)$ and therefore $e_\xi d \in C(R)$, which implies by (*) that $e_\xi d = 0$. Therefore $0 = \xi d = e_\tau d^3$.

(4) Now we decompose d as two elements, one of type (iii) and the other one of type (iv). There exists an idempotent $w' \in \mathrm{Sym}(C(R^{(3)}), *)$ such that $d = w'd + (1-w')d$ with $(1-w')d^2 = 0$, $w'd^3 = 0$: Given d^2, by 2.15 there exists an idempotent $w' \in \mathrm{Sym}(C(R^{(3)}), *)$ such that $w'd^2 = d^2$ and $\mathrm{Ann}_{R^{(3)}}(\mathrm{Id}_{R^{(3)}}(d^2)) = (1-w')R$. Therefore,

$$R^{(3)} = w'R^{(3)} \oplus (1-w')R^{(3)}$$

and $d = w'd + (1-w')d$ with $(1-w')d^2 = 0$ and $w'd^3 = 0$.

(4.1) If $w'd^2 \neq 0$ then the local algebra associated to $w'd^2$ is abelian and therefore R satisfies a nonzero GPI: since $0 = \mathrm{ad}^4_{w'd}(K^{(3)}) = 6w'd^2K^{(3)}d^2$ and R is free of 2 and 3-torsion, $w'd^2K^{(3)}d^2 = 0$. Now, for every $x \in R^{(3)}$,

$$w'd^2(x - x^*)w'd^2 = 0, \text{ i.e.,}\quad w'd^2xw'd^2 = w'd^2x^*w'd^2.$$

Let us consider the local ring $R^{(3)}_{w'd^2} = R_{w'd^2}$ of $R^{(3)}$ at $w'd^2$, which is a semiprime ring such that

$$\begin{aligned}w'd^2(xw'd^2y)w'd^2 &= w'd^2(xw'd^2y)^*w'd^2 = w'd^2(y^*)w'd^2(x^*)w'd^2\\ &= w'd^2yw'd^2xw'd^2.\end{aligned}$$

Then $R_{w'd^2}$ is a semiprime commutative ring, hence a PI ring and therefore R satisfies the nonzero GPI

$$w'd^2xw'd^2yw'd^2 = w'd^2yw'd^2xw'd^2.$$

(5) Finally we gather all the previous information together. Take the orthogonal idempotents e_1, e_2, $e_3 := (1-e_1-e_2)w + (1-e_1-e_2-w)(1-w')$, $e_4 := (1-e_1-e_2-w)w'$. We have that $e_1+e_2+e_3+e_4 = 1$. Given the set $\{a\}$, by 2.15 there exists an idempotent $e \in \mathrm{Sym}(C(R), *)$ such that $ea = a$ and $\mathrm{Ann}_R(\mathrm{Id}_R(a)) = (1-e)R$. Define $\hat{e}_1 := ee_1, \hat{e}_2 := ee_2, \hat{e}_3 := ee_3$, $\hat{e}_4 := ee_4$ and $\hat{e}_5 := 1-e$ and decompose

$$R = \bigoplus_{i=1}^{5} R\hat{e}_i.$$

Notice that this decomposition satisfies the claims of the theorem:

- $\hat{e}_1 a$ is central in R by (1),
- $\hat{e}_2 a$ is a central (skew) element of $\mathrm{Skew}(R, *)$, $\hat{e}_2 R$ is a PI-algebra that satisfies the standard identity S_4, $\mathrm{Skew}(\hat{e}_2 R, *)$ is an abelian Lie algebra and $\mathrm{Skew}(\hat{e}_2 R, *)^2 \subseteq Z(\hat{R})$ by (2),

- for the skew element $\lambda := w\alpha$, $(\hat{e}_3 a - \hat{e}_3\lambda)^2 = e(e_3 a - \lambda)^2 = e(wc - w\alpha)^2 + e(1-w')d^2 = 0$ by (3) and (4),
- $\hat{e}_4 a^3 = ew'd^3 = 0$, and if $\hat{e}_4 \neq 0$, $\hat{e}_4 a^2 = \hat{e}_4 d^2$ generates an essential ideal of $\hat{e}_4 R$ because it has zero annihilator, and R satisfies a nonzero GPI by (4), and
- $a = ea = \hat{e}_1 a + \hat{e}_2 a + \hat{e}_3 a + \hat{e}_4 a$, $\hat{e}_5 a = 0$, and the ideal generated by a is essential in $\bigoplus_{i=1}^4 R\hat{e}_i$. □

If R is a centrally closed $*$-prime ring and a is a Jordan element of $\mathrm{Skew}(R,*)$, since the idempotent e_5 obtained in 4.2 belongs to the annihilator of $\mathrm{Id}_R(a)$ in R, which is a $*$-prime ideal of R, we get $e_5 = 0$. Hence in this context we obtain the four mutually exclusive possibilities of the next corollary.

Corollary 4.3. *Let R be a centrally closed $*$-prime ring with involution $*$ free of 2 and 3-torsion and let $K := \mathrm{Skew}(R,*)$. Let $a \in K$ be a Jordan element of K. Then we have four mutually exclusive possibilities:*

(i) $a \in Z(R)$.
(ii) $a \in Z(K)\backslash Z(R)$, *$R$ is a PI-algebra and* $\mathrm{Skew}(R,*)$ *is an abelian Lie algebra.*
(iii) *There exists* $\lambda \in \mathrm{Skew}(Z(R),*)$ *such that $a-\lambda$ is nilpotent of index 2.*
(iv) $a^3 = 0$, $a^2 \neq 0$ *and R satisfies a nonzero GPI.*

5. Jordan algebras of Lie algebras arising from associative rings

In this section we are going to describe the Jordan algebras associated to the Jordan elements of the Lie algebras R^- and $\mathrm{Skew}(R,*)$ for a centrally closed semiprime ring R (with involution $*$) using the characterizations of these elements given in 3.2 and 4.2.

Lemma 5.1. *Let R be a centrally closed semiprime ring free of 2 and 3-torsion and let $a \in R$ be a Jordan element. Then there exists a' in R such that $(R^-)_a \cong (R_{a'})^+$, i.e., the Jordan algebra of the Lie algebra R^- at a is isomorphic to the symmetrization of a local algebra of the ring R.*

Proof. It is enough to consider $a' := a - \lambda$ with λ as given in 3.2, which satisfies $(a')^2 = 0$. Then the map $\phi : (R^-)_a \to (R_{a'})^+$ given by $\phi(\overline{x}) := \tilde{x}$ is an isomorphism of Jordan algebras: ϕ is well defined, since if $\overline{x} = \overline{0}$ then $0 = [a,[a,x]] = [a',[a',x]] = -2a'xa'$ and therefore $\phi(\overline{x}) = \tilde{0}$. It is clear that ϕ is a $C(R)$-linear map. Now

$$\phi(\overline{x} \bullet \overline{y}) = \frac{1}{2}\phi(\overline{[x,[a,y]]}) = \frac{1}{2}\phi(\overline{[x,[a',y]]}) = \phi(\overline{x}) \bullet \phi(\overline{y}).$$

Clearly ϕ is onto, and if $\phi(\overline{x}) = \tilde{0}$ we have $0 = a'xa' = [a',a',x]] = [a,[a,x]]$ and therefore $\overline{x} = \overline{0}$. □

Now we turn to the description of the Jordan algebras at Jordan elements of the skew elements of semiprime rings with involution. First we need some previous results.

Lemma 5.2. *Let R be a semiprime ring with involution $*$ free of 2-torsion, and let $h \in \mathrm{Sym}(R,*)$ and $k \in K := \mathrm{Skew}(R,*)$ be such that $hKh = hKk = 0$ and h generates an essential ideal in R. Then $k = 0$.*

Proof. Note that $kKh = (hKk)^* = 0$. Let $x, y \in R$ and decompose $x = x_k + x_s$ and $y = y_k + y_s$ where $x_k, y_k \in K$ and $x_s, y_s \in \mathrm{Sym}(R, *)$; then

$$hxkyh + hykxh = hx_sky_sh + hy_skx_sh = h(x_sky_s + y_skx_s)h = 0$$

since $x_sky_s+y_skx_s \in K$ and $hKh = 0$. Therefore by 2.14, since $\mathrm{Id}_R(hxk) \subset \mathrm{Id}_R(h)$, there exists $\lambda_x \in C(R)$ such that

$$hxk = \lambda_x h.$$

Now suppose that $x \in \mathrm{Sym}(R, *)$. Then $xkx \in K$ so $hxkxk = 0$ by hypothesis. But $0 = (hxk)xk = \lambda_x hxk = \lambda_x^2 h$, and since h generates an essential ideal in R, $\lambda_x^2 = 0$ so $\lambda_x = 0$ and $hxk = 0$. Therefore $2hRk = h\,\mathrm{Sym}(R, *)k + hKk = 0$ and we get $k = 0$ by using again that h generates an essential ideal of R and that R is free of 2-torsion. □

In the following proposition, although the results are valid for R, some of the computations are actually carried in $\hat{R}$.

Proposition 5.3. *Let R be a semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element such that $a^3 = 0$ and a^2 generates an essential ideal of R. Then*

(a) *For every $k \in K$ we have that $aka^2 = a^2ka$.*
(b) *For every $k \in K$ we have $a^2ka^2 = 0$.*
(c) *R_{a^2} is a semiprime commutative ring.*
(d) *For every $x \in R$ there exists a unique $\lambda_x \in \mathrm{Sym}(C(R), *)$ such that $a^2xa^2 = \lambda_x a^2$. Moreover, if $e_x := \lambda_x$ is an idempotent then there exists $c_x \in \mathrm{Sym}(\hat{R}, *)$ such that $c_x = e_xc_x$, $e_xa^2c_xa^2 = e_xa^2$, $c_xa^2c_x = c_x$, $c_x^2 = 0$ and $e_xa = e_x(a^2c_xa + ac_xa^2)$.*

In addition, if $Q := Q_m^s(R)$ is the symmetric Martindale ring of quotients of R, I is an ideal of R and we denote $\Delta_I := \{\lambda_x \in C(R) \mid \exists x \in I \text{ with } a^2xa^2 = \lambda_x a^2\}$, then

(e) *Δ_I is a subring of $C(R)$. Moreover, if $C(R)I \subset I$ then Δ_I is a von Neumann regular ring.*
(f) *If I is an essential ideal of R then $\Delta_I q = 0$ implies $q = 0$ for every $q \in Q$.*

Proof. (a) Since a is a Jordan element, for every $k \in K$ we have that $0 = \mathrm{ad}_a^3(k) = a^3k - 3a^2ka + 3aka^2 - ka^3$. Therefore, since R is free of 3-torsion and $a^3 = 0$, $aka^2 = a^2ka$.

(b) For every $k \in K$, by (a), $a^2ka^2 = aka^3 = 0$.

(c) Let $\tilde{x} \in R_{a^2}$ be such that $\tilde{x} \circ R_{a^2} \circ \tilde{x} = \tilde{0}$. Then for every $\tilde{y} \in R_{a^2}$ we have that $\tilde{0} = \tilde{x} \circ \tilde{y} \circ \tilde{x}$ and so $a^2xa^2ya^2xa^2 = 0$ which implies, since R is semiprime, that $a^2xa^2 = 0$ and $\tilde{x} = \tilde{0}$, i.e., that R_{a^2} is a semiprime associative ring. Moreover, given $\tilde{x}, \tilde{y} \in R_{a^2}$ we have that $\tilde{x} \circ \tilde{y} - \tilde{y} \circ \tilde{x} = (xa^2y - ya^2x) + \mathrm{Ker}(a^2) = \tilde{0}$ because $xa^2y - ya^2x \in K$ and, by (2), it is contained in $\mathrm{Ker}(a^2)$.

(d) By (c), for every $x, y \in R$ we have that

$$a^2xa^2ya^2 = a^2ya^2xa^2. \tag{D}$$

Therefore by 2.14, since $\mathrm{Id}_R(a^2xa^2) \subset \mathrm{Id}_R(a^2)$ we have that there exists $\lambda_x \in C(R)$ such that

$$a^2xa^2 = \lambda_x a^2. \tag{D'}$$

Notice that λ_x is unique for every $x \in R$ because a^2 generates an essential ideal of R. Since $a^2Ka^2 = 0$ we also obtain $\lambda_x = \lambda_x^* \in \mathrm{Sym}(C(R), *)$ by taking involutions in formula $a^2xa^2 = \lambda_x a^2$.

Now suppose that $a^2xa^2 = e_x a^2$ where $e_x := \lambda_x$ is an idempotent of $C(R)$. Then

$$(e_x a^2)x(e_x a^2) = e_x a^2 x a^2 = (e_x)^2 a^2 = e_x a^2,$$

and since $e_x a^2$ is a nilpotent element of index 2, by 2.12 there exists $c_x \in \mathrm{Sym}(\hat{R}, *)$ such that $c_x^2 = 0$, $(e_x a^2)c_x(e_x a^2) = e_x a^2$ and $c_x(e_x a^2)c_x = c_x$. Moreover $c_x = c_x(e_x a^2)c_x = e_x(c_x(e_x a^2)c_x) = e_x c_x$. Finally, for every $k \in K$,

$$\begin{aligned} e_x(a^2c_xa + ac_xa^2 - a)ka^2 &= e_x(a^2c_xaka^2 + ac_xa^2ka^2 - aka^2) \\ &= e_x(a^2c_xa^2ka + 0 - aka^2) \\ &= e_x(a^2ka - aka^2) = 0. \end{aligned}$$

Therefore, since $ac_xa + ac_xa^2 - a \in K$, by 5.2 we have $e_x(a^2c_xa + ac_xa^2 - a) = 0$ and $e_x a = e_x(a^2c_xa + ac_xa^2)$.

(e) Let us consider now the symmetric Martindale ring of quotients $Q := Q_m^s(R)$ of R and an ideal I of R, and denote $\Delta_I := \{\lambda_x \in C(R) \mid \exists x \in I \text{ with } a^2xa^2 = \lambda_x a^2\}$. Given $\lambda_x, \lambda_y \in \Delta_I$ there exist $x, y \in I$ such that $a^2xa^2 = \lambda_x a^2$ and $a^2ya^2 = \lambda_y a^2$. Then

$$\begin{aligned} (\lambda_x + \lambda_y)a^2 &= a^2xa^2 + a^2ya^2 = a^2(x+y)a^2 = \lambda_{x+y}a^2, \\ (\lambda_x \cdot \lambda_y)a^2 &= \lambda_x a^2ya^2 = a^2xa^2ya^2 = \lambda_{xa^2y}a^2. \end{aligned}$$

Therefore $\lambda_x + \lambda_y = \lambda_{x+y} \in \Delta_I$ and $\lambda_x \cdot \lambda_y = \lambda_{xa^2y} \in \Delta_I$. Now, if $\lambda_x \in \Delta_I$, $a^2xa^2 = \lambda_x a^2$ and for the unique element λ_x' associated to λ_x in 2.10 we have that

$$\lambda_x' a^2 = \lambda_x'\lambda_x'\lambda_x a^2 = (\lambda_x')^2 a^2xa^2 = a^2((\lambda_x')^2 x)a^2$$

and $\lambda_x' \in \Delta_I$ since $C(R)I \subseteq I$.

(f) Suppose that I is an essential ideal of R and take $q \in Q$ such that $\Delta_I q = 0$. For such q there exists an essential ideal J of R such that $Jq + qJ \subset R$. Let $y \in I \cap J$ and $t := yq$ or $t := qy$. Given any $x \in R$ and $z \in I$ we have that $a^2za^2xt = \lambda_y a^2xt = a^2x\lambda_y t = 0$. Therefore, since z is arbitrary in the essential ideal I, the ideal generated by a^2xt is a nilpotent ideal of R and hence $\mathrm{Id}_R(a^2xt) = 0$. Now $a^2xt = 0$ for every $x \in R$ implies that the ideal generated by t is orthogonal to the ideal generated by a^2, which is essential, so $t = 0$. Finally, since $t = qy$ or $t = yq$ for an arbitrary y in an essential ideal of R, $q = 0$. □

Proposition 5.4. *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element such that $a^3 = 0$ and a^2 generates an essential ideal of R. Let $Q := Q_m^s(R)$ be the symmetric Martindale ring of quotients of R. Then:*

(a) *Q is a semiprime ring with involution $*$ such that a is a Jordan element of $\mathrm{Skew}(Q, *)$, $a^3 = 0$ and a^2 generates an essential ideal in Q.*
(b) *There exists $c \in \mathrm{Sym}(Q, *)$ such that $a^2ca^2 = a^2$, $ca^2c = c$, $c^2 = 0$ and $a = a^2ca + aca^2$. In particular a^2 is a von Neumann regular element of Q.*
(c) *The Jordan element a is von Neumann regular. In particular, the Jordan algebra $\mathrm{Skew}(Q, *)_a$ at the Jordan element a is unital with unit element $1_a := \overline{ac + ca}$.*

(d) *For every* $x \in \mathrm{Skew}(Q,*)$ *there exists a unique* $\mu_x \in \mathrm{Sym}(C(R),*)$ *such that* $axa = \mu_x a$.

(e) Q_{a^2} *is isomorphic to* $C(R)$.

Proof. (a) By 2.8, Q is a semiprime ring with involution $*$. Let us prove some properties:

(1) Let $k \in \mathrm{Skew}(Q,*)$ and let I be an essential $*$-ideal of R such that $Ik + kI \subset R$. By 5.3(d), given any $y \in I$ there exists $\lambda_y \in \mathrm{Sym}(C(R),*)$ such that $a^2ya^2 = \lambda_y a^2$, therefore if $y = y_s + y_k$ where $y_s \in \mathrm{Sym}(R,*)$ and $y_k \in K$, by 5.3(b), then

$$\begin{aligned}\lambda_y a^2ka^2 &= (a^2ya^2)ka^2 = a^2(y_sa^2k)a^2 = a^2(y_sa^2k)^*a^2 = -a^2ka^2y_sa^2\\ &= -a^2ka^2ya^2 = -\lambda_y a^2ka^2.\end{aligned}$$

Thus $\Delta_I a^2ka^2 = 0$, which implies by 5.3(f) that $a^2ka^2 = 0$.

(2) Let us prove that a is a Jordan element of $\mathrm{Skew}(Q,*)$: Since R is a subring of Q, $a^3 = 0$ (in Q). Let $k \in \mathrm{Skew}(Q,*)$ and let I be an essential $*$-ideal of R such that $Ik + kI \subset R$. By 5.3(d), given any $y \in I$ there exists $\lambda_y \in C(R)$ such that $a^2ya^2 = \lambda_y a^2$. Therefore, by (1) and 5.3(a),

$$\begin{aligned}\lambda_y a^2ka &= a^2ya^2ka = a^2(ya^2k + ka^2y^*)a = a(ya^2k + ka^2y^*)a^2 = aka^2y^*a^2\\ &= aka^2ya^2 = \lambda_y aka^2.\end{aligned}$$

Thus $\Delta_I(a^2ka - aka^2) = 0$, which implies, by 5.3(f), that $a^2ka = aka^2$. Finally, for every $k \in \mathrm{Skew}(Q,*)$ we have

$$\mathrm{ad}_a^3(k) = a^3k - 3a^2ka + 3aka^2 - ka^3 = 0$$

and hence a is a Jordan element of $\mathrm{Skew}(Q,*)$. Furthermore, given a nonzero ideal I of Q, $I \cap R$ is nonzero. Therefore $I \cap R \cap Ra^2R \neq 0$ since Ra^2R is an essential ideal of R, which implies that Qa^2Q is an essential ideal of Q.

(b) By 5.3(e) we know that $\Delta_R := \{\lambda \in C(R) \mid \exists x \in R \text{ with } a^2xa^2 = \lambda a^2\}$ is a von Neumann regular subring of $C(R)$. Now consider a maximal family of orthogonal idempotents $\{e_i\}_{i\in I}$ of Δ_R.

(b_1) $J := \bigoplus_{i\in I} Re_i$ is an essential ideal of R: First notice that given any $\lambda \in \Delta_R$ such that $\lambda e_i = 0$ for every $i \in I$ we have that $\lambda = 0$, since otherwise, since Δ_R is von Neumann regular, there exists $\lambda' \in \Delta_R$ such that $\lambda\lambda'$ is an idempotent of Δ_R orthogonal to every e_i, a contradiction. Now, if $z \in R$ satisfies $zJ = 0$, then $zRe_i = 0$ for every $i \in I$ and therefore $e_iz = 0$. Moreover $a^2RzRa^2e_i = a^2Rze_iRa^2 = 0$, which implies, since by 5.3(d) $a^2RzRa^2 = \Lambda a^2$ with $\Lambda \subset C(R)$, that $\Lambda e_i a^2 = 0$ and hence that $\Lambda e_i = 0$ for all e_i since a^2 generates an essential ideal in R. Hence $\Lambda = 0$. Therefore $a^2RzRa^2 = 0$ and, since a^2 generates an essential ideal, $a^2Rz = 0$ (because it is orthogonal to the ideal generated by a^2) and again $z = 0$. Therefore J is an essential ideal of R.

(b_2) Let us consider $f : \bigoplus_{i\in I} Re_i \to R$ defined by $f(\sum a_ie_i) = \sum a_ic_i$, where c_i is a twin of e_ia^2 as given in 5.3(d). Then $c := [\bigoplus_{i\in I} Re_i, f] \in Q$ satisfies $a^2ca^2 = a^2$, since by construction $ce_i = c_i$ and therefore $(a^2ca^2 - a^2)e_i = 0$ for each e_i, so $(a^2ca^2 - a^2)I = 0$ with I being essential. Similarly $c^2 = 0$, $ca^2c = c \in \mathrm{Sym}(Q,*)$ and $a = a^2ca + aca^2$.

(c) Notice that for any $\bar{x} \in \mathrm{Skew}(Q,*)_a$ we have that $1_a \bullet \overline{x} = \overline{[[ac+ca,a],x]} = \overline{x}$ because $\mathrm{ad}_a^2([[ac+ca,a],x]) = \mathrm{ad}_a^2(x)$, since

$$\begin{aligned}[a,[a,[[ac+ca,a],x]]] &= [a,[[a,[ac+ca,a]],x]] + [a,[[ac+ca,a],[a,x]]] \\ &=^{(1)} [a,[[a,[ac+ca,a]],x]] =^{(2)} [[a,[ac+ca,a],[a,x]]] \\ &=^{(3)} [a,[a,x]],\end{aligned}$$

where $^{(1)}$ $[a,[[ac+ca,a],[a,x]]] = 0$ by [13, Lemma 2.3(iv)], $^{(2)}$ because a is Jordan and $^{(3)}$ $[a,[ac+ca,a]] = a$ follows from (b).

(d) Given any $t \in \mathrm{Skew}(Q,*)$ we have

$$\begin{aligned}ata &= (a^2ca + aca^2)t(a^2ca + aca^2) \\ &= a^2cata^2ca + aca^2ta^2ca + a^2cataca^2 + aca^2taca^2 \\ &= a^2(cat+tca)a^2ca + 0 + 0 + aca^2(cat+tca)a^2 \\ &= \mu_t a^2ca + \mu_t aca^2 = \mu_t a\end{aligned}$$

because $a^3 = 0$, $a^2\,\mathrm{Skew}(Q,*)a^2 = 0$, $a^2taca^2 = a^2(tac)^*a^2 = a^2cata^2$ and there exists a unique $\mu_t \in \mathrm{Sym}(C(R),*)$ such that $a^2(cat+tca)a^2 = \mu_t a^2$ by (a) and 5.3(b),(d) taking Q as the ring of the statement.

(e) Let us consider the homomorphism of rings $\Phi : C(R) \to Q_{a^2}$ defined by $\Phi(\lambda) := \widetilde{\lambda c}$, where c is as given in (b). Let us prove that Φ is an isomorphism: if $\Phi(\lambda) = \tilde{0}$ then $\tilde{0} = \widetilde{\lambda c}$ and therefore $0 = a^2(\lambda c)a^2 = \lambda a^2$, which implies that $a^2R\lambda = 0$ and therefore λ is orthogonal to the ideal generated by a^2, i.e., $\lambda = 0$. Finally, given any $\tilde{x} \in Q_{a^2}$, by (a) and 5.3(d) taking Q as the ring of the statement, we have that $a^2xa^2 = \lambda_x a^2 = a^2(\lambda_x c)a^2$ and therefore $\tilde{x} = \widetilde{\lambda_x c} = \Phi(\lambda_x)$ in Q_{a^2}. □

Corollary 5.5. *Let R be a centrally closed semiprime ring free of 2 and 3-torsion and let $Q_m^s(R)$ be the symmetric Martindale ring of quotients of R. Then:*

1. *If $a \in R$ is a Jordan element of R then a is a Jordan element of $Q_m^s(R)$.*
2. *If $*$ is an involution on R, $K := \mathrm{Skew}(R,*)$ and a is a Jordan element of K, then a is a Jordan element of $\mathrm{Skew}(Q_m^s(R),*)$.*

Proof. (1) If a is a Jordan element of R then by 3.2 there exists $\alpha \in C(R)$ such that $(a-\alpha)^2 = 0$. Therefore a is a Jordan element of $Q_m^s(R)$.

(2) Following the notation of Theorem 4.2 there exist four orthogonal idempotents $\{e_i\}_{i=1}^4$ contained in $\mathrm{Sym}(C(R),*)$ such that $a = e_1a + e_2a + e_3a + e_4a$ and:

(2.1) $e_1a \in C(R)$ and therefore $e_1a \in Z(Q_m^s(R))$ and e_1a is a Jordan element of $\mathrm{Skew}(Q_m^s(R),*)$ (in fact it is a Jordan element of $Q_m^s(R)$).

(2.2) $e_2a \in Z(\mathrm{Skew}(R,*))$, e_2R is a PI-algebra that satisfies the standard identity S_4, and $\mathrm{Skew}(e_2R,*)$ is an abelian Lie algebra. Let us prove that the Lie algebra $\mathrm{Skew}(e_2Q_m^s(R),*)$ is abelian too: Since $e_2Q_m^s(R)$ is the Martindale symmetric ring of quotients of e_2R, we have that $e_2Q_m^s(R)$ is a general left (or right) ring of quotients of e_2R, see [23, Definition 13.10 and Proposition 14.7]. Therefore, since e_2R is a PI-algebra, by Rowen's Theorem ([31, Theorem 2]), $0 \neq [p,q] \in \mathrm{Skew}(e_2Q_m^s(R),*)$ implies that there exists $\lambda \in Z(e_2R) \subset C(R)$ such that $\lambda p, \lambda q \in \mathrm{Skew}(e_2R,*)$ and $\lambda[p,q] \neq 0$. Pick a partner $\lambda' \in Z(R)$ of λ as in 2.10. Then $\lambda[p,q] = \lambda\lambda'\lambda[p,q] = \lambda'[\lambda p, \lambda q] = 0$, a contradiction. Therefore $\mathrm{Skew}(e_2Q_m^s(R),*)$ is an abelian Lie algebra and e_2a is a Jordan element of $\mathrm{Skew}(Q_m^s(R),*)$.

(2.3) There exists a unique $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(e_3 a - e_3\lambda)^2 = 0$ and $e_3 a$ is a Jordan element of $\mathrm{Skew}(Q_m^s(R), *)$ (in fact, it is a Jordan element of $Q_m^s(R)$).

(2.4) If $e_4 a \neq 0$ then $(e_4 a)^3 = 0$ satisfies 5.4(a) and therefore it is a Jordan element of $\mathrm{Skew}(Q_m^s(R), *)$.

In conclusion $a = \sum_{1=1}^{4} e_i a$ is a Jordan element of $\mathrm{Skew}(Q_m^s(R), *)$. □

The next result is a generalization of [13, Theorem 3.3].

Proposition 5.6. *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element. Let us suppose that there exists $\lambda \in Z(R)$ such that $(a - \lambda)^2 = 0$. Then $K_a \cong K_{a-\lambda} \cong \mathrm{Sym}(R_{a-\lambda}, *)$.*

Proof. Clearly, by construction $K_a \cong K_{a-\lambda}$. Moreover, if we denote $b := a - \lambda$ then the map $\Phi : K_b \to \mathrm{Sym}(R_b, *)$ defined by $\Phi(\overline{x}) := \widetilde{x}$ is an isomorphism of Jordan algebras:

• Φ is well defined: If $\overline{x} = \overline{y} \in K_b$ then $x - y \in \mathrm{Ker}(b)$ and therefore $0 = [b, [b, x - y]] = -2b(x - y)b$ (because $b^2 = 0$), which implies that $\widetilde{x} = \widetilde{y} \in R_b$. In addition, if $x \in K$ then $(\widetilde{x})^* = \widetilde{-x^*} = \widetilde{x}$ and therefore $\Phi(\overline{x}) \in \mathrm{Sym}(R_b, *)$.

• Φ is a homomorphism of Jordan algebras:

(1) $\Phi(\overline{x} + \overline{y}) = \Phi(\overline{x + y}) = \widetilde{x + y} = \widetilde{x} + \widetilde{y}$

(2) $\Phi(\overline{x} \bullet \overline{y}) = \Phi(\overline{[x, [b, y]]}) = \widetilde{xby} - \widetilde{xyb} - \widetilde{byx} + \widetilde{ybx} = \widetilde{xby} + \widetilde{ybx} = \widetilde{x} \bullet \widetilde{y}$

because $\widetilde{xyb} + \widetilde{byx} = \widetilde{0}$ since $b^2 = 0$.

• Φ is an isomorphism: If $\Phi(\overline{x}) = \widetilde{0}$ then $0 = -2bxb = [b, [b, x]]$ and therefore $\overline{x} = \overline{0} \in K_b$. Now let $\widetilde{x} \in \mathrm{Sym}(R_b, *)$. Then $\widetilde{x} = \widetilde{x}^* = \widetilde{-x^*}$ and hence $bxb = -bx^*b$ and $x + x^* \in \mathrm{Ker}(b)$. Therefore $\Phi(\overline{\frac{x-x^*}{2}}) = \widetilde{\frac{x-x^*}{2}} = \frac{1}{2}\widetilde{x} + \frac{1}{2}(\widetilde{x})^* = \widetilde{x}$. □

Proposition 5.7. *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element such that $a^3 = 0$, a^2 generates an essential ideal and there exists $c \in \mathrm{Sym}(R, *)$ such that $a^2ca^2 = a^2$, $ca^2c = c$, $c^2 = 0$ and $a = a^2ca + aca^2$. Then K_a is a nondegenerate Jordan algebra of quadratic form.*

Proof. Note that K_a is a unital Jordan algebra with unit element $1_a := \overline{ac + ca}$ (see the proof of 5.4(c)).

The map $\langle \cdot, \cdot \rangle' : K \times K \to Z(R)a$ defined by $\langle x, y \rangle' := a\{x, y, a\}a = axya^2 + a^2yxa$ is bilinear and symmetric: By the proof of 5.4(d), $a\{x, y, a\}a \in Z(R)a$, and it is bilinear. Moreover, by 5.3(a)

$$\begin{aligned} a\{x, y, a\}a &= axya^2 + a^2yxa = a[x, y]a^2 + a^2[y, x]a + ayxa^2 + a^2xya \\ &= a[x, y]a^2 + a[y, x]a^2 + ayxa^2 + a^2xya \\ &= 0 + a\{y, x, a\}a. \end{aligned}$$

(1) The map $\langle \cdot, \cdot \rangle : K_a \times K_a \to Z(R)$ defined by $\langle \overline{x}, \overline{y} \rangle := \mu_{xy}$ where μ_{xy} is the unique element of $Z(R)$ that satisfies $a\{x, y, a\}a = \mu_{xy}a$ is a nondegenerate symmetric bilinear form on K_a:

In order to prove that the map is well defined let us show that, if $x \in K$, then $[a, [a, x]] = 0$ if and only if $a^2x + xa^2 = 0$ or, equivalently, if and only if $axa = 0$: If $[a, [a, x]] = 0$ then $2axa = a^2x + xa^2$, so if we multiply by a on the left we get $2a^2xa = axa^2$ and hence by the proof of 5.4(d) it is $2\mu_x a^2 = \mu_x a^2$, which implies, since a^2 generates an essential ideal, that $\mu_x = 0$, i.e., $axa = 0$ and therefore $a^2x + xa^2 = 0$. Now, if $a^2x + xa^2 = 0$, multiplying by a on the left we get that $axa^2 = 0$. Therefore $\mu_x a^2 = 0$ and $\mu_x = 0$. Then $axa = 0$ and $[a, [a, x]] = 0$.

Now let $y \in K$ be such that $[a, [a, y]] = 0$. Hence $aya = 0$ and $a^2y + ya^2 = 0$. Thus

$$\begin{aligned} a\{x, y, a\}a &= ax[y, a]a + axaya + a[a, y]xa + ayaxa = ax[y, a]a + a[a, y]xa \\ &= ax[[y, a], a] + axa[y, a] + [a, [a, y]]xa + [a, y]axa \\ &= axa[y, a] + [a, y]axa = -axa^2y - ya^2xa = -axa(a^2y + ya^2) = 0. \end{aligned}$$

Therefore, since $\langle\cdot,\cdot\rangle'$ is bilinear and symmetric, if $\overline{y} = \overline{z}$ then $\langle\overline{a}, \overline{y}\rangle = \langle\overline{a}, \overline{z}\rangle$ and $\langle\overline{y}, \overline{a}\rangle = \langle\overline{a}, \overline{y}\rangle = \langle\overline{a}, \overline{z}\rangle = \langle\overline{z}, \overline{a}\rangle$, which implies that this form is well defined and therefore bilinear and symmetric.

Finally, if there exists $\overline{b} \in K_a$ such that for every $\overline{x} \in K_a$ we have that $\langle\overline{b}, \overline{x}\rangle = 0$, then $\overline{b} = \overline{0}$: By hypothesis $a^2bxa + axba^2 = 0$ and if we multiply by a we get $a^2xba^2 = 0$ for every $x \in K$. Now, $a^2Ka^2 = 0$ by 5.3(b) and, since we also have $a^2K(ba^2 + a^2b) = 0$, by 5.2 $ba^2 + a^2b = 0$ and hence $[a, [a, b]] = 0$.

(2) For every $x \in K$ we can define $T(\overline{x}) := \langle\overline{x}, 1_a\rangle$. Note that $\langle\overline{x}, 1_a\rangle = \langle\overline{x}, \overline{ac + ca}\rangle = \mu_x$ where μ_x is the unique element of $Z(R)$ such that $axa = \mu_x a$:

$$\begin{aligned} \langle x, ac + ca\rangle' &= ax(ca + ac)a^2 + a^2(ca + ac)xa = axaca^2 + a^2caxa \\ &= \mu_x(aca^2 + a^2ca) = \mu_x a. \end{aligned}$$

In particular $T(1_a) = 1$ because $\mu_{ac+ca}a = a(ac + ca)a = a^2ca + aca^2 = a$.

(3) For every $\lambda \in Z(R)$ we have that $\lambda^* = \lambda$. In particular K is a $Z(R)$-submodule of R: if this is not the case there exists $0 \neq \lambda \in \mathrm{Skew}(Z(R), *)$. Since we know that $a^2Ka^2 = 0$ by 5.3(b), we have that $\lambda a^2(\mathrm{Sym}(R, *)\lambda)a^2 \subseteq \lambda a^2Ka^2 = 0$. Hence $\lambda a^2R\lambda a^2 = 0$ and so $\lambda a^2 = 0$. Since a^2 generates an essential ideal this implies $\lambda = 0$.

(4) Now, $W := \{\overline{x} \in K_a \mid T(\overline{x}) = 0\}$ is a submodule of K_a such that

$$K_a = W \oplus Z(R)1_a$$

because for any $\overline{x} \in K_a$ we have that $\overline{x} = (\overline{x} - T(x)1_a) + T(x)1_a$.

(5) If $\overline{v}, \overline{w} \in W$, then $\overline{v} \circ \overline{w} = \mu_{wv}1_a$: since $ava = awa = 0$ we have

$$[a, [a, [[v, a], w]]] = a^2vwa + awva^2 = a\{w, v, a\}a = \mu_{wv}a = \mu_{wv}[a, [a, ac + ca]].$$

Therefore K_a is a nondegenerate Jordan algebra of quadratic form associated to the nondegenerate symmetric bilinear form given in (1). □

Theorem 5.8. *Let R be a centrally closed semiprime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element. Let $Q_m^s(R)$ be the symmetric Martindale ring of quotients of R. Then there exist two idempotents $e, f \in \mathrm{Sym}(C(R), *)$ that decompose $Q_m^s(R)$ as a sum of three orthogonal ideals $Q_m^s(R) = eQ_m^s(R) \oplus fQ_m^s(R) \oplus (1 - e - f)Q_m^s(R)$, and an element $\lambda \in e\,\mathrm{Skew}(C(R), *)$ such that $a = ea + fa + (1 - e - f)a \in \mathcal{K} := \mathrm{Skew}(Q_m^s(R), *)$, $\mathcal{K}_a \cong \mathcal{K}_{ea} \oplus \mathcal{K}_{(1-e-f)a}$ and*

(i) *$\mathcal{K}_{ea} \cong \mathcal{K}_{ea-e\lambda} \cong \mathrm{Sym}(Q_m^s(R)_{ea-e\lambda}, *)$.*
(ii) *$\mathcal{K}_{(1-e-f)a}$ is a nondegenerate Jordan algebra of quadratic form.*
(iii) *$\mathcal{K}_{fa} = 0$.*

Proof. Let us denote $Q := Q^s_m(R)$. By 5.5, a is a Jordan element of Q and therefore by 4.2 and following its notation there exist two idempotents $e := e_3, f := e_1 + e_2 \in \mathrm{Sym}(C(R), *)$ and $\lambda \in e\,\mathrm{Skew}(C(R), *)$ such that $fa \in Z(\mathcal{K})$, $(ea - e\lambda)^2 = 0$ and, if $e + f \neq 1$, then $(1 - e - f)a$ is nilpotent of index 3 and $((1 - e - f)a)^2$ generates an essential ideal in $(1 - e - f)Q$.

(i) Consider $Q_1 := eQ$, which is again a semiprime ring with involution $*$ free of 2 and 3-torsion. Notice that it is the symmetric Martindale ring of quotients of eR. Then $ex \in \mathrm{Skew}(Q_1, *)$ is a Jordan element. Now $(ea - e\lambda)^2 = 0$ implies by 5.6 that $\mathrm{Skew}(Q_1, *)_{ex} \cong \mathrm{Skew}(Q_1, *)_{ea-e\lambda} \cong \mathrm{Sym}((Q_1)_{ea-e\lambda}, *)$. Moreover $(Q_1)_{ea-e\lambda} \cong Q_{ea-e\lambda}$, $\mathrm{Skew}(Q_1, *)_{ea} \cong \mathcal{K}_{ea}$ and $\mathrm{Skew}(Q_1, *)_{ea-e\lambda} \cong \mathcal{K}_{ea-e\lambda}$.

(ii) Consider $Q_2 := (1-e-f)Q$, which is again a semiprime ring with involution $*$ free of 2 and 3-torsion. Notice that it is the symmetric Martindale ring of quotients of $(1-e-f)R$ and therefore is centrally closed. Then $(1-e-f)a \in \mathrm{Skew}(Q_2, *)$ is a Jordan element such that $(1-e-f)a$ is nilpotent of index 3 and $((1-e-f)a)^2$ generates an essential ideal in Q_2. Now, by 5.4(b) there exists $c \in \mathrm{Sym}(Q_2, *)$ such that $(1-e-f)a^2ca^2 = (1-e-f)a^2$, $(1-e-f)ca^2c = c$, $c^2 = 0$ and $(1-e-f)a = (1-e-f)(a^2ca + aca^2)$ and thus, by 5.7, $\mathrm{Skew}(Q_2, *)_{(1-e-f)a}$ is a nondegenerate Jordan algebra of quadratic form. Moreover $\mathrm{Skew}(Q_2, *)_{(1-e-f)a} \cong \mathcal{K}_{(1-e-f)a}$.

(iii) Since $fa \in Z(\mathcal{K})$ we have that $\mathrm{Ker}(fa) = \mathcal{K}$ and hence $\mathcal{K}_{fa} = 0$. $\square$

The next result is a corollary in the case that R is a prime ring. Note that in this case it is not necessary to extend the ring to its Martindale ring of quotients to determine the structure of the Jordan algebra associated to the Jordan element. Case (ii) corresponds to [9, Theorem 4.7].

Corollary 5.9. *Let R be a centrally closed $*$-prime ring with involution $*$ free of 2 and 3-torsion and let $a \in K := \mathrm{Skew}(R, *)$ be a Jordan element. Then we have one of the next mutually exclusive possibilities:*

(i) *There exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a - \lambda)^2 = 0$ and therefore $K_a \cong K_{a-\lambda} \cong \mathrm{Sym}(R_{a-\lambda}, *)$.*
(ii) *$a^3 = 0$, $a^2 \neq 0$ and K_a is a Clifford Jordan algebra.*
(iii) *$a \in Z(K)$ but $a \notin Z(R)$ and therefore $K_a = \{0\}$.*

Proof. By 4.3 we have three possibilities:

(1) Cases (i) and (iii) of 4.3: There exists $\lambda \in \mathrm{Skew}(C(R), *)$ such that $(a-\lambda)^2 = 0$ and therefore by 5.6 we have that $K_a \cong K_{a-\lambda} \cong \mathrm{Sym}(R_{a-\lambda}, *)$. Note that case (i) implies that $K_a = \{0\}$.

(2) Case (iv) of 4.3: We have that $a^3 = 0$ and a^2 generates an essential ideal of R. By 5.3(d), since R is semiprime, there exist $x \in R$ and $\lambda_x \in \mathrm{Sym}(C(R), *)$ such that $0 \neq a^2xa^2 = \lambda_x a^2$. Consider the partner λ'_x of λ_x given in 2.10. Now, $a^2\lambda'_x xa^2 = \lambda'_x\lambda_x a^2 = a^2$ (because 1 is the only nonzero symmetric idempotent in $C(R)$, since R is $*$-prime). Now, again by the proof of 5.3(d) there exists $c \in \mathrm{Sym}(R, *)$ such that $a^2ca^2 = a^2$, $ca^2c = c$, $c^2 = 0$ and $a = (a^2ca + aca^2)$. Finally, since $C(R)$ is a field, 5.7 proves that K_a is a Clifford Jordan algebra (see 2.2).

(3) Case (ii) of 4.3: We have that $a \in Z(K)$ and it is not contained in $Z(R)$. Then R is a PI-algebra that satisfies the standard identity S_4 and K is an abelian Lie algebra. In particular $K_a = \{0\}$. $\square$

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